MULTICUT AND INTEGRAL MULTIFLOW IN RINGS

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Abstract. We show how to solve in polynomial time the multicut and the maximum integral multiflow problems in rings. For the latter problem we generalize an approach proposed by Sai Anand and Erlebach for special cases of the call control problem in ring networks. Moreover, we give linear-time procedures to solve both problems in rings with uniform capacities.

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1. Introduction

Let $\mathfrak{R} = (V, E)$ be a ring, i.e. a connected graph where all vertices have degree 2, with a positive integral capacity, or weight, u_e on each edge e of E, and let $\mathfrak{L}$ be a list of K pairs of terminals $\{s_k, t_k\}$, $k \in \{1, \dots, K\}$, set at the vertices of $\mathfrak{R}$. Such structures are encountered for instance in telecommunications because of the deployment of fiber equipment (SONET: Synchronous Optical Networks [3, 10]). The multicut problem MCP is to find a minimum weight set of edges whose removal separates s_k from t_k for each pair $\{s_k, t_k\}$ of $\mathfrak{L}$. Associate a commodity with each pair $\{s_k, t_k\}$: the maximum integral multiflow problem $IMFP$ consists in maximizing the sum over all commodities of the integral flow corresponding to a commodity subject to capacity and flow conservation requirements. For $K = 1$ the problems are the classical min cut-max flow problems solvable in polynomial time but both problems are known to be NP-hard and APX-hard for $K \geq 3$, even in planar graphs [7].

In an undirected (or bidirectional) ring the flow routed from s_k to t_k is splittable into two parts. One part is routed in a clockwise direction and the other in a counterclockwise direction. In order to separate s_k from t_k, the multicut must contain at least one edge between s_k and t_k in each direction. In a directed (or unidirectional) ring the flow from s_k to t_k is routed entirely in a clockwise direction; the multicut must contain at least one arc on the unique path from s_k to t_k. We will see that an instance of $IMFP$ or MCP in an undirected ring can be reduced to an instance of the same problem in a directed one.

Consider a directed ring $\mathfrak{R} = (V, A)$. Let p_k be the only path from s_k to t_k, let f_k, $k \in \{1, \ldots, K\}$, be the flow routed on p_k, and let c_e, $e \in A$, be a binary variable such that $c_e = 1$ if the arc e belongs to the cut, $c_e = 0$ otherwise. MCP and $IMFP$ can be stated as two integer linear programs whose continuous relaxations are dual [6]:

$$\left| \begin{array}{lll} (P - IMFP) & & \\ \text{Max} & \displaystyle\sum_{k=1}^{K} f_k & \\ \text{s. t.} & \displaystyle\sum_{k\ st\ e \in p_k} f_k \ \leq u_e & \forall e \in A \quad (1) \\ & f_k \in \mathbb{N} & \forall k \in \{1, \ldots, K\} \end{array} \right.$$

$$\left| \begin{array}{lll} (P - MCP) & & \\ \text{Min} & \displaystyle\sum_{e \in A} u_e\, c_e & \\ \text{s. t.} & \displaystyle\sum_{e \in p_k} c_e \ \geq 1 & \forall k \in \{1, \ldots, K\} \quad (2) \\ & c_e \in \{0, 1\} & \forall e \in A \end{array} \right.$$

Except for some special cases like directed trees [4], there is generally a gap between the optimal values of MCP and $IMFP$. This is also the case in rings. An example is given by a directed ring with 3 vertices v_1, v_2, v_3, 3 arcs of weight/capacity 5 and 3 pairs $\{s_k, t_k\}$ such that $s_1 = t_2 = v_1$, $s_2 = t_3 = v_2$ and $s_3 = t_1 = v_3$. Here, the optimal values of MCP and $IMFP$ are 10 and 7 respectively.

If we consider the special case of $IMFP$ where all the values on the edges are equal to 1 we get $MEDP$, the maximum edge disjoint paths problem, which is polynomial in rings [11]. Some authors consider the multicommodity flow problem with demands in which one wishes to send d_k units of flow from s_k to t_k, $k \in \{1, \ldots, K\}$. Let n be the number of vertices and edges (or arcs) of $\mathfrak{R}$. Vachani et al. [10] proposed an $O\left(n^3\right)$ algorithm for finding an integer multiflow with demands (whenever a solution exists) on bidirectional rings with uniform capacities. The unsplittable flow problem in ring networks, where each demand must be routed entirely in a clockwise or a counterclockwise direction was shown to be NP-hard by Cosares and Saniee in [3]. Note that the problem with demands cannot be reduced to the corresponding maximization problem without loosing the ring structure [5].

