Bounded Discrete Walks

Pierre Nicodème

CNRS - LIX École polytechnique, Palaiseau
and Amib Project, INRIA-Saclay

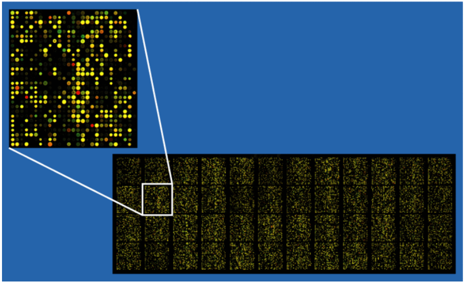
Joint work with Cyril Banderier

CNRS - Team "CALIN", LIPN, University Paris 13

Tortoiseshell cats - Chats Isabelle

The patchy colours of a tortoiseshell cat are the result of different levels of expression of pigmentation genes in different areas of the skin.

DNA micro-array and diagnostic in genetics?



measuring fluorescence of a spot provides the level of expression of the corresponding gene.

- $\Gamma :=$ set of genes of a background population $(|\Gamma| = G)$
- $\gamma :=$ set of genes of a particular family $(|\gamma| = g)$

Question. Are the levels of expression of the genes of γ with respect to the level of expressions of the genes of Γ characteristic of an exceptional behaviour? (Disease, *etc*)

Random walks model

- Keller, Backes and Lenhof (2007)

Order the genes by level of expression.

Build a walk $(B_i)_{0\leq i\leq G+g}$ such that $B_0 = 0$ and

$$B_i = \begin{cases} +G & \text{if the gene at rank } i \text{ belongs to } \gamma, \\ -g & \text{if the gene at rank } i \text{ belongs to } \Gamma \end{cases}$$

These walks are therefore **bridges** as $B_{G+g} = Gg - gG = 0$.

exceptional overexpression of the genes of $\gamma \quad \Longrightarrow$ **exceptional height** of the bridge with respect to the height of a bridge chosen **at random** among the $\binom{G+g}{g}$ possible bridges.

Example

$$G = 6, \quad g = 3$$

represent by $\begin{cases} \circ & \text{genes of } \Gamma \\ \bullet & \text{genes of } \gamma \end{cases}$

Pattern of level expression

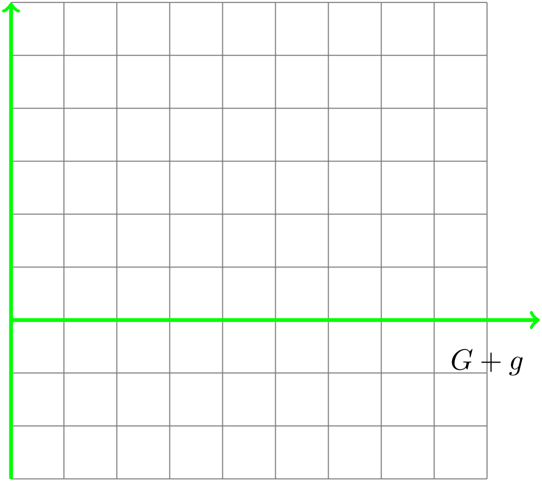
High level $\longrightarrow$ Low level

$\circ \quad \bullet \quad \bullet \quad \circ \quad \bullet \quad \circ \quad \circ \quad \circ \quad \circ$

○ ● ● ○ ● ○ ○ ○ ○

● $\rightsquigarrow X_j = +2 \quad (+6/3)$

○ $\rightsquigarrow X_j = -1 \quad (-3/3)$

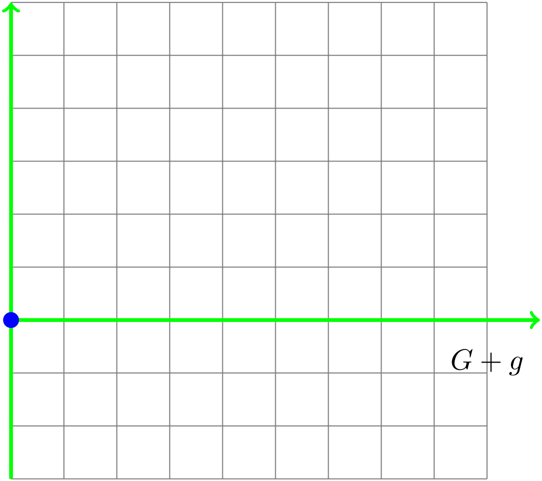


$$B_i = \sum_{1 \le j \le i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

● $\rightsquigarrow X_j = +2 \quad (+6/3)$

○ $\rightsquigarrow X_j = -1 \quad (-3/3)$

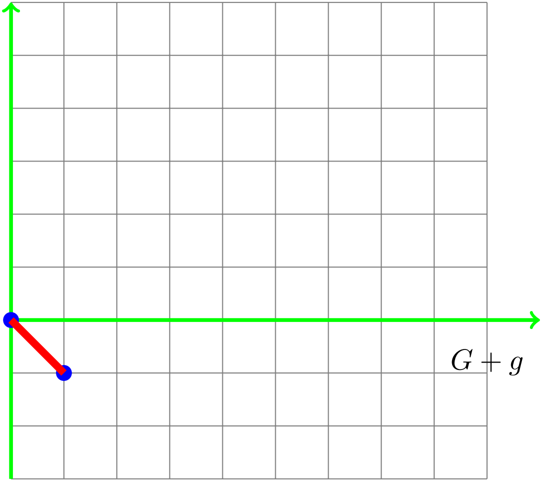


$$\color{red}{B_i} = \sum_{1 \le j \le i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

$\bullet \rightsquigarrow X_j = +2 \quad (+6/3)$

$\circ \rightsquigarrow X_j = -1 \quad (-3/3)$

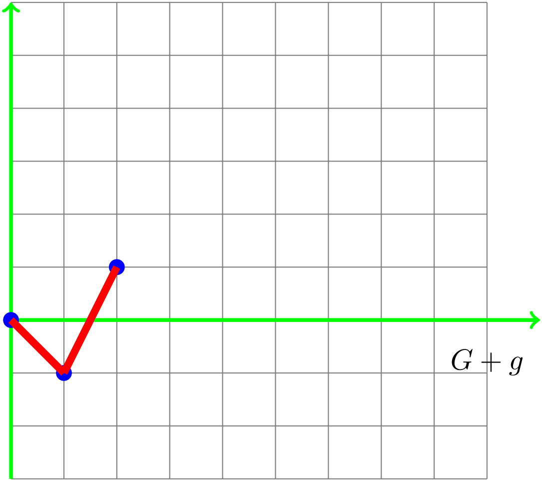


$$\color{red}{B_i} = \sum_{1\le j\le i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

$\bullet \rightsquigarrow X_j = +2 \quad (+6/3)$

$\circ \rightsquigarrow X_j = -1 \quad (-3/3)$

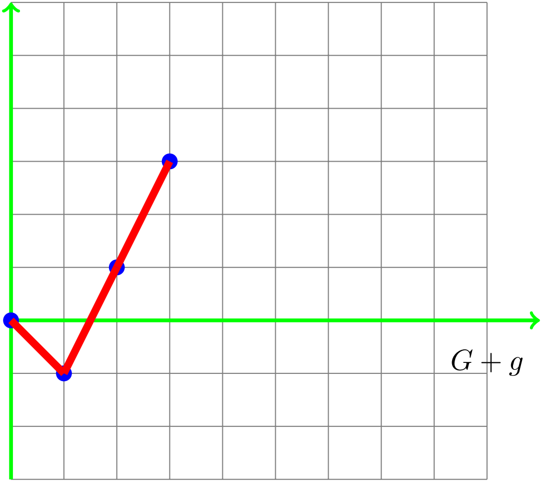


$$\color{red}{B_i} = \sum_{1 \leq j \leq i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

● $\rightsquigarrow X_j = +2 \quad (+6/3)$

○ $\rightsquigarrow X_j = -1 \quad (-3/3)$

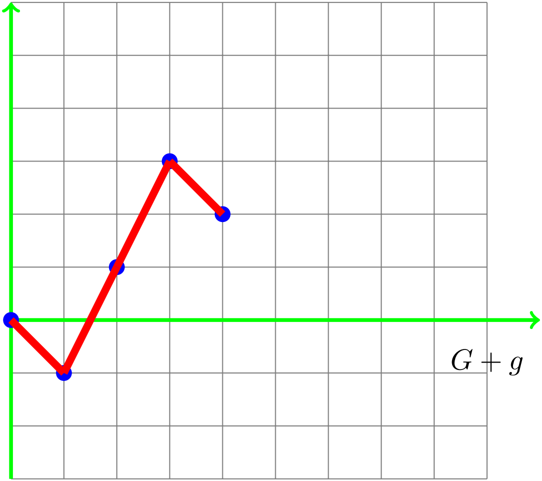


$$\color{red}{B_i} = \sum_{1 \leq j \leq i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

