Bounded Discrete Walks

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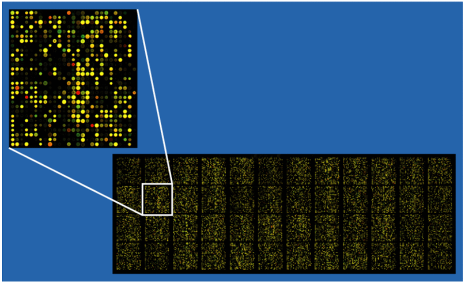
Tortoiseshell cats - Chats Isabelle

The patchy colours of a tortoiseshell cat are the result of different levels of expression of pigmentation genes in different areas of the skin.

DNA micro-array and diagnostic in genetics?



measuring fluorescence of a spot provides the level of expression of the corresponding gene.

- $\Gamma :=$ set of genes of a background population $(|\Gamma| = G)$
- $\gamma :=$ set of genes of a particular family $(|\gamma| = g)$

Question. Are the levels of expression of the genes of γ with respect to the level of expressions of the genes of Γ characteristic of an exceptional behaviour? (Disease, *etc*)

Random walks model

- Keller, Backes and Lenhof (2007)

Order the genes by level of expression.

Build a walk $(B_i)_{0 \le i \le G+g}$ such that $B_0 = 0$ and

$$B_i = \begin{cases} +G & \text{if the gene at rank } i \text{ belongs to } \gamma, \\ -g & \text{if the gene at rank } i \text{ belongs to } \Gamma \end{cases}$$

These walks are therefore **bridges** as $B_{G+g} = Gg - gG = 0$.

exceptional overexpression of the genes of $\gamma \quad \Longrightarrow$
exceptional height of the bridge with respect to the height of a bridge chosen **at random** among the $\binom{G+g}{g}$ possible bridges.

Example

$$G = 6, \quad g = 3$$

$$\text{represent by} \begin{cases} \circ & \text{genes of } \Gamma \\ \bullet & \text{genes of } \gamma \end{cases}$$

Pattern of level expression

High level $\longrightarrow$ Low level

$\circ \quad \bullet \quad \bullet \quad \circ \quad \bullet \quad \circ \quad \circ \quad \circ \quad \circ$

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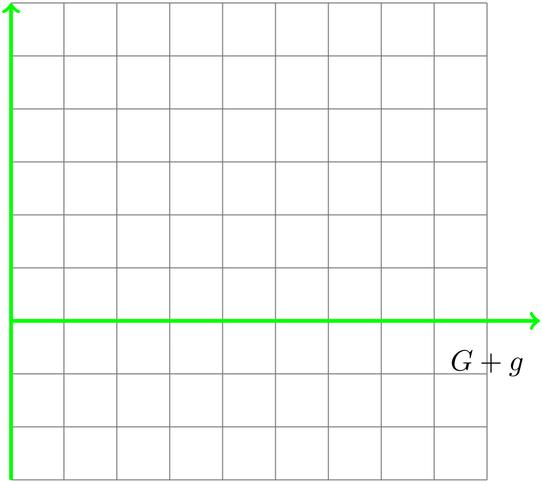
● $\rightsquigarrow X_j = +2 \quad (+6/3)$

○ $\rightsquigarrow X_j = -1 \quad (-3/3)$



$$\color{red}{B_i} = \sum_{1 \leq j \leq i} X_j$$

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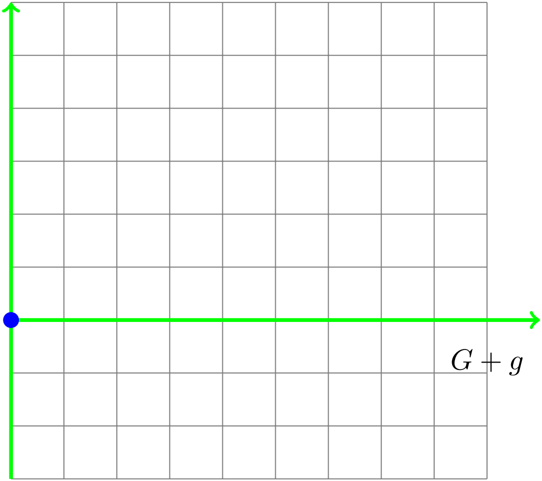

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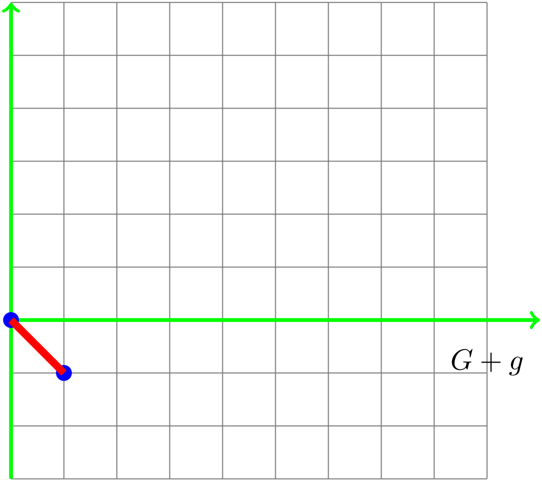

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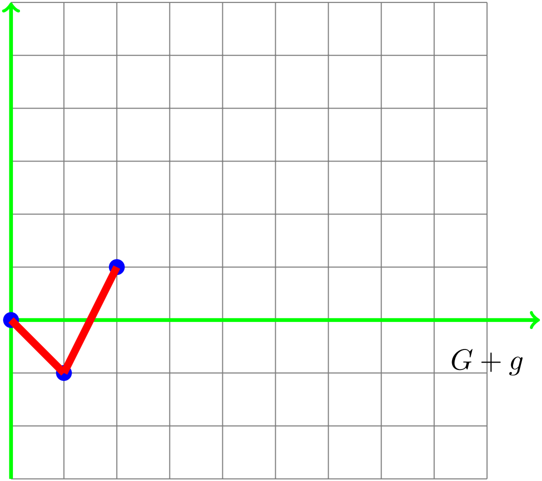

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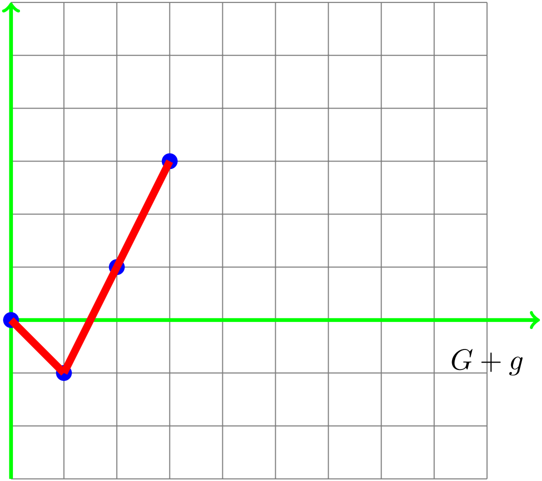

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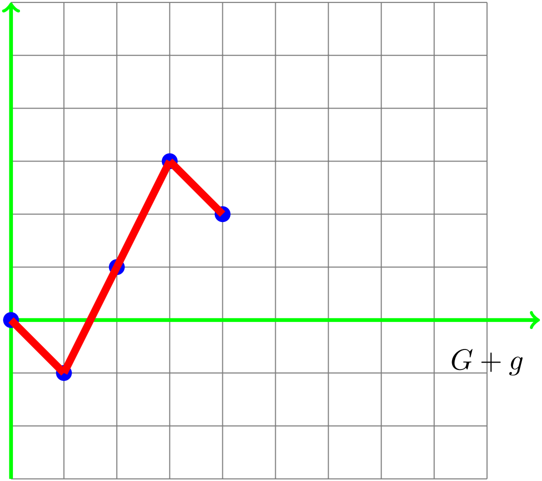