In this paper, we first propose some reductions (in Section 2). Then, in Section 3, we show that in a ring MCP can be solved by using a polynomial algorithm for chain networks [4] and that an approach proposed by Sai Anand and Erlebach [2] for the call control problem can be used to solve $IMFP$ in rings; we also prove that the integrality gap is strictly smaller than 1. In Section 4 we propose $O(n)$ algorithms to solve $IMFP$ and MCP in uniform (reduced) rings, i.e. in rings where all the capacities are equal and where all the reductions have been made.

2. Simplifications and reductions

To any instance of MCP or $IMFP$ in an undirected ring we can associate an equivalent instance in a directed ring by doubling the number of terminal pairs. To each pair $\{s_k, t_k\}$, $k \in \{1, \ldots, K\}$, we associate a new pair $\{s_{k+K}, t_{k+K}\}$ where s_{k+K} (resp. t_{k+K}) is located at the same vertex as t_k (resp. s_k). The path from s_{k+K} to t_{k+K} in the directed ring corresponds to the path from s_k to t_k in a counterclockwise direction of the undirected ring. It is clear that to any solution of MCP or $IMFP$ obtained in the directed (resp. undirected) ring corresponds a solution with the same value in the undirected (resp. directed) ring. Now, we show how to simplify a *directed* instance in such a way that the resulting instance has a source and/or a sink located at each vertex, and contains only proper pairs, i.e. no path p_i is included into p_j for $i \neq j$.

Contracting paths. A path without terminals, except for its endpoints, can be replaced by a single arc, which is the lowest weighted arc of the path. This arc is the one which limits any flow routed on this path and which would possibly be selected in a minimum multicut: the other arcs do not play any role and can be suppressed. Now, consider two adjacent arcs (u, v) and (v, w) such that there is only a source s_k located at v. If the capacity of (u, v) is greater than or equal to the one of (v, w) then the arc (u, v) can be contracted into a single vertex $(u = v)$, since (v, w) can be preferred to (u, v) in any minimal cut and since (v, w) is more constraining for the flow than (u, v). Now, s_k is located at u. In the same way, if there is only a sink located at v and if the capacity of (u, v) is smaller than or equal to the one of (v, w) then (v, w) can be contracted into a single vertex.

Suppressing pairs. Each pair $\{s_k, t_k\}$ corresponding to a path p_k which contains as subpath (or which is equivalent to) a path p_j can be removed from the list $\mathfrak{L}$. Indeed, first, a multicut for $\mathfrak{L} - \{s_k, t_k\}$ is a multicut for $\mathfrak{L}$ since if s_j is separated from t_j, so is s_k from t_k. Second, any multiflow f with a positive subflow f_k routed from s_k to t_k can be replaced by an equivalent multiflow f' with $f'_k = 0$, $f'_j = f_j + f_k$ and $f'_i = f_i$ for all $i \neq j, k$. We call a set of pairs satisfying this property a "proper set" of pairs (as in [2]). Note that a proper set of pairs contains at most n pairs. Indeed, two pairs in $\mathfrak{L}$ cannot have their sources at the same vertex.

The three reductions (contracting paths or arcs and suppressing pairs) must be iterated recursively until no more reduction can be made. Since at least one arc or one pair is suppressed by each reduction, the ring can be reduced in $O(K + n)$ time. Note that the suppressing pairs reduction can be efficiently implemented by using a stack (each pair will be seen twice).

Therefore, for the remainder of the paper, it is no loss of generality to consider a reduced ring, denoted by $\mathfrak{R} = (V, A)$, directed in a clockwise direction, and to assume that there is a source and/or a sink located at each vertex and that the list $\mathfrak{L}$ is a proper set of pairs. The vertices and the arcs are numbered from 1 to n, and the sources and the sinks are numbered from 1 to K, $K \leq n$, in a clockwise direction.

3. A polynomial algorithm in general rings

In this section, we give polynomial algorithms to solve MCP and $IMFP$.