● $\rightsquigarrow X_j = +2 \quad (+6/3)$

○ $\rightsquigarrow X_j = -1 \quad (-3/3)$

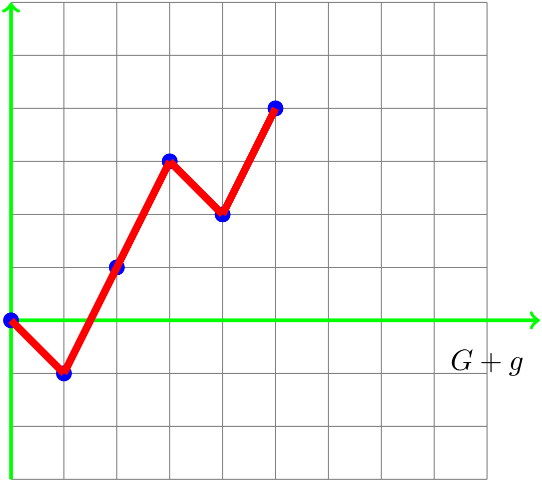


$$\color{red}{B_i} = \sum_{1 \leq j \leq i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

● $\rightsquigarrow X_j = +2 \quad (+6/3)$

○ $\rightsquigarrow X_j = -1 \quad (-3/3)$

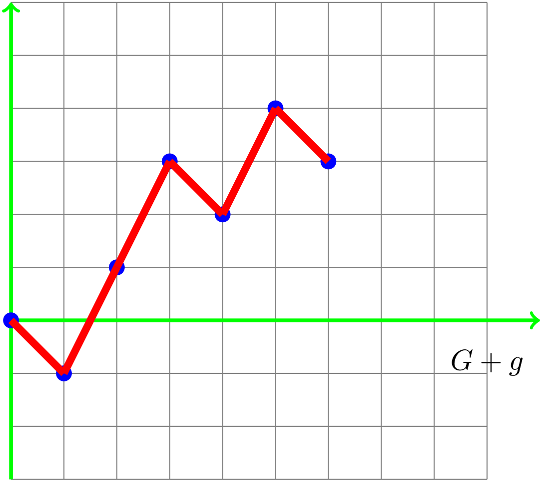


$$B_i = \sum_{1 \leq j \leq i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

● $\rightsquigarrow X_j = +2 \quad (+6/3)$

○ $\rightsquigarrow X_j = -1 \quad (-3/3)$

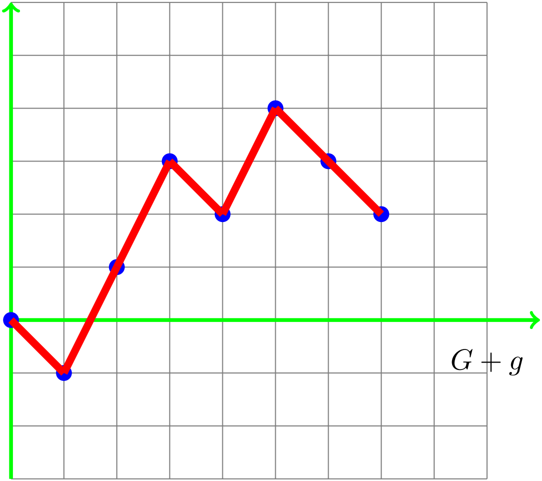


$$\color{red}{B_i} = \sum_{1 \leq j \leq i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

● $\rightsquigarrow X_j = +2 \quad (+6/3)$

○ $\rightsquigarrow X_j = -1 \quad (-3/3)$

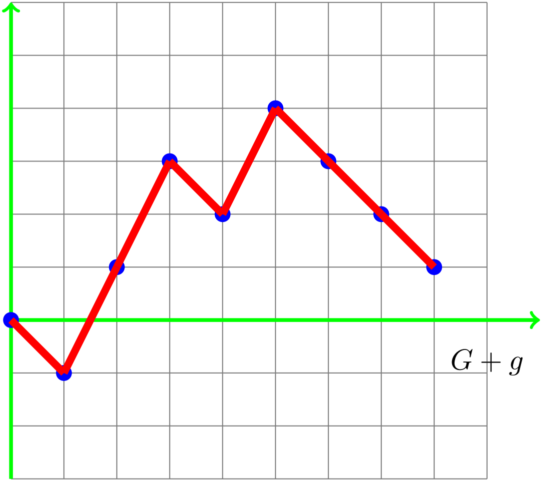


$$B_i = \sum_{1 \le j \le i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

$\bullet \rightsquigarrow X_j = +2 \quad (+6/3)$

$\circ \rightsquigarrow X_j = -1 \quad (-3/3)$

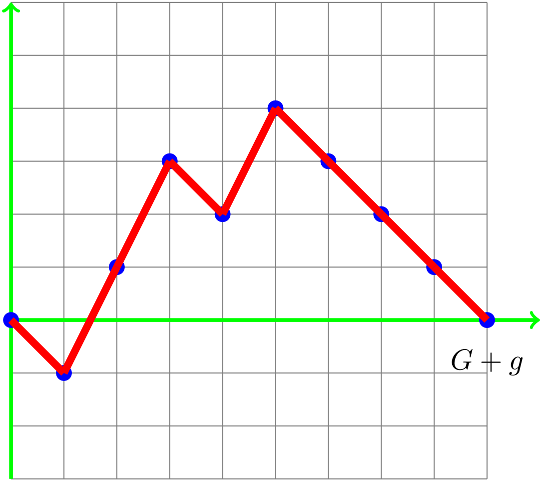


$$B_i = \sum_{1 \leq j \leq i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

● $\rightsquigarrow X_j = +2 \quad (+6/3)$

○ $\rightsquigarrow X_j = -1 \quad (-3/3)$

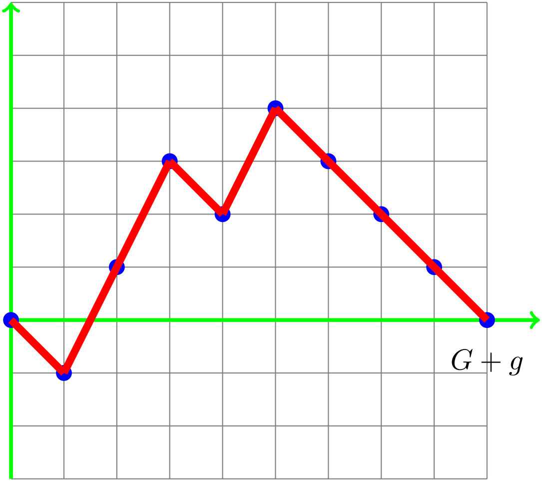


$$B_i = \sum_{1 \leq j \leq i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