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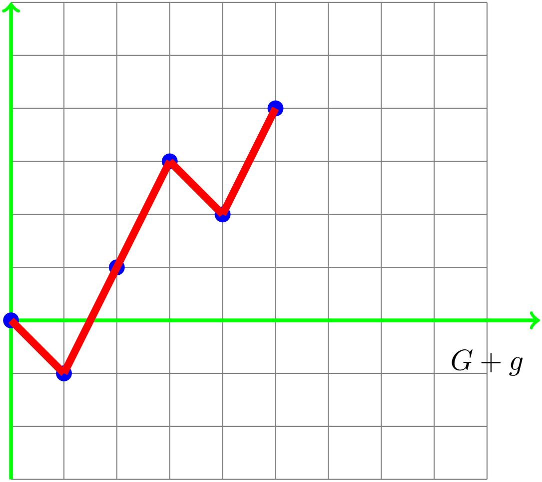

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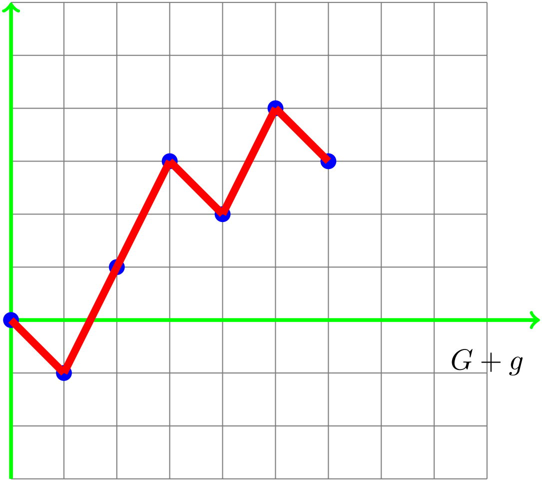

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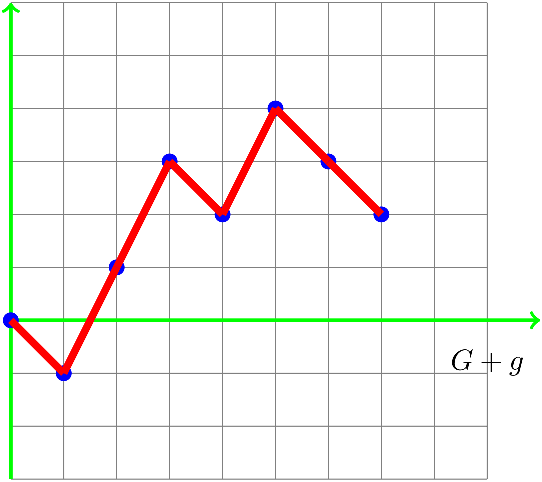

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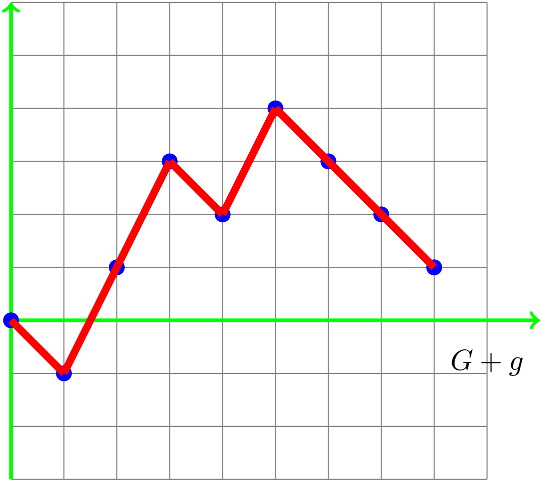

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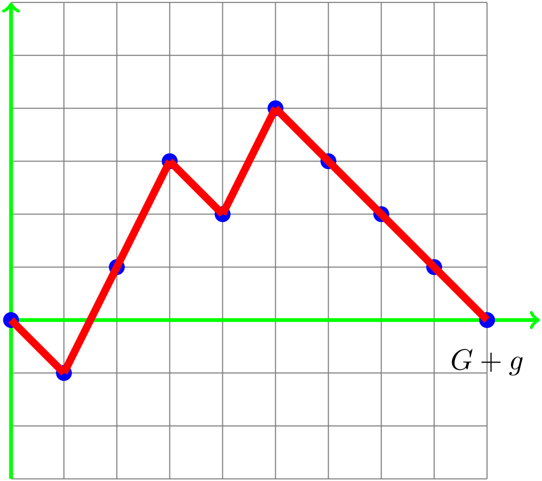

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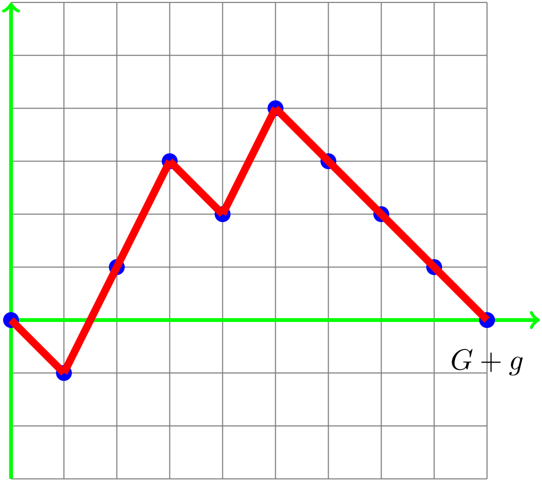


$$B_i = \sum_{1 \leq j \leq i} X_j$$

Number of discrete bridges of length n

- i.i.d. **integer jumps** $X_i \in \{-c, \dots, +d\}$
- characteristic polynomial $P(u) = p_{-c}u^{-c} + \cdots + p_d u^d$
- $P(1) = 1$
- $\mathbf{E}(X_i) = P'(1) = 0$

Typical generating function

$$F(z,u) = \sum_{n=0}^{\infty} f_n(u) z^n = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} f_{n,k} u^k z^n$$

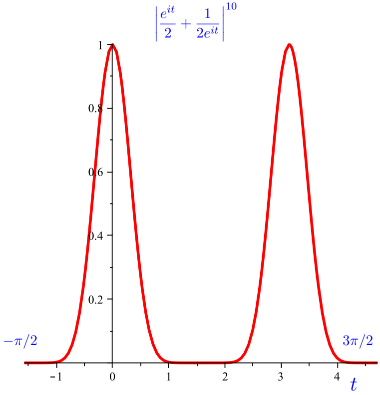
where

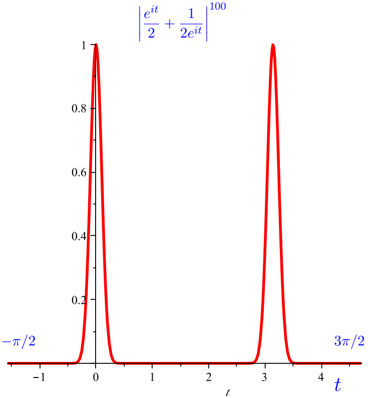
- $f_{n,k}$ is the probability that a walk remaining in an horizontal strip (by instance $]-\infty, +h]$) has **altitude** k at **time** n.

$[z^n][u^0]F(z,u)$ corresponds to **bridges** of **length** n

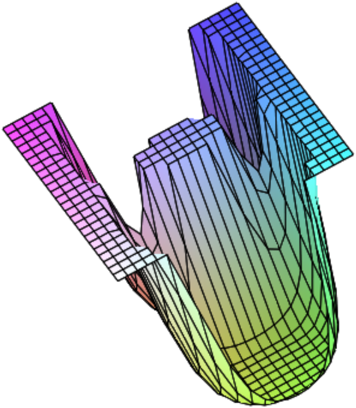
Asymptotics of the number of bridges

$$B_n = [u^0]P^n(u) = \frac{1}{2i\pi}\oint_{|u|=1} \frac{P^n(u)}{u}du$$





Saddle-points and 3D-landscape of $\left|\frac{u}{2}+\frac{1}{2u}\right|^{10}$



Saddle-point expansion

Recall $P(1) = 1$ and $P'(1) = 0$

$$\begin{aligned}
B_n &= \frac{1}{2i\pi} \oint_{|u|=1} \frac{P^n(u)}{u} du \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(n \log(P(e^{it})))dt \quad (u = e^{it}) \\
&\approx \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \exp(-n\sigma^2 t^2/2) \exp(n(i\alpha_3 t^3 + \alpha_4 t^4 + \dots))dt \quad (\sigma^2 = P''(1)) \\
&\approx \frac{1}{2\pi\sigma\sqrt{n}} \int_{-\infty}^{+\infty} e^{-s^2/2} \left(1 + \frac{\beta_1}{\sqrt{n}} s^3 + \frac{\beta_2}{n} s^4 + \dots \right) ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi n}} + \gamma_3 \times \frac{1}{n^{3/2}} + \gamma_5 \times \frac{1}{n^{5/2}} + \dots
\end{aligned}$$