Theorem 1. *MCP can be solved in $O\left(n^2\right)$ time in rings.*

Proof. The initial reduction of the ring runs in $O\left(K+n\right)$. MCP in a directed path can be solved by recursive algorithms in $O(K+n)$ (e.g. [9]), and thus in $O(n)$ here. Any path p_k of $\mathfrak{R}$ must contain a cut arc. The idea is to select a path, say p_l, preferably the shortest one, and to compute for each arc e_i of p_l the value of a minimum multicut on a directed path μ_i obtained by removing e_i from $\mathfrak{R}$ (and all the pairs $\{s_k, t_k\}$ such that $e_i \in p_k$). If C_i denotes the minimal multicut on μ_i, then $C_i \cup \{e_i\}$ is a multicut for the initial ring and its value is $v(C_i) + u_{e_i}$, where $v(C_i)$ is the value of C_i. A minimal multicut C^* in $\mathfrak{R}$ is given by $C_{i^*} \cup \{e_{i^*}\}$ with $e_{i^*} = \text{argmin}_{e_i \in p_l}(v(C_i) + u_i)$. It is obtained in $O\left(n^2\right)$ steps. □

Now, we show that IMFP is polynomial-time solvable in rings by using ideas similar to the ones given in [2, pp. 26–29] to solve special cases of the Pre-routed Call Admission Control problem (PCAC) in rings: roughly speaking, PCAC in rings is equivalent (at least in the unweighted case) to IMFP in directed rings when the flow routed from s_k to t_k can be either 0 or 1 for each k. We show the following theorem:

Theorem 2. *IMFP can be solved in polynomial time in rings.*

Proof. We make a binary search on the value of $\sum_{i=1}^{K} f_i \leq K u_{max}$, where u_{max} is the maximum arc capacity of the ring. Once the value of this sum is fixed and equal to an integer F, we can add to $(P - IMFP)$ the constraint (Λ_0): $\sum_{i=1}^{K} f_i = F$. Our problem is now a decision problem: does there exist a feasible solution? Consider the constraint matrix of $(P - IMFP)$. There are two kinds of rows: either the 1's are consecutive or they are not, and in this case the 0's are consecutive (this property is known as the *circular 1's property*). For each constraint (Λ) involving non consecutive flows (i.e. where the 1's are not consecutive), we define a new constraint $(\Lambda') \leftarrow (\Lambda_0) - (\Lambda)$ and remove (Λ): in (Λ'), all the 1's are consecutive. Therefore, we obtain an equivalent problem whose $0-1$ constraint matrix verifies the consecutive 1's property (interval matrix) and thus is totally unimodular [1], with an integer vector on the right-hand side. Hence, there exists an integer solution if and only if there exists a fractional solution. Since each interval matrix is a network matrix, the solution can be found in strongly polynomial time [8]. □

Now, note that our approach also implies that:

Corollary 1. *In rings, the gap between the value of the maximum fractional multiflow and the value of the maximum integer multiflow is strictly smaller than* 1.

Proof. The existence of a fractional solution of value F^* implies the existence of a fractional solution of value F for each real $F \leq F^*$ and, in particular, for $\lfloor F^* \rfloor$. Now, apply the transformation of constraints described in the proof of Theorem 2 with (Λ_0): $\sum_{i=1}^{K} f_i = \lfloor F^* \rfloor$. The resulting (equivalent) program has a totally unimodular constraint matrix with an integral vector on the right-hand side. This implies the existence of an integer solution of value $\lfloor F^* \rfloor$. □

4. The case of uniform rings

In this section, we propose $O(n)$ algorithms to solve MCP and $IMFP$ in uniform (reduced) rings, i.e. where all the arcs have the same capacity, denoted by U. First, note that, in uniforms rings, we can assume w.l.o.g. that there is exactly one source and one sink at each vertex of the ring. This results from the reductions made in Section 2. Indeed, if only one terminal is located at a vertex v then one of the two arcs indicent to v is contracted. Hence, the number of terminal pairs is now equal to the number of vertices, i.e. $K = n$, and all the paths have the same length, denoted by L. As previously, let the terminal pairs be numbered from 1 to n in a clockwise direction: there are exactly L successive flows routed through each arc, assuming that f_1 follows f_n.

Now, we claim that optimal solutions for the continuous relaxations of $(P-IMFP)$ and $(P-MCP)$ in rings with a uniform capacity U are easy to obtain. Indeed, one has:

Lemma 1. *$f_k = \frac{U}{L}$ for all $k \in \{1, \ldots, n\}$ and $c_e = \frac{1}{L}$ for all $e \in A$ are two optimal solutions of value $\frac{nU}{L}$ of the continuous relaxations of $(P-IMFP)$ and $(P-MCP)$ respectively.*

Proof. First, each arc belongs to exactly L paths p_k, thus the constraints (1) in $(P-IMFP)$ are satisfied since $\sum_{k\, st\, e\in p_k} f_k = L\frac{U}{L} = U\ \forall e \in A$. Second, every path p_k has L arcs, thus the constraints (2) in $(P-MCP)$ are also satisfied since here $\sum_{e\in p_k} c_e = L\frac{1}{L} = 1\ \forall k \in \{1, \ldots, n\}$. Finally, the values of both objective functions at these points are equal to $\frac{nU}{L}$, thus, by linear programming duality, they are optimal. □