● ⇝ $X_j = +2 \quad (+6/3)$

○ ⇝ $X_j = -1 \quad (-3/3)$

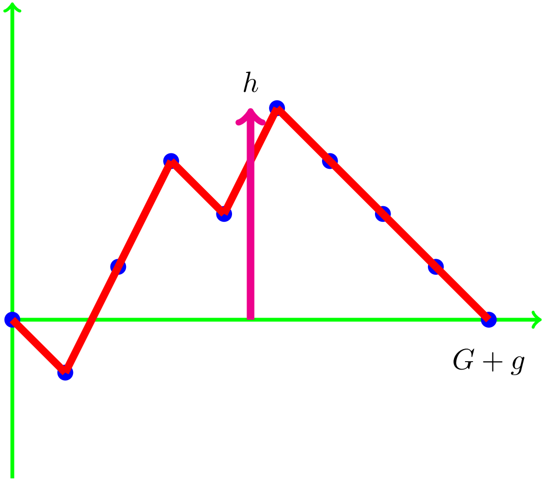


$$B_i = \sum_{1 \le j \le i} X_j$$

○ ● ● ○ ● ○ ○ ○ ○

$\bullet \rightsquigarrow X_j = +2 \quad (+6/3)$

$\circ \rightsquigarrow X_j = -1 \quad (-3/3)$

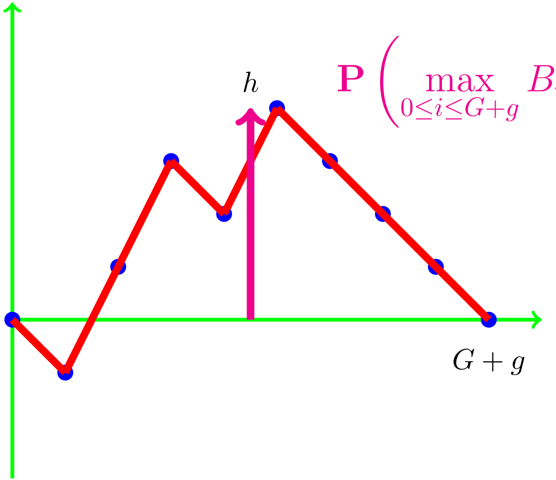


$$B_i = \sum_{1 \leq j \leq i} X_j$$

$\circ \bullet \bullet \circ \bullet \circ \circ \circ \circ$

$\bullet \rightsquigarrow X_j = +2 \quad (+6/3)$

$\circ \rightsquigarrow X_j = -1 \quad (-3/3)$



$$\mathbf{P}\left(\max_{0\leq i\leq G+g} B_i\right) > h\ ?$$

$$B_i = \sum_{1\leq j\leq i} X_j$$

Tail distribution for the height - Keller *et al.*

Keller, Backes and Lenhof (2007) compute

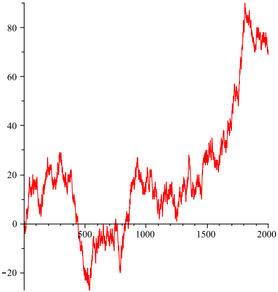
$$p\text{-value}(G, g, h) := \mathbf{P}\left(\max_{0 \le i \le G+g} B_i > h\right)$$

by a dynamical programming method in **complexity**

$$O(G \times g)$$

Our result: heuristics in **complexity** $O(1)$

Brownian motion as limit of discrete walks



Strong approximations of discrete walks by Brownian motion

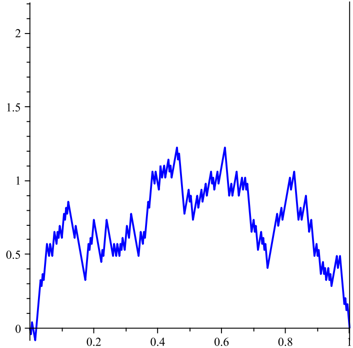
- Komlós, Major, Tusnády (1976), Chatterjee (2010)

$$\mathbf{P}\left\{\max_{1\le k\le n} |S_k - W(k)| > C \log n + x\right\} < K e^{-\lambda x} \quad (W(k) \text{ Wiener process})$$

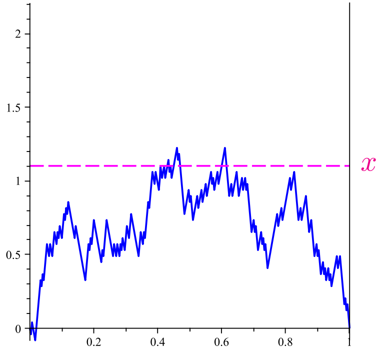
Little done for **approximations of bridges**

- Kaigh (1976)

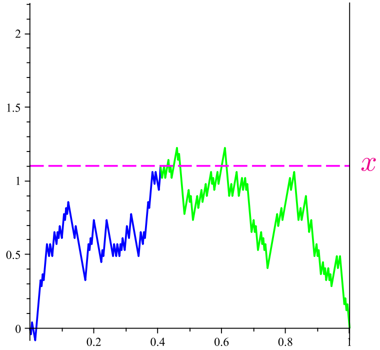
Désiré André reflexion principle



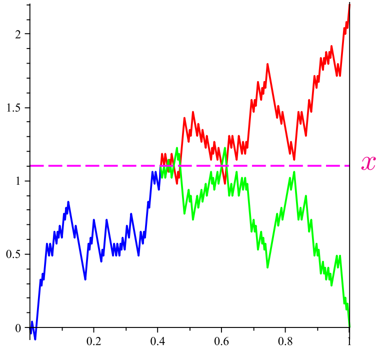
Désiré André reflexion principle



Désiré André reflexion principle

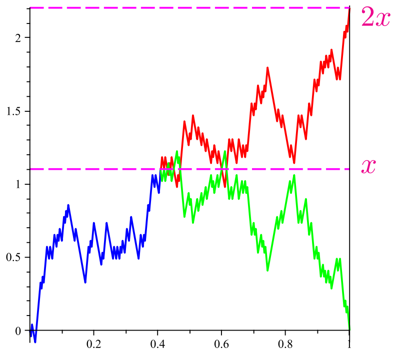


Désiré André reflexion principle



Caveat: This nice reflexion trick won't work in the discrete case if the walk has a drift or its jumps are other than $+1, 0, -1$. Another approach is needed then! (=our work).

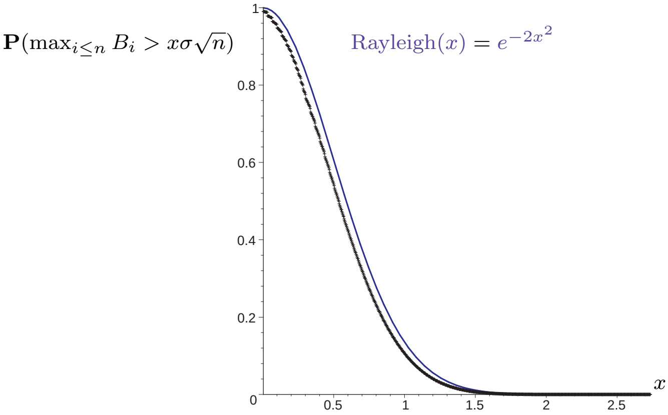
Désiré André reflexion principle



$$\Phi(2x) = \mathbf{P}(W(1) = 2x) = \frac{1}{\sqrt{2\pi}} e^{-2x^2}$$

$$\mathbf{P}\left(\max_{t\in[0,1]} B_t \geq x\right) = \frac{\Phi(2x)}{\Phi(0)} = \text{Rayleigh}(x) = e^{-2x^2}$$