I am looking for the asymptotics expansion of [u^0]P^n(u) with jumps (+d,-1) by saddle-point (P(1)=1; P'(1)=0)

```
> P:=u->u^d/(d+1)+d/u/(d+1);
```

$$P := u \to \frac{u^d}{d+1} + \frac{d}{u\,(d+1)} \qquad (1)$$

u=exp(I*s) I expand the log

```
> SS:=simplify(series(n*log(P(exp(I*s))),s=0,5));
```

$$SS := -\frac{1}{2}\,d\,n\,s^2 - \frac{1}{6}\,\mathrm{I}\,(d-1)\,d\,n\,s^3 + \frac{1}{24}\,(d^2-4\,d+1)\,d\,n\,s^4 + \mathrm{O}(s^5) \qquad (2)$$

ds=dt/sqrt(d)/sqrt(n)

```
> PP:=subs(s=t/sqrt(d)/sqrt(n),SS);
```

$$PP := -\frac{1}{2}\,t^2 - \frac{\frac{1}{6}\,\mathrm{I}\,(d-1)\,t^3}{\sqrt{d}\,\sqrt{n}} + \frac{1}{24}\,\frac{(d^2-4\,d+1)\,t^4}{d\,n} + \mathrm{O}\left(\frac{t^5}{d^{5/2}\,n^{5/2}}\right) \qquad (3)$$

I compute the expansion of the left exponential at t=0

```
> QQ:=convert(series(exp(PP+t^2/2),t=0,6),polynom);
```

$$QQ := 1 - \frac{\frac{1}{6}\,\mathrm{I}\,(d-1)\,t^3}{\sqrt{d}\,\sqrt{n}} + \frac{1}{24}\,\frac{(d^2-4\,d+1)\,t^4}{d\,n} \qquad (4)$$

```
> f:=k->int(exp(-x^2/2)*x^k,x=-infinity..+infinity);
```

$$f := k \to \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\,x^2}\,x^k\,\mathrm{d}x \qquad (5)$$

I put back the coefficient of dt in ds

```
> AA:=asympt(1/2/Pi/sqrt(d)/sqrt(n)*add(coeff(QQ,t,i)*f(i),i=0..6),
  n,10);
```

$$AA := \frac{1}{2}\,\frac{\sqrt{2}\,\sqrt{\frac{1}{n}}}{\sqrt{\pi}\,\sqrt{d}} + \frac{1}{16}\,\frac{(d^2-4\,d+1)\,\sqrt{2}\,\left(\frac{1}{n}\right)^{3/2}}{\sqrt{\pi}\,d^{3/2}} \qquad (6)$$

I check with jumps (+1,-1)

```
> subs(d=1,AA);
```

$$\frac{1}{2}\,\frac{\sqrt{2}\,\sqrt{\frac{1}{n}}}{\sqrt{\pi}} - \frac{1}{8}\,\frac{\sqrt{2}\,\left(\frac{1}{n}\right)^{3/2}}{\sqrt{\pi}} \qquad (7)$$

```
> subs(n=2*n,%);
```

I compute directly the asymptotics of bridges (+1,-1) !!!! I miss a factor 2

```
> [n!/(n/2)!/(n/2)!/2^n,asympt(n!/(n/2)!/(n/2)!/2^n,n,3)];
```

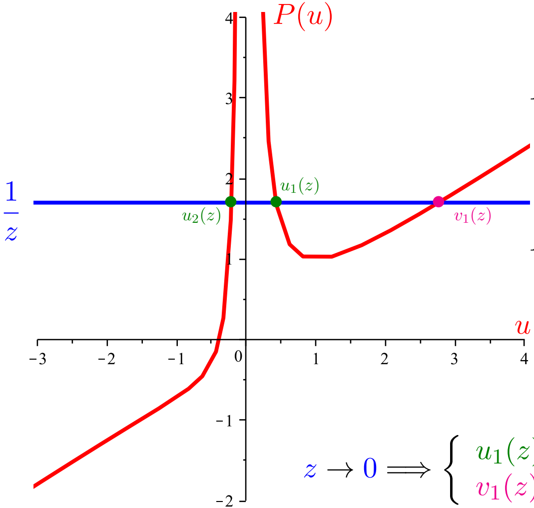
$$\left[\frac{n!}{\left(\frac{1}{2}\,n\right)!^2\,2^n},\ \frac{\sqrt{2}\,\sqrt{\frac{1}{n}}}{\sqrt{\pi}} - \frac{1}{4}\,\frac{\sqrt{2}\,\left(\frac{1}{n}\right)^{3/2}}{\sqrt{\pi}} + \mathrm{O}\left(\left(\frac{1}{n}\right)^{5/2}\right)\right] \qquad (8)$$

Generating function of the bridges

$$F(z,u) = \sum_{n\geq 0} f_{n,k} u^k z^n = \sum_{n\geq 0} z^n P^n(u) = \frac{1}{1 - zP(u)}$$

$f_{n,k}$ is the probability that the walk is at height k at time n.
The poles $u_i(z)$ and $v_j(z)$ of $F(z,u)$ $u_i(z)$ and $v_j(z)$ verify

$$1 - zP(u_i(z)) = 0, \qquad 1 - zP(v_j(z)) = 0$$



$$P(u) = \frac{4}{7}u + \frac{2}{7u} + \frac{1}{7u^2}$$

$$K(z,u) = 1 - z \times P(u)$$

$$z \to 0 \Longrightarrow \begin{cases} u_1(z), u_2(z) \to 0 & \text{small roots} \\ v_1(z) \to +\infty & \text{large root} \end{cases}$$

Getting the generating function of bridges

$$B(z) = [u^0]\frac{1}{1 - zP(u)} = \frac{1}{2i\pi}\oint \frac{1}{u} \times \frac{1}{1 - zP(u)}du$$

We consider the integrand as a function of u

For a punctured contour close to the origin, the $u_i(z)$ are the poles

The residues are $R_i = -\dfrac{1}{zu_i(z)P'_u(u(z))}$

But

$$K(z,u) = 1 - zP(u(z)) = 0 \Longrightarrow \frac{d}{dz}K(z,u) = -P(u(z)) - zP'_u(u(z))u'(z) = 0$$

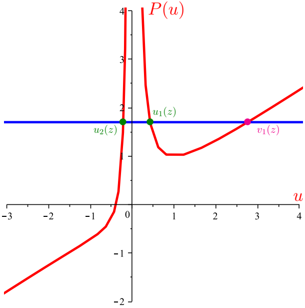
Therefore (**bridges**) $B(z) = [u^0]\dfrac{1}{1 - zP(u)} = z\displaystyle\sum_{1\leq i\leq c}\frac{u'_i(z)}{u_i(z)}$

Similarly, walks terminating at **altitude** k with $k < c$ verify

$$W_k(z) = [u^k]\frac{1}{1 - zP(u)} = z\sum_{1\leq i\leq c}\frac{u'_i(z)}{u_i^{k+1}(z)}$$

Domination properties of the roots $u_i(z)$ and $v_j(z)$

$\frac{1}{z}$



$u_1(z) < v_1(z)$ for $z < 1$

$u_i(z), v_j(z)$ $(i, j > 1)$ complex most of the time

$$\frac{1}{z} = P(u_1(z)) = P(u_j(z)) = |P(u_j(z))| < P(|u_j(z)|)$$

$P(u)$ **decreases** from $+\infty$ to 1 as u **increases** from 0 to 1
Therefore $|u_j(z)| < u_1(z)$ for $0 < z < 1$