Theorem 3. *MCP can be solved in $O(n)$ in uniforms rings.*

Proof. From Lemma 1 and since c_e is integral for each arc e, we obtain $\sum_{e\in A} c_e \geq \lceil \frac{n}{L} \rceil$. This implies $\sum_{e\in A} u_e c_e \geq \lceil \frac{n}{L} \rceil U$. Eventually, one can define a multicut of value $\lceil \frac{n}{L} \rceil U$ (and thus optimal). Indeed, for $j \in \{1, \ldots, n\}$, set $c_{e_j} = 1$ if $j \in \{1 + pL, p \in \{0, \ldots, \lceil \frac{n}{L} \rceil - 1\}\}$, and $c_{e_j} = 0$ otherwise. This provides a feasible multicut with $\lceil \frac{n}{L} \rceil$ arcs in $O(n)$. □

Consider now the following algorithm:

Algorithm 1 Maxmultiflow_uniform_rings

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Ensure: An integral multiflow f̂ = { f̂_j }_{j∈{1,...,n}} of value ⌊nU/L⌋
  f̂_1 = ⌊U/L⌋;
  for j = 1 to n − 1 do
    if j U/L − ⌊j U/L⌋ ≥ ⌈U/L⌉ − U/L then
      f̂_{j+1} = ⌈U/L⌉;
    else
      f̂_{j+1} = ⌊U/L⌋;
    end if
  end for
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The main idea of this algorithm is to round down or up the values of the variables in the optimal continuous solution of Lemma 1 in order to keep at each step j a gap smaller than one between the sum of the $j+1$ first integral flows and $(j+1)\frac{U}{L}$, the corresponding sum of the continuous flows. The aim is to obtain an integral multiflow of total value $\lfloor \frac{nU}{L} \rfloor$, and thus optimal.

Lemma 2. *For $1 \leq k \leq n$, one has $\sum_{j=1}^{k} \hat{f}_j = \lfloor k\frac{U}{L} \rfloor$.*

Proof. The result is obtained by induction on k. The property is true when $k = 1$. Let k be in $\{1, \ldots, n-1\}$ and assume that $\sum_{j=1}^{k} \hat{f}_j = \lfloor k\frac{U}{L} \rfloor$. We have $\sum_{j=1}^{k+1} \hat{f}_j = \lfloor k\frac{U}{L} \rfloor + \hat{f}_{k+1}$. If in Algorithm 1 one has $(k+1)\,\frac{U}{L} \geq \lfloor k\frac{U}{L} \rfloor + \lceil \frac{U}{L} \rceil$ then, the right-hand part being integral, this implies $\lfloor (k+1)\frac{U}{L} \rfloor \geq \lfloor k\frac{U}{L} \rfloor + \lceil \frac{U}{L} \rceil$. In this case we have $\hat{f}_{k+1} = \lceil \frac{U}{L} \rceil$. We get $\lfloor (k+1)\frac{U}{L} \rfloor \leq \lfloor k\frac{U}{L} \rfloor + \lceil \frac{U}{L} \rceil = \sum_{j=1}^{k+1} \hat{f}_j \leq \lfloor (k+1)\frac{U}{L} \rfloor$ and finally $\sum_{j=1}^{k+1} \hat{f}_j = \lfloor (k+1)\frac{U}{L} \rfloor$. Otherwise one has $(k+1)\,\frac{U}{L} < \lfloor k\frac{U}{L} \rfloor + \lceil \frac{U}{L} \rceil$. That implies $\lfloor (k+1)\frac{U}{L} \rfloor < \lfloor k\frac{U}{L} \rfloor + \lceil \frac{U}{L} \rceil$ and thus $\lfloor (k+1)\frac{U}{L} \rfloor \leq \lfloor k\frac{U}{L} \rfloor + \lfloor \frac{U}{L} \rfloor$. In this case we have $\hat{f}_{k+1} = \lfloor \frac{U}{L} \rfloor$ and since $\lfloor k\frac{U}{L} \rfloor + \lfloor \frac{U}{L} \rfloor \leq \lfloor (k+1)\frac{U}{L} \rfloor$ we obtain $\sum_{j=1}^{k+1} \hat{f}_j = \lfloor k\frac{U}{L} \rfloor + \lfloor \frac{U}{L} \rfloor = \lfloor (k+1)\frac{U}{L} \rfloor$. □