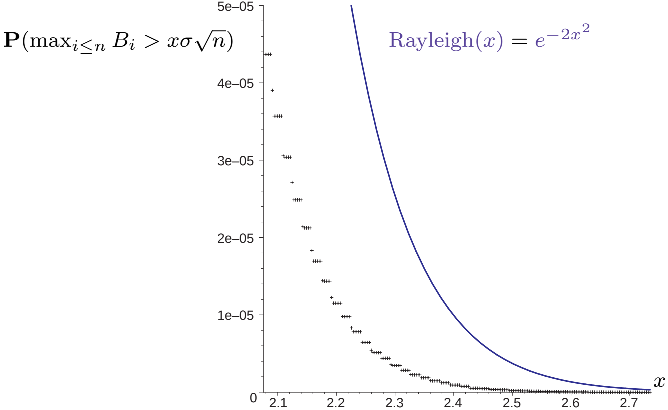
Brownian bridge versus simulations



bridge length $n = G + g = 104$ $\quad G = +93 \quad - g = -11$

10^8 simulations

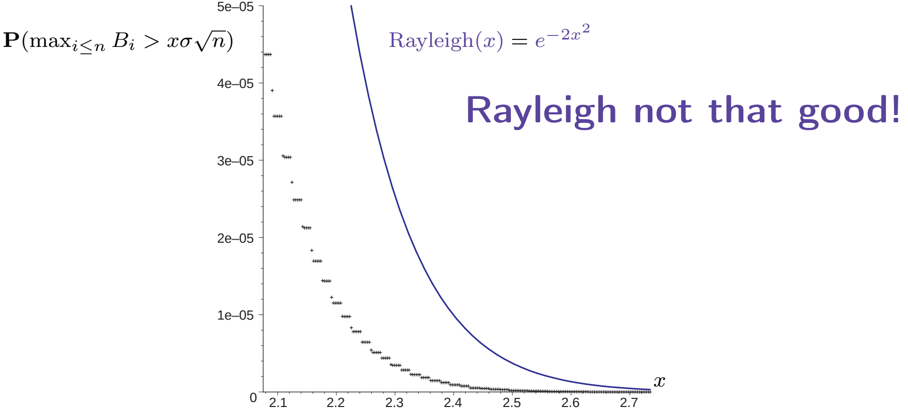
Brownian bridge versus simulations



bridge length $n = G + g = 104$ $G = +93$ $-g = -11$

$\sigma = \sqrt{G \times g}$ 10^8 simulations

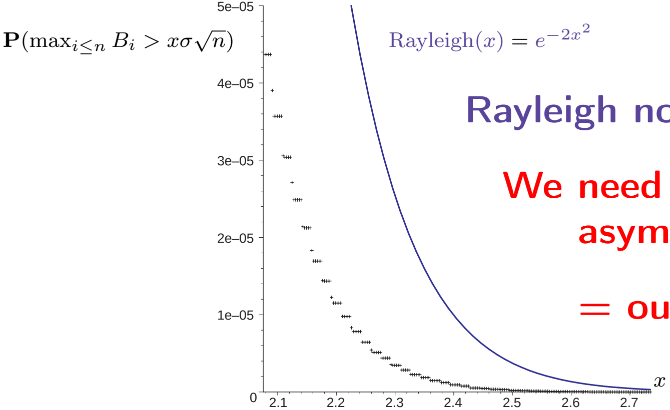
Brownian bridge versus simulations



bridge length $n = G + g = 104$ $\quad G = +93 \quad -g = -11$

$\sigma = \sqrt{G \times g}$ $\quad 10^8$ simulations

Brownian bridge versus simulations



Rayleigh not that good!

We need more precise asymptotics!

= our work!

bridge length $n = G + g = 104$ $\quad G = +93 \quad -g = -11$

$\sigma = \sqrt{G \times g}$ $\quad 10^8$ simulations

Summary of the talk

- Model
- Generating functions for bounded walks
- Asymptotic limit law for bridges
- Łukasiewicz bridges
- Back to simulations

Directed discrete walks

Model = generalized Dyck walks
= discrete "excursions/bridges/meanders"

- Knuth (1973)
- Gessel (1980)
- Labelle & Yeah (1990), Merlini & al. (1996), Duchon (2000)
- Banderier and Flajolet (2002)
- Bousquet-Mélou (2009)
- Bousquet-Mélou and Ponty (2009)

Model

- i.i.d. **integer jumps** $X_i \in \{-c, \ldots, +d\}$
- characteristic polynomial $P(u) = p_{-c}u^{-c} + \cdots + p_d u^d$
- $\mathbf{E}(X_i) = P'(1) = 0$

Model

- i.i.d. **integer jumps** $X_i \in \{-c, \dots, +d\}$
- characteristic polynomial $P(u) = p_{-c}u^{-c} + \cdots + p_d u^d$
- $\mathbf{E}(X_i) = P'(1) = 0$

Typical generating function

$$F(z,u) = \sum_{n=0}^{\infty} f_n(u) z^n = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} f_{n,k} u^k z^n$$

where

- $f_{n,k}$ is the probability that a walk remaining in an horizontal strip (by instance $]-\infty, +h]$) has **altitude** k at **time** n.

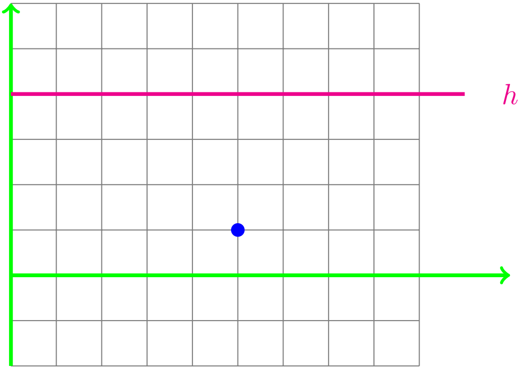
$[z^n][u^0]F(z,u)$ corresponds to **bridges** of **length** n

Getting the generating function

$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u}$

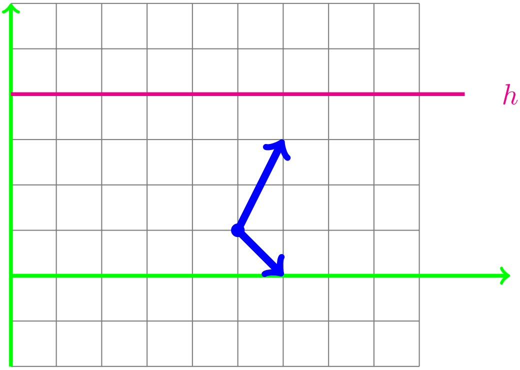
Getting the generating function

$X_i \in \{-1, +2\}$ $\qquad P(u) = u^2 + \frac{1}{u}$



Getting the generating function

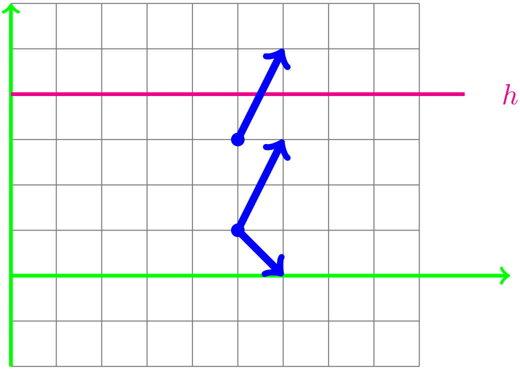
$X_i \in \{-1, +2\}$ $\qquad P(u) = u^2 + \frac{1}{u}$



$$f_{k+1}(u) = f_k(u) \times P(u)$$

Getting the generating function

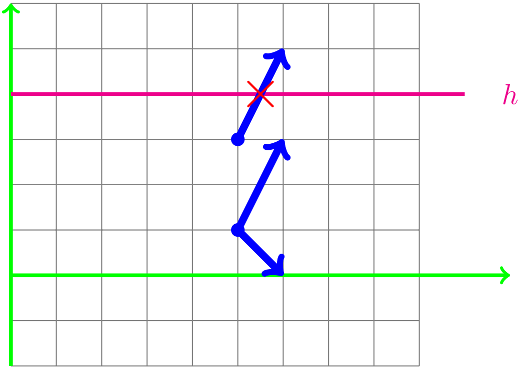
$X_i \in \{-1, +2\}$ $\qquad P(u) = u^2 + \frac{1}{u}$



$$f_{k+1}(u) = f_k(u) \times P(u)$$

Getting the generating function

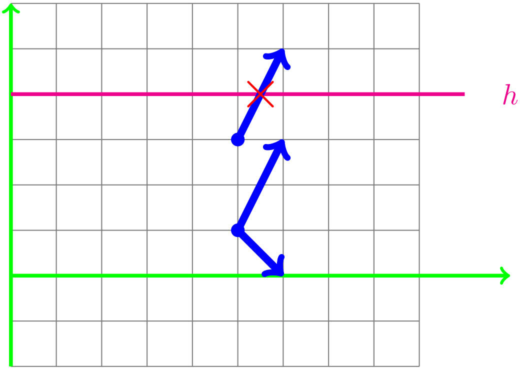
$X_i \in \{-1, +2\}$ $\qquad P(u) = u^2 + \frac{1}{u}$



$f_{k+1}(u) = f_k(u) \times P(u)$

Getting the generating function

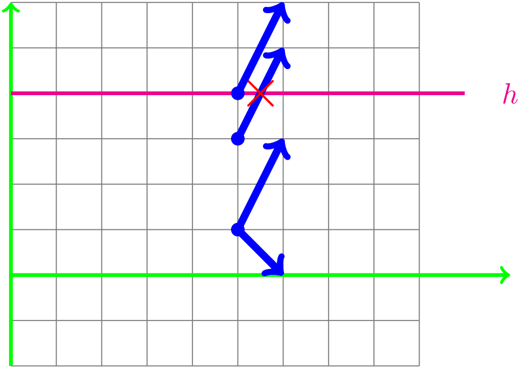
$X_i \in \{-1, +2\}$ $\qquad P(u) = u^2 + \dfrac{1}{u}$