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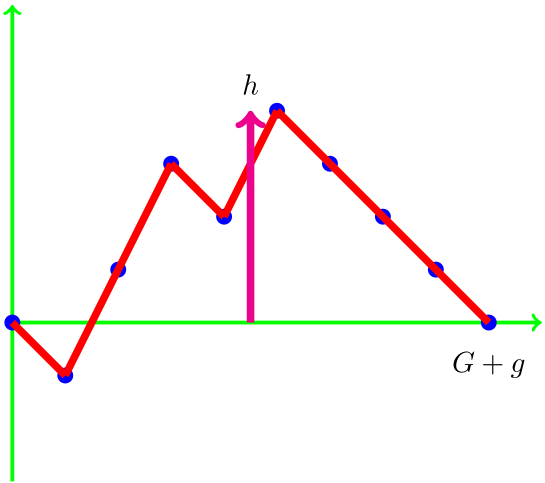

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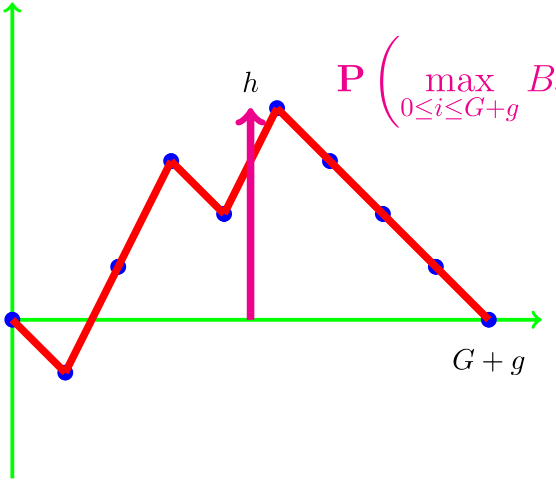
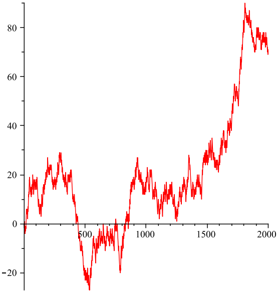


$$\mathbf{P}\left(\max_{0\leq i\leq G+g} B_i\right) > h\ ?$$

$$B_i = \sum_{1\leq j\leq i} X_j$$

Brownian motion as limit of discrete walks



Strong approximations of discrete walks by Brownian motion

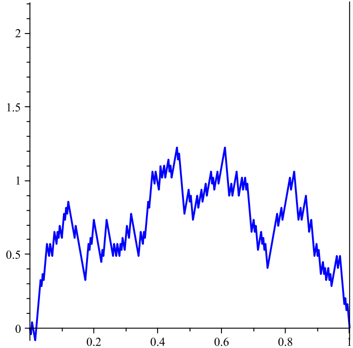
- Komlós, Major, Tusnády (1976), Chatterjee (2010)

$$\mathbf{P}\left\{\max_{1\le k\le n}|S_k - W(k)| > C\log n + x\right\} < Ke^{-\lambda x} \quad (W(k) \text{ Wiener process})$$

Little done for **approximations of bridges**

- Kaigh (1976)

Désiré André reflexion principle



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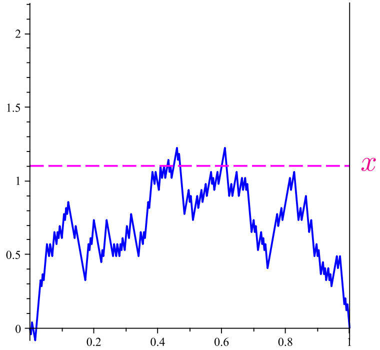

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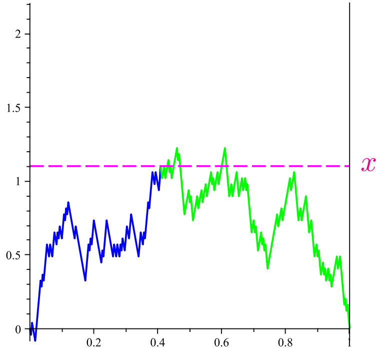

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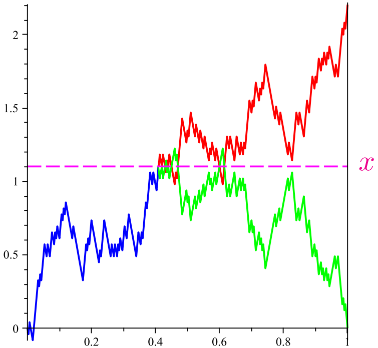
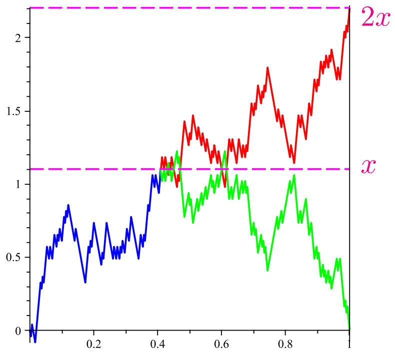


Caveat: This nice reflexion trick won't work in the discrete case if the walk has a drift or its jumps are other than $+1, 0, -1$. Another approach is needed then!

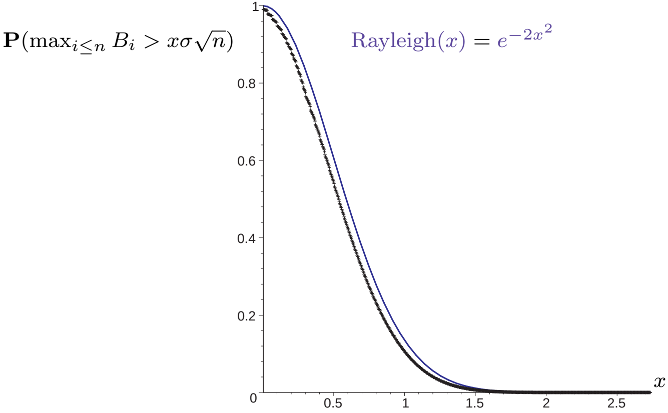
Désiré André reflexion principle



$$\Phi(2x) = \mathbf{P}(W(1) = 2x) = \frac{1}{\sqrt{2\pi}} e^{-2x^2}$$

$$\mathbf{P}\left(\max_{t\in[0,1]} B_t \geq x\right) = \frac{\Phi(2x)}{\Phi(0)} = \text{Rayleigh}(x) = e^{-2x^2}$$

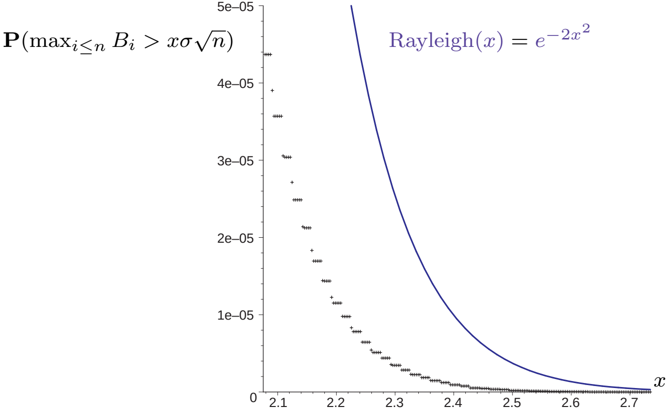
Brownian bridge versus simulations



bridge length $n = G + g = 104$ $\quad G = +93 \quad -g = -11$

10^8 simulations

Brownian bridge versus simulations



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$\sigma = \sqrt{G \times g}$ $\quad 10^8$ simulations

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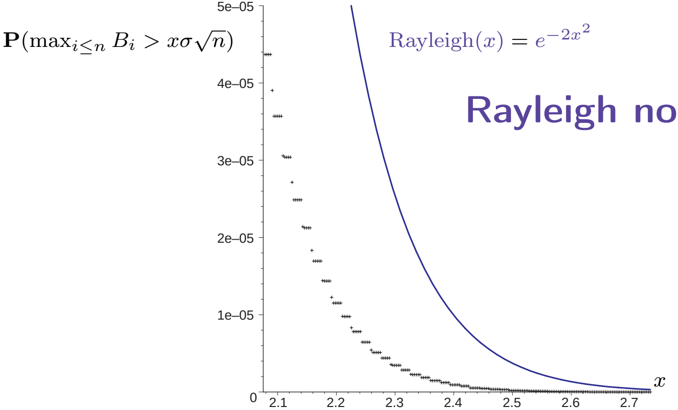

Rayleigh not that good!