Theorem 4. *IMFP can be solved in $O(n)$ time in uniforms rings.*

Proof. The complexity of Algorithm 1 is obviously $O(n)$, and the integral multiflow $\hat{f}$ has a total value of $\lfloor \frac{nU}{L} \rfloor$ (it is the particular case of Lemma 2 where $k = n$). Thus we have just to prove that the multiflow $\hat{f}$ actually satisfies the capacity constraints on the arcs. Since any arc e belongs to L successive commodities, we are going to prove that the sum of any L consecutive integral flows is always less than or equal to U. We consider two cases. First, assume that $\hat{f}_1$ or $\hat{f}_n$ does not belong to the sequence of flows. Thanks to Lemma 2, we have $\sum_{j=1}^{L} \hat{f}_j = \lfloor L\frac{U}{L} \rfloor = U$ and thus the property is satisfied for the sequence $\hat{f}_1, \ldots, \hat{f}_L$. Moreover, for $1 \leq j \leq L-1$, we have $(L+j)\frac{U}{L} - \lfloor (L+j)\frac{U}{L} \rfloor = j\frac{U}{L} - \lfloor j\frac{U}{L} \rfloor$. This implies that the test condition in Algorithm 1 will be the same for $\hat{f}_j$, $\hat{f}_{L+j}, \ldots, \hat{f}_{\alpha L+j}$ (α integer s.t. $\alpha L + j \leq n$). Let $a_k \in \mathbb{N}$, $b_k \in \mathbb{N}$, $b_k < L$ be such that $k = a_k L + b_k$; then $\hat{f}_{k+j} = \hat{f}_{a_k L + b_k + j} = \hat{f}_{b_k + j}$. For any sequence $\hat{f}_{k+1}, \ldots, \hat{f}_{k+L}$ where $1 \leq k \leq n-L$ we get $\sum_{j=1}^{L} \hat{f}_{k+j} = \sum_{j=1}^{L} \hat{f}_{b_k+j} = \sum_{j=b_k+1}^{L} \hat{f}_j + \sum_{j=L+1}^{L+b_k} \hat{f}_j = \sum_{j=b_k+1}^{L} \hat{f}_j + \sum_{j=1}^{b_k} \hat{f}_j = \sum_{j=1}^{L} \hat{f}_j = U$.

Second, assume that $\hat{f}_1$ and $\hat{f}_n$ belong to the sequence, which thus can be written as $\hat{f}_{n-L+a+1}, \ldots, \hat{f}_n, \hat{f}_1, \ldots, \hat{f}_a$ for some a. Applying Lemma 2 we get $\sum_{j=1}^{a} \hat{f}_j = \lfloor a\frac{U}{L} \rfloor$ (for $k = a$), $\sum_{j=1}^{n-L+a} \hat{f}_j = \lfloor (n-L+a)\,\frac{U}{L} \rfloor = \lfloor (n+a)\,\frac{U}{L} \rfloor - U$ (for $k = n-L+a$), and $\sum_{j=1}^{n-L+a} \hat{f}_j + \sum_{j=n-L+a+1}^{n} \hat{f}_j = \lfloor n\frac{U}{L} \rfloor$ (for $k = n$). Thus we obtain $\sum_{j=1}^{a} \hat{f}_j + \sum_{j=n-L+a+1}^{n} \hat{f}_j = \lfloor a\frac{U}{L} \rfloor + \lfloor n\frac{U}{L} \rfloor - \lfloor (n+a)\,\frac{U}{L} \rfloor + U$. Since $\lfloor a\frac{U}{L} \rfloor + \lfloor n\frac{U}{L} \rfloor \leq \lfloor (n+a)\,\frac{U}{L} \rfloor$, we finally get $\sum_{j=1}^{a} \hat{f}_j + \sum_{j=n-L+a+1}^{n} \hat{f}_j \leq U$.

□

5. Conclusion

We have proposed polynomial-time algorithms to solve MCP and $IMFP$ in rings, and linear-time algorithms for reduced rings with uniform capacities. Generally the complexity and resolution of MCP and $IMFP$ are affected by considering directed or undirected graphs [5], but we have shown that it is not the case in rings. Note that in these graphs the Multiterminal Flow and Cut problems are trivially solved in $O\,(K+n)$. Recall that in these particular cases of $IMFP$ and MCP, the pairs of terminals are $\{s_k, s_{k'}\}$ for all $k \neq k'$ in $\{1, \ldots, K\}$. Here, both optimum values of these problems are equal to the sum of the values of the edges remaining in the ring after simplifications.

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