$$f_{k+1}(u) = f_k(u) \times P(u)$$
$$-u^{h+1}[u^{h+1}]f_k(u) \times P(u)$$

Getting the generating function

$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u}$

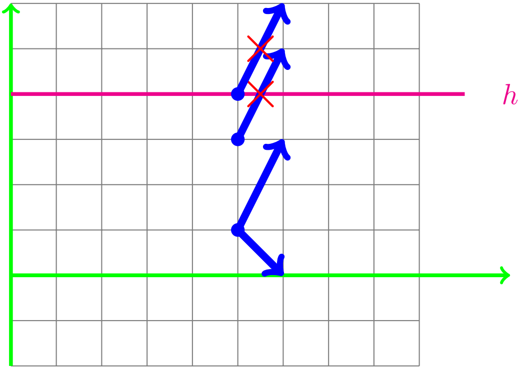


$$f_{k+1}(u) = f_k(u) \times P(u)$$

$$-u^{h+1}[u^{h+1}]f_k(u) \times P(u)$$

Getting the generating function

$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u}$

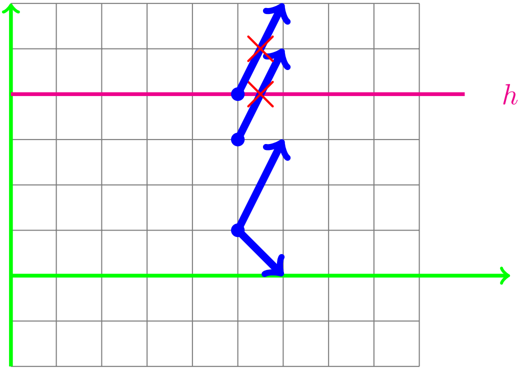


$$f_{k+1}(u) = f_k(u) \times P(u)$$

$$-u^{h+1}[u^{h+1}]f_k(u) \times P(u)$$

Getting the generating function

$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u}$



$$f_{k+1}(u) = f_k(u) \times P(u)$$
$$-u^{h+1}[u^{h+1}]f_k(u) \times P(u)$$
$$-u^{h+2}[u^{h+2}]f_k(u) \times P(u)$$

Getting the generating function

$$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u} \qquad f_0(u) = 1$$

$$f_{k+1}(u) = f_k(u) \times P(u) - \begin{cases} u^{h+1}[u^{h+1}]f_k(u) \times P(u) \\ + u^{h+2}[u^{h+2}]f_k(u) \times P(u) \end{cases}$$

Getting the generating function

$$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u} \qquad f_0(u) = 1$$

$$f_{k+1}(u) = f_k(u) \times P(u) - \left\{ \begin{array}{r} u^{h+1}[u^{h+1}]f_k(u) \times P(u) \\ + u^{h+2}[u^{h+2}]f_k(u) \times P(u) \end{array} \right.$$

$$\begin{aligned} \sum_{k=0}^{\infty} f_{k+1}(u) z^{k+1} &= F(z,u) - f_0(u) \\ &= zP(u) \sum_{k=0}^{\infty} f_k(u) z^k - \left\{ \begin{array}{r} zu^{h+1} F_{h+1}(z) \\ + zu^{h+2} F_{h+2}(z) \end{array} \right. \end{aligned}$$

Getting the generating function

$$X_i \in \{-1, +2\} \qquad P(u) = u^2 + \frac{1}{u} \qquad f_0(u) = 1$$

$$f_{k+1}(u) = f_k(u) \times P(u) - \left\{ \begin{array}{r} u^{h+1}[u^{h+1}]f_k(u) \times P(u) \\ + u^{h+2}[u^{h+2}]f_k(u) \times P(u) \end{array} \right.$$

$$\begin{aligned} \sum_{k=0}^{\infty} f_{k+1}(u) z^{k+1} &= F(z,u) - f_0(u) \\ &= zP(u) \sum_{k=0}^{\infty} f_k(u) z^k - \left\{ \begin{array}{r} zu^{h+1} F_{h+1}(z) \\ + zu^{h+2} F_{h+2}(z) \end{array} \right. \end{aligned}$$

$$F(z,u)(1 - zP(u)) = 1 - zu^{h+1} F_{h+1}(z) - zu^{h+2} F_{h+2}(z)$$

Kernel method

Knuth, Tutte, Brown, Bousquet-Mélou, Petkovšek, *etc.*

$X_i \in \{-1, +2\}$ $\qquad P(u) = \frac{1}{u} + u^2$

$$F(z,u)(1 - zP(u)) = 1 - zu^{h+1}F_{h+1}(z) - zu^{h+2}F_{h+2}(z)$$

$F(z,u), F_{h+1}(z), F_{h+2}(z)$ **unknown functions**

but the **roots** $u(z)$ of $1 - zP(u) = 0$ **cancel** the left member of the equation

two roots $u(z)$ provide a linear system of **two** equations whose solutions are $F_{h+1}(z)$ and $F_{h+2}(z)$

General case - Any finite set of integer jumps

$$P(u) = p_{-c}u^{-c} + p_{-c+1}u^{-c+1} + \cdots + p_{d-1}u^{d-1} + p_d u^d$$

$$F(z,u)(1 - zP(u)) = 1 - zu^{h+1}F_{h+1}(z) - \cdots - zu^{h+d}F_{h+d}(z)$$

d unknown functions $F_{h+j}(z)$, but the equation $1 - zP(u) = 0$ has

$$\begin{cases} d \text{ large roots } v_i(z) & \text{such that } v_i(z) \sim \frac{1}{z}^{1/d} & \text{as } z \to 0 \\ c \text{ small roots } u_j(z) & \text{such that } u_j(z) \sim z^{1/c} & \text{as } z \to 0 \end{cases}$$

General case - Any finite set of integer jumps

$$P(u) = p_{-c}u^{-c} + p_{-c+1}u^{-c+1} + \cdots + p_{d-1}u^{d-1} + p_d u^d$$

$$F(z,u)(1 - zP(u)) = 1 - zu^{h+1}F_{h+1}(z) - \cdots - zu^{h+d}F_{h+d}(z)$$

d unknown functions $F_{h+j}(z)$, but the equation $1 - zP(u) = 0$ has

$$\begin{cases} d \text{ large roots } v_i(z) & \text{such that } v_i(z) \sim \frac{1}{z}^{1/d} \quad \text{as } z \to 0 \\ c \text{ small roots } u_j(z) & \text{such that } u_j(z) \sim z^{1/c} \quad \text{as } z \to 0 \end{cases}$$

$$\begin{cases} v_1(z)^{h+1}F_{h+1}(z) + \cdots + v_1(z)^{h+d}F_{h+d}(z) = 1/z, \\ \cdots \\ v_d(z)^{h+1}F_{h+1}(z) + \cdots + v_d(z)^{h+d}F_{h+d}(z) = 1/z \end{cases}$$

General case - Any finite set of integer jumps

$$P(u) = p_{-c}u^{-c} + p_{-c+1}u^{-c+1} + \cdots + p_{d-1}u^{d-1} + p_d u^d$$

$$F(z,u)(1 - zP(u)) = 1 - zu^{h+1}F_{h+1}(z) - \cdots - zu^{h+d}F_{h+d}(z)$$

d unknown functions $F_{h+j}(z)$, but the equation $1 - zP(u) = 0$ has

$$\begin{cases} d \text{ large roots } v_i(z) & \text{such that } v_i(z) \sim \frac{1}{z}^{1/d} \quad \text{as } z \to 0 \\ c \text{ small roots } u_j(z) & \text{such that } u_j(z) \sim z^{1/c} \quad \text{as } z \to 0 \end{cases}$$

$$\begin{cases} v_1(z)^{h+1}F_{h+1}(z) + \cdots + v_1(z)^{h+d}F_{h+d}(z) = 1/z, \\ \ldots \\ v_d(z)^{h+1}F_{h+1}(z) + \cdots + v_d(z)^{h+d}F_{h+d}(z) = 1/z \end{cases}$$

Vandermonde determinants $\mathbb{V}(\ldots)$, Schur functions,
occurring also when considering the area (Banderier-Gittenberger)

Nice expression for the generating function $F^{]-\infty,h]}$

$$\begin{cases} v_1(z)^{h+1}F_{h+1}(z) + \cdots + v_1(z)^{h+d}F_{h+d}(z) = 1/z, \\ \dots \\ v_d(z)^{h+1}F_{h+1}(z) + \cdots + v_d(z)^{h+d}F_{h+d}(z) = 1/z \end{cases}$$

$$F(z,u)(1-zP(u))$$

$$= 1 - \sum_{j=1}^{d} u^{h+j} \frac{\begin{vmatrix} v_1^{h+d} & \dots & v_1^{h+d-(j-1)} & 1 & v_1^{h+d-(j+1)} & \dots & v_1^{h+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ v_d^{h+d} & \dots & v_d^{h+d-(j-1)} & 1 & v_d^{h+d-(j+1)} & \dots & v_d^{h+1} \end{vmatrix}}{v_1^h \dots v_d^h \mathbb{V}(v_1, \dots, v_d)}$$