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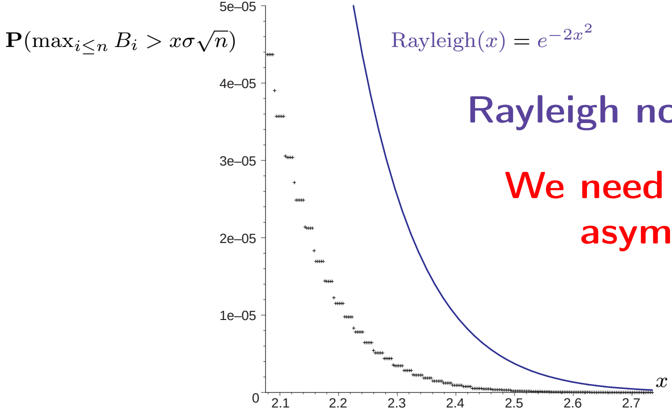

Rayleigh not that good!

We need more precise asymptotics!

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$\sigma = \sqrt{G \times g}$ $\quad 10^8$ simulations

Model for bounded walks

- i.i.d. **integer jumps** $X_i \in \{-c, \ldots, +d\}$
- characteristic polynomial $P(u) = p_{-c}u^{-c} + \cdots + p_d u^d$
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- i.i.d. **integer jumps** $X_i \in \{-c, \dots, +d\}$
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- $\mathbf{E}(X_i) = P'(1) = 0$

Typical generating function

$$F(z,u) = \sum_{n=0}^{\infty} f_n(u) z^n = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} f_{n,k} u^k z^n$$

where

- $f_{n,k}$ is the probability that a walk remaining in an **horizontal strip** (by instance $]-\infty, +h]$) has **altitude** k at **time** n.

Getting the generating function

$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u}$

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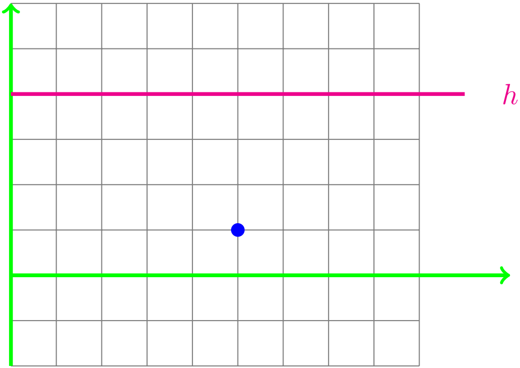

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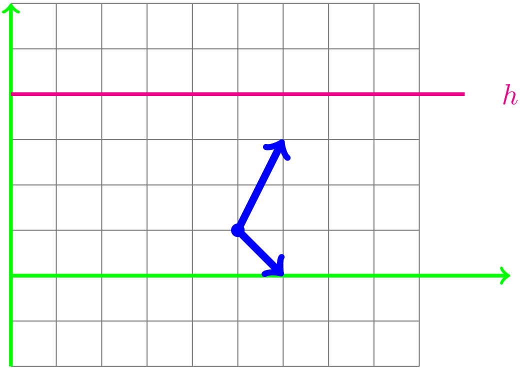

$$f_{k+1}(u) = f_k(u) \times P(u)$$

Getting the generating function

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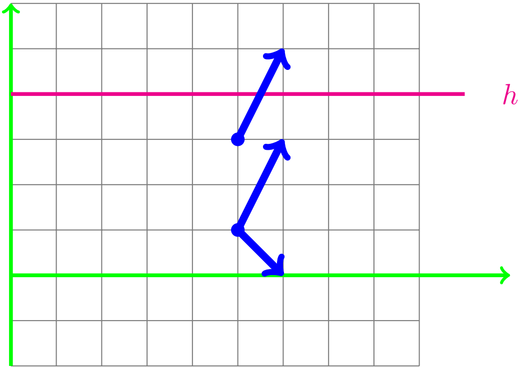

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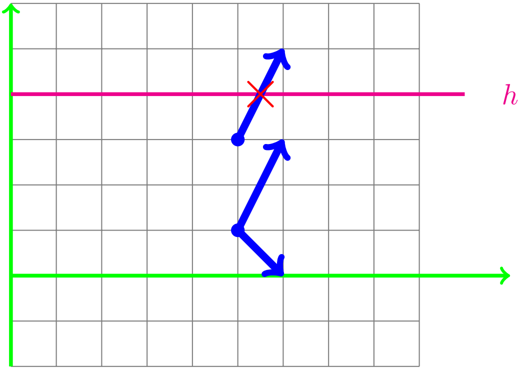

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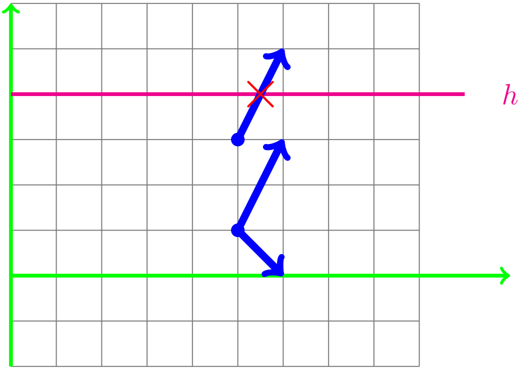

$$f_{k+1}(u) = f_k(u) \times P(u)$$
$$-u^{h+1}[u^{h+1}]f_k(u) \times P(u)$$

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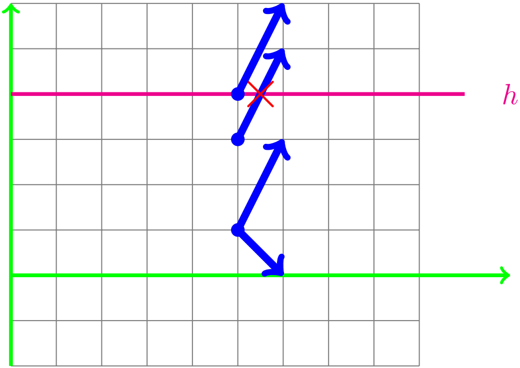

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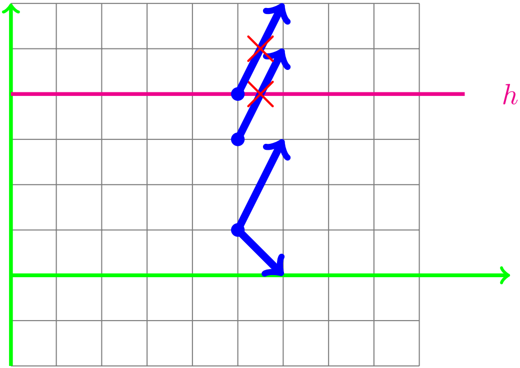

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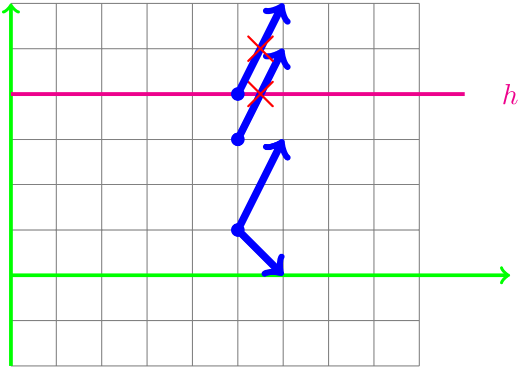

$$f_{k+1}(u) = f_k(u) \times P(u)$$
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Getting the generating function

$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u} \qquad f_0(u) = 1$

$$f_{k+1}(u) = f_k(u) \times P(u) - \begin{cases} \quad u^{h+1}[u^{h+1}]f_k(u) \times P(u) \\ +\, u^{h+2}[u^{h+2}]f_k(u) \times P(u) \end{cases}$$

Getting the generating function

$$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u} \qquad f_0(u) = 1$$

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$$\begin{aligned} \sum_{k=0}^{\infty} f_{k+1}(u) z^{k+1} &= F(z,u) - f_0(u) \\ &= zP(u) \sum_{k=0}^{\infty} f_k(u) z^k - \left\{ \begin{array}{r} zu^{h+1} F_{h+1}(z) \\ +\, zu^{h+2} F_{h+2}(z) \end{array} \right. \end{aligned}$$

Getting the generating function

$$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u} \qquad f_0(u) = 1$$

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$$= zP(u) \sum_{k=0}^{\infty} f_k(u) z^k - \begin{cases} zu^{h+1} F_{h+1}(z) \\ + zu^{h+2} F_{h+2}(z) \end{cases}$$

$$F(z,u)(1 - zP(u)) = 1 - zu^{h+1} F_{h+1}(z) - zu^{h+2} F_{h+2}(z)$$

Kernel method

Knuth, Tutte, Brown, Bousquet-Mélou, Petkovšek, *etc.*

$X_i \in \{-1, +2\}$ $\qquad P(u) = \frac{1}{u} + u^2$

$$F(z,u)(1 - zP(u)) = 1 - zu^{h+1}F_{h+1}(z) - zu^{h+2}F_{h+2}(z)$$

$F(z,u), F_{h+1}(z), F_{h+2}(z)$ **unknown functions**

but the **roots** $u(z)$ of $1 - zP(u) = 0$ **cancel** the left member of the equation

two roots $u(z)$ provide a linear system of **two** equations whose solutions are $F_{h+1}(z)$ and $F_{h+2}(z)$

General case - Any finite set of integer jumps

$$P(u) = p_{-c}u^{-c} + p_{-c+1}u^{-c+1} + \cdots + p_{d-1}u^{d-1} + p_d u^d$$

$$F(z,u)(1 - zP(u)) = 1 - zu^{h+1}F_{h+1}(z) - \cdots - zu^{h+d}F_{h+d}(z)$$

d unknown functions $F_{h+j}(z)$, but the equation $1 - zP(u) = 0$ has

$$\begin{cases} d \text{ large roots } v_i(z) & \text{such that } v_i(z) \sim \frac{1}{z}^{1/d} \quad \text{as } z \to 0 \\ c \text{ small roots } u_j(z) & \text{such that } u_j(z) \sim z^{1/c} \quad \text{as } z \to 0 \end{cases}$$

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$$\begin{cases} v_1(z)^{h+1}F_{h+1}(z) + \cdots + v_1(z)^{h+d}F_{h+d}(z) = 1/z, \\ \ldots \\ v_d(z)^{h+1}F_{h+1}(z) + \cdots + v_d(z)^{h+d}F_{h+d}(z) = 1/z \end{cases}$$

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Vandermonde determinants $\mathbb{V}(\dots)$

Nice expression for the generating function $F^{]-\infty,h]}$

$$\begin{cases} v_1(z)^{h+1}F_{h+1}(z)+\dots+v_1(z)^{h+d}F_{h+d}(z)=1/z, \\ \dots \\ v_d(z)^{h+1}F_{h+1}(z)+\dots+v_d(z)^{h+d}F_{h+d}(z)=1/z \end{cases}$$

$$F(z,u)(1-zP(u))$$

$$=1-\sum_{j=1}^{d} u^{h+j}\frac{\begin{vmatrix} v_1^{h+d} & \dots & v_1^{h+d-(j-1)} & 1 & v_1^{h+d-(j+1)} & \dots & v_1^{h+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ v_d^{h+d} & \dots & v_d^{h+d-(j-1)} & 1 & v_d^{h+d-(j+1)} & \dots & v_d^{h+1} \end{vmatrix}}{v_1^h\dots v_d^h\mathbb{V}(v_1,\dots,v_d)}$$

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Nice expression for the generating function $F^{]-\infty,h]}$

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$$= 1 - \sum_{j=1}^{d} \frac{u^{h+1}}{v_j^{h+1}} \prod_{\substack{1\le i\le d \\ i\neq j}} \frac{u - v_i}{v_j - v_i}$$

Nice expression for the generating functions

$$F^{]-\infty,h]}(z,u) = \frac{1}{1-zP(u)} - \frac{1}{1-zP(u)} \sum_{j=1}^{d} \frac{u^{h+1}}{v_j^{h+1}} \prod_{\substack{1\le i\le d\\ i\neq j}} \frac{u-v_i}{v_j-v_i}$$

N.B.: $\frac{1}{1-zP(u)}$ counts **all** the walks.

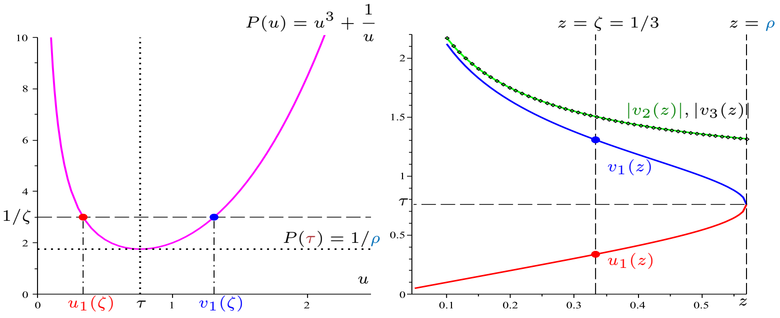
Theorem (Banderier-N. 2010)

Walks going **beyond** *the barrier* $+h$ *verify*

$$F^{[>h]}(z,u) = \frac{1}{1-zP(u)} \sum_{j=1}^{d} \frac{u^{h+1}}{v_j(z)^{h+1}} \prod_{\substack{1\le i\le d\\ i\neq j}} \frac{u-v_i}{v_j-v_i}$$

Gives fast computation scheme for the n-th coefficients via holonomy theory.

Roots properties (Banderier-Flajolet)



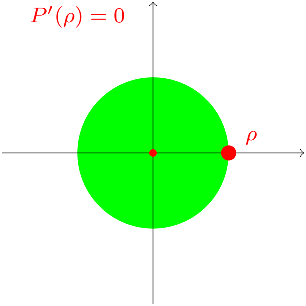
Left: behaviour of the characteristic polynomial $P(u) = u^3 + \frac{1}{u}$.

Right: domination property of the roots of $1 - zP(u) = 1 - z\left(u^3 + \frac{1}{u}\right)$ in $]0, \rho]$, where τ is the unique positive solution of $P'(z) = 0$ and $\rho = 1/P(\tau)$.

$$P'(\tau) = 0 \quad \Longrightarrow u_1(\rho) = v_1(\rho).$$

$u_1(z) < v_1(z) < |v_2(z)| = |v_3(z)|$ for $z \in]0, \rho[$.

Roots properties (Banderier-Flajolet)



for $\epsilon < |z| < \rho$

$$\max_{i\geq 2} |u_i(z)| < |u_1(z)| < |v_1(z)| < \min_{j\geq 2} |v_j(z)|$$

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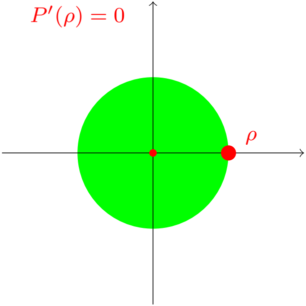
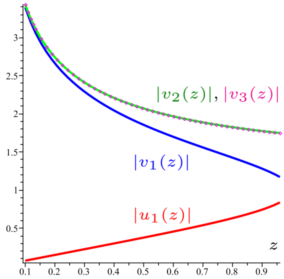




for $\epsilon < |z| < \rho$

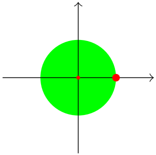
$$\max_{i\geq 2} |u_i(z)| < |u_1(z)| < |v_1(z)| < \min_{j\geq 2} |v_j(z)|$$

$$X_i \in \{+3, -1\} \quad \begin{cases} \mathbf{E}(X) = P'(1) = 0 \ (\rho = 1) \\ P(1) = 1 \end{cases}$$

$$z \sim 1^- \begin{cases} u_1(z) = 1 - \sqrt{\frac{2}{P''(1)}(1-z)} + O(1-z) \\ v_1(z) = 1 + \sqrt{\frac{2}{P''(1)}(1-z)} + O(1-z) \end{cases}$$

Asymptotics simplifications for $F^{[>h]}$ as $h \longrightarrow \infty$

$$\frac{u^{h+1}}{v_j(z)^{h+1}} = \frac{u^{h+1}}{v_1(z)^{h+1}} \left(\frac{v_1(z)}{v_j(z)}\right)^{h+1} = O(A^h)$$



$$\left(j \geq 2, \quad A = \max_{j \geq 2} \sup_{|z|<\rho-\epsilon} \frac{|v_1(z)|}{|v_j(z)|} < 1 \right)$$