Nice expression for the generating function $F^{]-\infty,h]}$

$$\begin{cases} v_1(z)^{h+1}F_{h+1}(z)+\cdots+v_1(z)^{h+d}F_{h+d}(z)=1/z, \\ \dots \\ v_d(z)^{h+1}F_{h+1}(z)+\cdots+v_d(z)^{h+d}F_{h+d}(z)=1/z \end{cases}$$

$$F(z,u)(1-zP(u))$$

$$=1-\sum_{j=1}^{d} u^{h+j}\frac{\begin{vmatrix} v_1^{h+d} & \dots & v_1^{h+d-(j-1)} & 1 & v_1^{h+d-(j+1)} & \dots & v_1^{h+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ v_d^{h+d} & \dots & v_d^{h+d-(j-1)} & 1 & v_d^{h+d-(j+1)} & \dots & v_d^{h+1} \end{vmatrix}}{v_1^h \dots v_d^h \mathbb{V}(v_1,\dots,v_d)}$$

$$=1-\sum_{j=1}^{d}\frac{\begin{vmatrix} v_1^{h+d} & \dots & v_1^{h+d-(j-1)} & u^{h+j} & v_1^{h+d-(j+1)} & \dots & v_1^{h+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ v_d^{h+d} & \dots & v_d^{h+d-(j-1)} & u^{h+j} & v_d^{h+d-(j+1)} & \dots & v_d^{h+1} \end{vmatrix}}{v_1^h \dots v_d^h \mathbb{V}(v_1,\dots,v_d)}$$

Nice expression for the generating function $F^{]-\infty,h]}$

$$F(z,u)(1-zP(u))$$

$$= 1 - \sum_{j=1}^{d} \frac{\begin{vmatrix} v_1^{h+d} & \dots & v_1^{h+d-(j-1)} & u^{h+j} & v_1^{h+d-(j+1)} & \dots & v_1^{h+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ v_d^{h+d} & \dots & v_d^{h+d-(j-1)} & u^{h+j} & v_d^{h+d-(j+1)} & \dots & v_d^{h+1} \end{vmatrix}}{v_1^h \dots v_d^h \mathbb{V}(v_1, \dots, v_d)}$$

Nice expression for the generating function $F^{]-\infty,h]}$

$$F(z,u)(1-zP(u))$$

$$= 1 - \sum_{j=1}^{d} \frac{\begin{vmatrix} v_1^{h+d} & \dots & v_1^{h+d-(j-1)} & u^{h+j} & v_1^{h+d-(j+1)} & \dots & v_1^{h+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ v_d^{h+d} & \dots & v_d^{h+d-(j-1)} & u^{h+j} & v_d^{h+d-(j+1)} & \dots & v_d^{h+1} \end{vmatrix}}{v_1^h \dots v_d^h \mathbb{V}(v_1, \dots, v_d)}$$

$$= 1 - \sum_{j=1}^{d} \frac{\mathrm{Subs}(v_j = u, \mathbb{V}(v_1, \dots, v_d))}{v_1^h \dots v_d^h \mathbb{V}(v_1, \dots, v_d)}$$

$$= 1 - \sum_{j=1}^{d} \frac{\mathrm{Subs}\left(v_j = u, \begin{vmatrix} \dots & \dots & \dots & \dots & \dots \\ v_j^{h+d} & \dots & v_j^{h+j} & \dots & v_j^{h+1} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}\right)}{v_1^h \dots v_d^h \mathbb{V}(v_1, \dots, v_d)}$$

Nice expression for the generating function $F^{]-\infty,h]}$

$$
\begin{aligned}
&F(z,u)(1-zP(u)) \\
&= 1 - \sum_{j=1}^{d} \frac{\begin{vmatrix} v_1^{h+d} & \dots & v_1^{h+d-(j-1)} & u^{h+j} & v_1^{h+d-(j+1)} & \dots & v_1^{h+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ v_d^{h+d} & \dots & v_d^{h+d-(j-1)} & u^{h+j} & v_d^{h+d-(j+1)} & \dots & v_d^{h+1} \end{vmatrix}}{v_1^h \dots v_d^h \mathbb{V}(v_1,\dots,v_d)} \\
&= 1 - \sum_{j=1}^{d} \frac{\operatorname{Subs}(v_j = u, \mathbb{V}(v_1,\dots,v_d))}{v_1^h \dots v_d^h \mathbb{V}(v_1,\dots,v_d)} \\
&= 1 - \sum_{j=1}^{d} \frac{\operatorname{Subs}\left(v_j = u, \begin{vmatrix} \dots & \dots & \dots & \dots & \dots \\ v_j^{h+d} & \dots & v_j^{h+j} & \dots & v_j^{h+1} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}\right)}{v_1^h \dots v_d^h \mathbb{V}(v_1,\dots,v_d)} \\
&= 1 - \sum_{j=1}^{d} \frac{u^{h+1}}{v_j^{h+1}} \prod_{\substack{1 \le i \le d \\ i \ne j}} \frac{u - v_i}{v_j - v_i}
\end{aligned}
$$

Nice expression for the generating functions

$$F^{]-\infty,h]}(z,u) = \frac{1}{1-zP(u)} - \frac{1}{1-zP(u)} \sum_{j=1}^{d} \frac{u^{h+1}}{v_j^{h+1}} \prod_{\substack{1\le i\le d \\ i\neq j}} \frac{u-v_i}{v_j-v_i}$$

N.B.: $\frac{1}{1-zP(u)}$ counts **all** the walks.

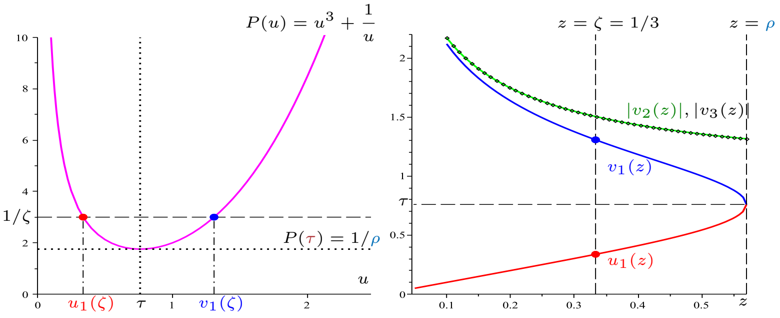
Theorem (Banderier-N. 2010)

Walks going ***beyond*** *the barrier* $+h$ *verify*

$$F^{[>h]}(z,u) = \frac{1}{1-zP(u)} \sum_{j=1}^{d} \frac{u^{h+1}}{v_j(z)^{h+1}} \prod_{\substack{1\le i\le d \\ i\neq j}} \frac{u-v_i}{v_j-v_i}$$

Gives fast computation scheme for the n-th coefficients via holonomy theory.

Roots properties (Banderier-Flajolet)



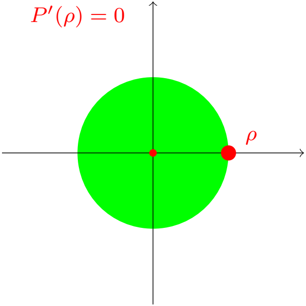
Left: behaviour of the characteristic polynomial $P(u) = u^3 + \frac{1}{u}$.

Right: domination property of the roots of $1 - zP(u) = 1 - z\left(u^3 + \frac{1}{u}\right)$ in $]0, \rho]$, where τ is the unique positive solution of $P'(z) = 0$ and $\rho = 1/P(\tau)$.

$P'(\tau) = 0 \quad \implies u_1(\rho) = v_1(\rho)$.

$u_1(z) < v_1(z) < |v_2(z)| = |v_3(z)|$ for $z \in]0, \rho[$.

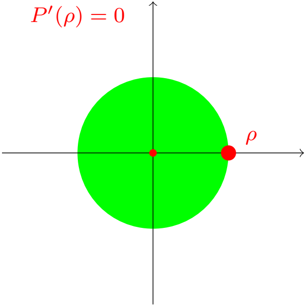
Roots properties (Banderier-Flajolet)

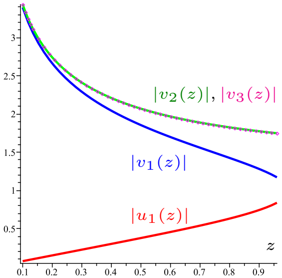


for $\epsilon < |z| < \rho$

$$\max_{i \geq 2} |u_i(z)|$$
$$< |u_1(z)|$$
$$< |v_1(z)|$$
$$< \min_{j \geq 2} |v_j(z)|$$

Roots properties (Banderier-Flajolet)





for $\epsilon < |z| < \rho$

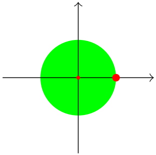
$$\max_{i\geq 2} |u_i(z)| < |u_1(z)| < |v_1(z)| < \min_{j\geq 2} |v_j(z)|$$

$$X_i \in \{+3, -1\} \quad \begin{cases} \mathbf{E}(X) = P'(1) = 0 \ (\rho = 1) \\ P(1) = 1 \end{cases}$$

$$z \sim 1^- \begin{cases} u_1(z) = 1 - \sqrt{\frac{2}{P''(1)}(1-z)} + O(1-z) \\ v_1(z) = 1 + \sqrt{\frac{2}{P''(1)}(1-z)} + O(1-z) \end{cases}$$

Asymptotics simplifications for $F^{[>h]}$ as $h \longrightarrow \infty$

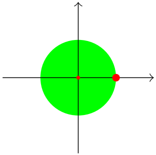
$$\frac{u^{h+1}}{v_j(z)^{h+1}} = \frac{u^{h+1}}{v_1(z)^{h+1}} \left(\frac{v_1(z)}{v_j(z)}\right)^{h+1} = O(A^h)$$



$$\left(j \geq 2, \quad A = \max_{j \geq 2} \sup_{|z| < \rho - \epsilon} \frac{|v_1(z)|}{|v_j(z)|} < 1 \right)$$

Asymptotics simplifications for $F^{[>h]}$ as $h \longrightarrow \infty$

$$\frac{u^{h+1}}{v_j(z)^{h+1}} = \frac{u^{h+1}}{v_1(z)^{h+1}} \left(\frac{v_1(z)}{v_j(z)}\right)^{h+1} = O(A^h)$$



$$\left(j \geq 2, \quad A = \max_{j\geq 2} \sup_{|z|<\rho-\epsilon} \frac{|v_1(z)|}{|v_j(z)|} < 1\right)$$

$$\Longrightarrow \quad F^{[>h]}(z,u) = \frac{1}{1-zP(u)} \sum_{j=1}^{d} \frac{u^{h+1}}{v_j(z)^{h+1}} \prod_{\substack{1\leq i\leq d \\ i\neq j}} \frac{u-v_i}{v_j-v_i}$$

$$= \frac{1}{1-zP(u)} \frac{u^{h+1}}{v_1(z)^{h+1}} \frac{Q(u)}{Q(v_1(z))} \left(1+O(A^h)\right)$$

where $Q(x) = \prod_{2\leq i\leq d} (x - v_i(z))$

Asymptotics simplifications for $[u^0]F^{[>h]}$ (bridges)

Thm. Banderier-Flajolet

$$(-k < -c) \qquad [u^{-k}]\frac{1}{1 - zP(u)} = z\sum_{j=1}^{c} \frac{u_j'(z)}{u_j(z)^{-k+1}}$$

Asymptotics simplifications for $[u^0]F^{[>h]}$ (bridges)

Thm. Banderier-Flajolet

$$(-k < -c) \qquad [u^{-k}]\frac{1}{1-zP(u)} = z\sum_{j=1}^{c}\frac{u_j'(z)}{u_j(z)^{-k+1}} = [u^0]\frac{u^k}{1-zP(u)}$$

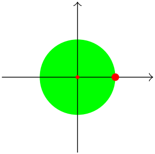
Asymptotics simplifications for $[u^0]F^{[>h]}$ (bridges)

Thm. Banderier-Flajolet

$(-k < -c)$ $$[u^{-k}]\frac{1}{1-zP(u)} = z\sum_{j=1}^{c}\frac{u_j'(z)}{u_j(z)^{-k+1}} = [u^0]\frac{u^k}{1-zP(u)}$$

$$Q(u) = \prod_{2\le j\le d}(u - v_j(z)) = \sum_{i=0}^{d-1} q_i(z)u^i$$

$$[u^0]F^{[>h]}(z,u) = [u^0]\frac{1}{1-zP(u)}\frac{u^{h+1}}{v_1(z)^{h+1}}\frac{Q(u)}{Q(v_1(z))}\left(1+O(A^h)\right)$$



$$= \frac{1}{v_1(z)^{h+1}Q(v_1(z))}\sum_{i=0}^{d-1} q_i(z)[u^0]\frac{u^{h+i+1}}{1-zP(u)}\left(1+O(A^h)\right)$$

$$= z\left(\frac{u_1(z)}{v_1(z)}\right)^h \times \frac{u_1'(z)Q(u_1(z))}{v_1(z)Q(v_1(z))} \times \left(1+O(C^h)\right)$$

$$\sup_{\substack{\epsilon<|z|<\rho\\ j\ge 2}} \frac{|u_1(z)|}{|u_j(z)|} < B \qquad C = \max(A, B)$$

Extracting asymptotically $[z^n][u^0]F^{[>x\sigma\sqrt{n}]}(z,u)$

$P(1) = 1, \quad P'(1) = 0, \quad \rho = 1, \quad \sigma^2 = P''(1)$

$$z \sim 1^- \begin{cases} u_1(z) = 1 - \sqrt{\frac{2}{\sigma^2}(1-z)} + O(1-z) \\ v_1(z) = 1 + \sqrt{\frac{2}{\sigma^2}(1-z)} + O(1-z) \\ \frac{Q(u_1(z))}{Q(v_1(z))} = \frac{Q(1) + O(\sqrt{1-z})}{Q(1) + O(\sqrt{1-z})} = 1 + O(\sqrt{1-z}) \end{cases}$$

$$[u^0]F^{[>x\sigma\sqrt{n}]}(z,u) = z\left(\frac{u_1(z)}{v_1(z)}\right)^{x\sigma\sqrt{n}} \times \frac{u_1'(z)Q(u_1(z))}{v_1(z)Q(v_1(z))} \times (1 + O(C^n))$$

Extracting asymptotically $[z^n][u^0]F^{[>x\sigma\sqrt{n}]}(z,u)$

$P(1)=1, \quad P'(1)=0, \quad \rho=1, \quad \sigma^2=P''(1)$

$$z \sim 1^- \begin{cases} u_1(z) = 1 - \sqrt{\frac{2}{\sigma^2}(1-z)} + O(1-z) \\ v_1(z) = 1 + \sqrt{\frac{2}{\sigma^2}(1-z)} + O(1-z) \\ \dfrac{Q(u_1(z))}{Q(v_1(z))} = \dfrac{Q(1)+O(\sqrt{1-z})}{Q(1)+O(\sqrt{1-z})} = 1 + O(\sqrt{1-z}) \end{cases}$$

$$[u^0]F^{[>x\sigma\sqrt{n}]}(z,u) = z\left(\frac{u_1(z)}{v_1(z)}\right)^{x\sigma\sqrt{n}} \times \frac{u_1'(z)Q(u_1(z))}{v_1(z)Q(v_1(z))} \times (1+O(C^n))$$

$$= \frac{z}{\sigma\sqrt{2}} \frac{\left(1-2\sqrt{\frac{2}{\sigma^2}(1-z)}\right)^{x\sigma\sqrt{n}}}{\sqrt{1-z}} \times \left(1+O(\sqrt{1-z})\right) \times (1+O(C^n))$$

Extracting asymptotically $[z^n][u^0]F^{[>x\sigma\sqrt{n}]}(z,u)$

$P(1)=1, \quad P'(1)=0, \quad \rho=1, \quad \sigma^2=P''(1)$

$$z \sim 1^- \begin{cases} u_1(z) = 1 - \sqrt{\frac{2}{\sigma^2}(1-z)} + O(1-z) \\ v_1(z) = 1 + \sqrt{\frac{2}{\sigma^2}(1-z)} + O(1-z) \\ \dfrac{Q(u_1(z))}{Q(v_1(z))} = \dfrac{Q(1)+O(\sqrt{1-z})}{Q(1)+O(\sqrt{1-z})} = 1 + O(\sqrt{1-z}) \end{cases}$$

$$[u^0]F^{[>x\sigma\sqrt{n}]}(z,u) = z\left(\frac{u_1(z)}{v_1(z)}\right)^{x\sigma\sqrt{n}} \times \frac{u_1'(z)Q(u_1(z))}{v_1(z)Q(v_1(z))} \times (1+O(C^n))$$

$$= \frac{z}{\sigma\sqrt{2}} \frac{\left(1-2\sqrt{\frac{2}{\sigma^2}(1-z)}\right)^{x\sigma\sqrt{n}}}{\sqrt{1-z}} \times \left(1+O(\sqrt{1-z})\right) \times (1+O(C^n))$$