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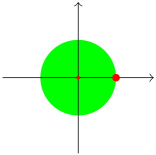

$$\left(j \geq 2, \quad A = \max_{j\geq 2} \sup_{|z|<\rho-\epsilon} \frac{|v_1(z)|}{|v_j(z)|} < 1 \right)$$

$$\Longrightarrow \quad F^{[>h]}(z,u) = \frac{1}{1-zP(u)} \sum_{j=1}^{d} \frac{u^{h+1}}{v_j(z)^{h+1}} \prod_{\substack{1\leq i\leq d \\ i\neq j}} \frac{u-v_i}{v_j-v_i}$$

$$= \frac{1}{1-zP(u)} \frac{u^{h+1}}{v_1(z)^{h+1}} \frac{Q(u)}{Q(v_1(z))} \left(1+O(A^h)\right)$$

where $Q(x) = \prod_{2\leq i\leq d} (x - v_i(z))$

Asymptotics simplifications for $[u^0]F^{[>h]}$ (bridges)

Thm. Banderier-Flajolet

$$(-k < -c) \qquad [u^{-k}]\frac{1}{1 - zP(u)} = z \sum_{j=1}^{c} \frac{u_j'(z)}{u_j(z)^{-k+1}}$$

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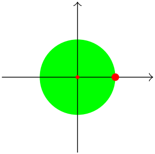
$$Q(u) = \prod_{2\le j\le d}(u - v_j(z)) = \sum_{i=0}^{d-1} q_i(z)u^i$$

$$[u^0]F^{[>h]}(z,u) = [u^0]\frac{1}{1-zP(u)}\frac{u^{h+1}}{v_1(z)^{h+1}}\frac{Q(u)}{Q(v_1(z))}(1+O(A^h))$$



$$= \frac{1}{v_1(z)^{h+1}Q(v_1(z))}\sum_{i=0}^{d-1} q_i(z)[u^0]\frac{u^{h+i+1}}{1-zP(u)}(1+O(A^h))$$

$$= z\left(\frac{u_1(z)}{v_1(z)}\right)^h \times \frac{u_1'(z)Q(u_1(z))}{v_1(z)Q(v_1(z))} \times (1+O(C^h))$$

$$\sup_{\substack{\epsilon<|z|<\rho \\ j\ge 2}} \frac{|u_1(z)|}{|u_j(z)|} < B \qquad C = \max(A,B)$$

Extracting asymptotically $[z^n][u^0]F^{[>x\sigma\sqrt{n}]}(z,u)$

$$P(1)=1, \quad P'(1)=0, \quad \rho=1, \quad \sigma^2=P''(1)$$

$$z\sim 1^- \left\{ \begin{array}{l} u_1(z) = 1-\sqrt{\frac{2}{\sigma^2}(1-z)}+O(1-z) \\ v_1(z) = 1+\sqrt{\frac{2}{\sigma^2}(1-z)}+O(1-z) \\ \dfrac{Q(u_1(z))}{Q(v_1(z))} = \dfrac{Q(1)+O(\sqrt{1-z})}{Q(1)+O(\sqrt{1-z})} = 1+O(\sqrt{1-z}) \end{array} \right.$$

$$[u^0]F^{[>x\sigma\sqrt{n}]}(z,u) = z\left(\frac{u_1(z)}{v_1(z)}\right)^{x\sigma\sqrt{n}} \times \frac{u_1'(z)Q(u_1(z))}{v_1(z)Q(v_1(z))} \times (1+O(C^n))$$

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$$= \frac{z}{\sigma\sqrt{2}} \frac{\left(1-2\sqrt{\frac{2}{\sigma^2}(1-z)}\right)^{x\sigma\sqrt{n}}}{\sqrt{1-z}} \times \left(1+O(\sqrt{1-z})\right) \times (1+O(C^n))$$

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$$= \frac{z}{\sigma\sqrt{2}} \frac{\left(1-2\sqrt{\frac{2}{\sigma^2}(1-z)}\right)^{x\sigma\sqrt{n}}}{\sqrt{1-z}} \times \left(1+O(\sqrt{1-z})\right) \times (1+O(C^n))$$

Semi-large powers Banderier-Flajolet-Soria-Schaeffer (2001)

Aiming to a Cauchy integral with Hankel contour

$$b_n^{>x\sigma\sqrt{n}} = \frac{1}{2i\pi}\frac{z}{\sigma\sqrt{2}}\frac{\left(1-2\sqrt{\frac{2}{\sigma^2}(1-z)}\right)^{x\sigma\sqrt{n}}}{\sqrt{1-z}} \times (1+O(\sqrt{1-z}))$$

$$= \frac{1}{2i\pi}\oint_{\Gamma'}\frac{1}{\sigma\sqrt{2}\sqrt{n}}\frac{e^t e^{-2x\sqrt{2t}}}{\sqrt{t}} \times \left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)dt,$$

This follows from the substitution $z = 1 - \frac{t}{n}$

Expand the term $e^{-2x\sqrt{2t}}$ and set $t = -r$

This gives integrals of the Hankel form, valid for all 4 s $\in \mathbb{C}4$

$$\frac{1}{2i\pi}\int_{+\infty}^{(0)}(-r)^s e^{-r}dr = \frac{1}{\pi}\sin(\pi s)\Gamma(1+s)$$

Gathering the terms of resulting sum provides

$$\frac{b_n^{>x\sigma\sqrt{n}}}{\sigma\sqrt{2\pi n}} = \sum_{k=0}^{\infty}(-1)^k\frac{(\sqrt{2}x)^{2k}}{k!} \times \left(1+O\left(\frac{1}{\sqrt{n}}\right)\right) = e^{-2x^2} \times \left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Asymptotics for upper bounded bridges

$$P(1) = 1, \quad P'(1) = 0, \quad \rho = 1, \quad \sigma^2 = P''(1)$$

$$[z^n][u^0]F^{[>x\sigma\sqrt{n}]} = \frac{\sqrt{n}}{\sigma\sqrt{2}} \times e^{-2x^2} \times \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

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but for unconditionned bridges (Banderier-Flajolet)

$$[z^n][u^0]F^{]-\infty,+\infty[} = \frac{\sqrt{n}}{\sigma\sqrt{2}} \times \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Theorem (Banderier-N. 2010)

$$\mathbf{P}\left(\max_{0\leq i\leq n} B_i > x\sigma\sqrt{n}\right) = e^{-2x^2} \times \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Full asymptotics for Łukasiewicz bridges

$X_i \in \{-1, \dots, +d\}$ only **one small root**

$Q(u_1(z))$ and $Q(v_1(z))$ expressible as functions of $u_1(z)$ and $v_1(z)$ only

$$Q(u) = \prod_{j=2}^{d} (u - v_j(z)) = \frac{u(1 - zP(u))}{p_d z(u - u_1(z))(u - v_1(z))}$$

$P'(u(z)) = -1/\big(z^2 u'(z)\big)$ for **any root** $u(z)$ of the kernel

$$Q(u_1(z)) = \frac{1}{p_d z} \frac{\partial}{\partial u} \frac{u(1 - zP(u))}{u - v_1(z)} \bigg|_{u=u_1(z)} = \frac{1}{p_d z^2} \frac{u_1(z)}{u_1'(z)(u_1(z) - v_1(z))}$$

Asymptotics at higher order

$$b_n^{>x\sigma\sqrt{n}} = \frac{z}{\sigma\sqrt{2}} \frac{\left(1 - 2\sqrt{\frac{2}{\sigma^2}(1-z)}\right)^{x\sigma\sqrt{n}}}{\sqrt{1-z}} \times \sum_{i\geq 1} \alpha_i (1-z)^{i/2}$$

$$= \frac{1}{2i\pi} \oint_{\Gamma'} \frac{1}{\sigma\sqrt{2}\sqrt{n}} \frac{e^t e^{-2x\sqrt{2t}}}{\sqrt{t}} \times \sum_{i\geq 1} \alpha_i \left(\frac{t}{n}\right)^{i/2} dt,$$

Hankel integral again

$$\frac{1}{2i\pi} \int_{+\infty}^{(0)} (-r)^s e^{-r} dr = \frac{1}{\pi} \sin(\pi s) \Gamma(1+s)$$

Full asymptotics for Łukasiewicz bridges

Proposition (Banderier-N. 2010)

Łukasiewicz bridges verify asymptotically

$$[u^0]F^{[>h]}(z,u) = z\left(\frac{u_1(z)}{v_1(z)}\right)^h \times \frac{-v_1'(z)u_1(z)}{v_1(z)^2} \times (1 + O(C^h))$$

Full asymptotics for Łukasiewicz bridges

Proposition (Banderier-N. 2010)

Łukasiewicz bridges verify asymptotically

$$[u^0]F^{[>h]}(z,u) = z\left(\frac{u_1(z)}{v_1(z)}\right)^h \times \frac{-v_1'(z)u_1(z)}{v_1(z)^2} \times (1 + O(C^h))$$

use `Newton iterations` for expansions of $u_1(z)$ and $v_1(z)$

$$\frac{\beta_n^{>x\sigma\sqrt{n}}}{\exp(-2x^2)} = 1 + \frac{(-(2/3)x\xi/\zeta^{3/2} - 6x/\sqrt{\zeta})}{\sqrt{n}} + \frac{1}{n}\left((-2 - \frac{10}{9}\frac{\xi^2}{\zeta^3} + \frac{2}{3}\frac{\theta}{\zeta^2} - \frac{16}{3\zeta} - \frac{8}{3}\frac{\xi}{\zeta^2})x^4 \right.$$
$$\left. +(\frac{24}{\zeta} + \frac{5}{3}\frac{\xi^2}{\zeta^3} + 3 - \frac{\theta}{\zeta^2} + \frac{20}{3}\frac{\xi}{\zeta^2})x^2 - \frac{5}{\zeta} - \frac{3}{8} - \frac{7}{6}\frac{\xi}{\zeta^2} - \frac{5}{24}\frac{\xi^2}{\zeta^3} + \frac{1}{8}\frac{\theta}{\zeta^2} + \frac{5}{24}\frac{\xi^3}{\zeta^3} - \frac{1}{8}\frac{\theta^2 - 3\zeta^2}{\zeta^2}\right)$$
$$+ O\left(\frac{1}{n^{3/2}}\right)$$

$$\beta_n^{>x\sigma\sqrt{n}} = \mathbf{P}\left(\max_{0\le i\le n} B_i\right) > x\sigma\sqrt{n},$$

$$\begin{cases} \zeta = \sigma^2 = P''(1), \\ \xi = P'''(1), \quad \theta = P''''(1) \end{cases}$$

Simple Łukasiewicz walks with d as parameter

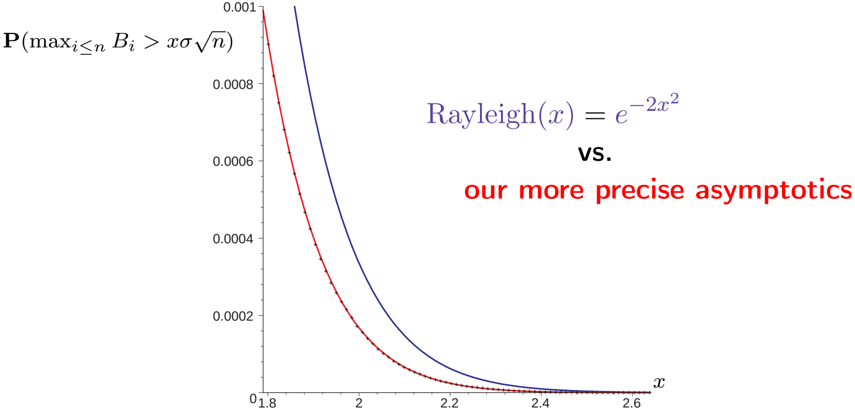
$$P(u) = \frac{u^d}{d+1} + \frac{1}{(d+1)u}$$

$$\begin{aligned}
b_n^{>x\sigma\sqrt{n}} \times \frac{\sigma\sqrt{2\pi n}}{e^{-2x^2}} &= 1 + \left(-\frac{2}{3}\,x\sqrt{d} - \frac{10}{3}\,\frac{x}{\sqrt{d}}\right)\frac{1}{\sqrt{n}} + \left(\left(\frac{2}{3}\,x^2 - \frac{4}{9}\,x^4 - 1/12\right)d - \frac{4}{9}\,x^4 + \frac{10}{3}\,x^2 - 3/4 + \left(-\frac{4}{9}\,x^4 - \frac{17}{12} + 6\,x^2\right)d^{-1}\right)\frac{1}{n} \\
&+ \left(\left(\frac{8}{27}\,x^5 - \frac{76}{135}\,x^3 + \frac{13}{90}\,x\right)d^{3/2} + \left(\frac{16}{9}\,x^5 - \frac{208}{45}\,x^3 + \frac{9}{5}\,x\right)\sqrt{d} + \left(\frac{16}{9}\,x^5 - \frac{48}{5}\,x^3 + \frac{83}{15}\,x\right)\frac{1}{\sqrt{d}} + \left(\frac{40}{27}\,x^5 - \frac{1244}{135}\,x^3 + \frac{497}{90}\,x\right)d^{-3/2}\right)\frac{1}{n^{3/2}} \\
&+ \left(\left(\frac{19}{27}\,x^4 + \frac{8}{81}\,x^8 - \frac{8}{15}\,x^6 - \frac{11}{54}\,x^2 + \frac{1}{288}\right)d^2 + \left(\frac{694}{135}\,x^4 - \frac{361}{135}\,x^2 - \frac{304}{135}\,x^6 + \frac{109}{720} + \frac{16}{81}\,x^8\right)d\right. \\
&- \frac{1051}{90}\,x^2 + \frac{469}{480} + \frac{727}{45}\,x^4 + \frac{8}{27}\,x^8 - \frac{232}{45}\,x^6 + \left(\frac{1469}{720} + \frac{16}{81}\,x^8 - \frac{2701}{135}\,x^2 - \frac{208}{45}\,x^6 + \frac{2854}{135}\,x^4\right)d^{-1} \\
&\left. + \left(\frac{8}{81}\,x^8 - \frac{392}{135}\,x^6 - \frac{3583}{270}\,x^2 + \frac{1957}{1440} + \frac{1871}{135}\,x^4\right)d^{-2}\right)\frac{1}{n^2} + O\left(n^{-5/2}\right) \qquad (23)
\end{aligned}$$

Conjecture

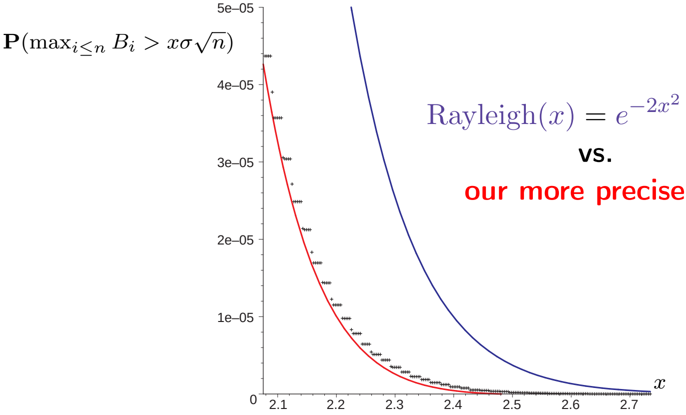
The error term of order r in the asymptotics development of $b_n^{\sigma x\sqrt{n}}$ is of the form $O\left(d^{r/2}x^{2r} \times n^{-r/2}\right)$.

Back to simulations



$$X \in \{-1, +19\} \qquad n = 400$$

Heuristics for bioinformatics - rational jumps



$$X \in \{-11, +93\} \rightsquigarrow X' \in \left\{-1, +\frac{93}{11}\right\} \qquad n = 104$$

Very Short Bibliography

- *Flajolet* and *Sedgewick*, "Analytic Combinatorics" book, 2009
- *Banderier* and *Flajolet*, "Basic analytic combinatorics of directed lattice paths", TCS 281, pages 37-80
- *Banderier* and *Nicodeme*, "Bounded discrete walks", Proceedings of AofA2010 conference, Pages 35-48