Semi-large powers Banderier-Flajolet-Soria-Schaeffer (2001)

Asymptotics for upper bounded bridges

$$P(1) = 1, \quad P'(1) = 0, \quad \rho = 1, \quad \sigma^2 = P''(1)$$

$$[z^n][u^0]F^{[>x\sigma\sqrt{n}]} = \frac{\sqrt{n}}{\sigma\sqrt{2}} \times e^{-2x^2} \times \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Asymptotics for upper bounded bridges

$$P(1) = 1, \quad P'(1) = 0, \quad \rho = 1, \quad \sigma^2 = P''(1)$$

$$[z^n][u^0]F^{[>x\sigma\sqrt{n}]} = \frac{\sqrt{n}}{\sigma\sqrt{2}} \times e^{-2x^2} \times \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

but for unconditionned bridges (Banderier-Flajolet)

$$[z^n][u^0]F^{]-\infty,+\infty[} = \frac{\sqrt{n}}{\sigma\sqrt{2}} \times \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Theorem (Banderier-N. 2010)

$$\mathbf{P}\left(\max_{0\le i\le n} B_i > x\sigma\sqrt{n}\right) = e^{-2x^2} \times \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Asymptotics for upper bounded bridges

$$P(1) = 1, \quad P'(1) = 0, \quad \rho = 1, \quad \sigma^2 = P''(1)$$

$$[z^n][u^0]F^{[>x\sigma\sqrt{n}]} = \frac{\sqrt{n}}{\sigma\sqrt{2}} \times e^{-2x^2} \times \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

but for unconditionned bridges (Banderier-Flajolet)

$$[z^n][u^0]F^{]-\infty,+\infty[} = \frac{\sqrt{n}}{\sigma\sqrt{2}} \times \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Theorem (Banderier-N. 2010)

$$\mathbf{P}\left(\max_{0\le i\le n} B_i > x\sigma\sqrt{n}\right) = e^{-2x^2} \times \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Moreover we get a concentration property around $\sigma\sqrt{n}$

Full asymptotics for Łukasiewicz bridges

$X_i \in \{-1, \dots, +d\}$ only **one small root**

$Q(u_1(z))$ and $Q(v_1(z))$ expressible as functions of $u_1(z)$ and $v_1(z)$ only

$$Q(u) = \prod_{j=2}^{d}(u - v_j(z)) = \frac{u(1 - zP(u))}{p_d z(u - u_1(z))(u - v_1(z))}$$

$P'(u(z)) = -1/\big(z^2 u'(z)\big)$ for **any root** $u(z)$ of the kernel

$$Q(u_1(z)) = \frac{1}{p_d z} \frac{\partial}{\partial u} \frac{u(1 - zP(u))}{u - v_1(z)}\bigg|_{u=u_1(z)} = \frac{1}{p_d z^2} \frac{u_1(z)}{u_1'(z)(u_1(z) - v_1(z))}$$

Full asymptotics for Łukasiewicz bridges

Proposition (Banderier-N. 2010)

Łukasiewicz bridges verify asymptotically

$$[u^0]F^{[>h]}(z,u) = z\left(\frac{u_1(z)}{v_1(z)}\right)^h \times \frac{-v_1'(z)u_1(z)}{v_1(z)^2} \times (1 + O(C^h))$$

Full asymptotics for Łukasiewicz bridges

Proposition (Banderier-N. 2010)

Łukasiewicz bridges verify asymptotically

$$[u^0]F^{[>h]}(z,u) = z\left(\frac{u_1(z)}{v_1(z)}\right)^h \times \frac{-v_1'(z)u_1(z)}{v_1(z)^2} \times (1+O(C^h))$$

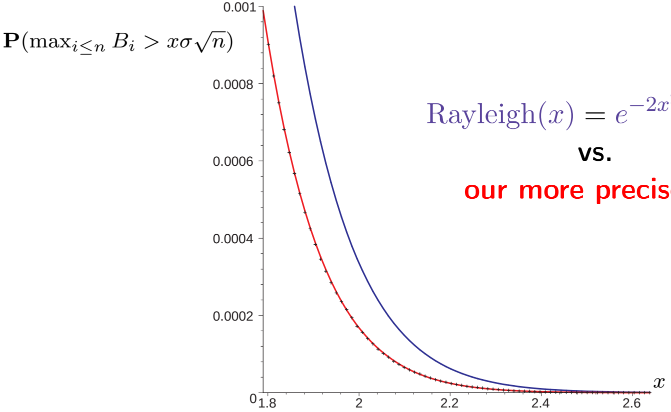
use gdev for expansions of $u_1(z)$ and $v_1(z)$

$$\frac{\beta_n^{>x\sigma\sqrt{n}}}{\exp(-2x^2)} = 1 + \frac{(-(2/3)x\xi/\zeta^{3/2} - 6x/\sqrt{\zeta})}{\sqrt{n}} + \frac{1}{n}\Bigg((-2 - \frac{10}{9}\frac{\xi^2}{\zeta^3} + \frac{2}{3}\frac{\theta}{\zeta^2} - \frac{16}{3\zeta} - \frac{8}{3}\frac{\xi}{\zeta^2})x^4$$
$$+(\frac{24}{\zeta} + \frac{5}{3}\frac{\xi^2}{\zeta^3} + 3 - \frac{\theta}{\zeta^2} + \frac{20}{3}\frac{\xi}{\zeta^2})x^2 - \frac{5}{\zeta} - \frac{3}{8} - \frac{7}{6}\frac{\xi}{\zeta^2} - \frac{5}{24}\frac{\xi^2}{\zeta^3} + \frac{1}{8}\frac{\theta}{\zeta^2} + \frac{5}{24}\frac{\xi^3}{\zeta^3} - \frac{1}{8}\frac{\theta^2 - 3\zeta^2}{\zeta^2}\Bigg)$$
$$+ O\left(\frac{1}{n^{3/2}}\right)$$

$$\beta_n^{>x\sigma\sqrt{n}} = \mathbf{P}\left(\max_{0\le i\le n} B_i\right) > x\sigma\sqrt{n},$$

$$\begin{cases} \zeta = \sigma^2 = P''(1), \\ \xi = P'''(1), \quad \theta = P''''(1) \end{cases}$$

Back to simulations



$\text{Rayleigh}(x) = e^{-2x^2}$

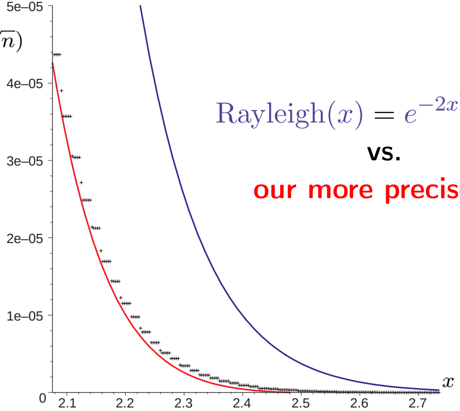
vs.

our more precise asymptotics

$$X \in \{-1, +19\} \qquad n = 400$$

Heuristics for bioinformatics - rational jumps

$\mathbf{P}(\max_{i\leq n} B_i > x\sigma\sqrt{n})$



$\mathrm{Rayleigh}(x) = e^{-2x^2}$

vs.

our more precise asymptotics

$$X \in \{-11, +93\} \rightsquigarrow X' \in \left\{-1, +\frac{93}{11}\right\} \qquad n = 104$$

Upper and lower bounded Brownian bridges

$$\mathbf{P}\left(\max_{t\in[0,1]}|B_t|>x\right)=2\sum_{k\geq 1}(-1)^{k+1}e^{-2k^2x^2}$$

Fourier theory:

$$G(x)=\sum_{k=-\infty}^{\infty}e^{-\pi k^2x^2} \quad\Longrightarrow\quad G(x)=\frac{1}{x}G\left(\frac{1}{x}\right)$$

$$\mathbf{P}\left(\max_{t\in[0,1]}|B_t|<x\right)=2G\left(\frac{4x}{\sqrt{2\pi}}\right)-G\left(\frac{2x}{\sqrt{2\pi}}\right)$$

for small x, use the functional equation relating $G(x)$ and $G(1/x)$