On the Diameter of Kneser Graphs *

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Abstract

Let n and k be positive integers. The Kneser graph K_n^{2n+k} is the graph with vertex set $[2n+k]^n$ and where two n-subsets $A, B \in [2n+k]^n$ are joined by an edge if $A \cap B = \emptyset$. In this note we show that the diameter of the Kneser graph K_n^{2n+k} is equal to $\lceil \frac{n-1}{k} \rceil + 1$.

Keywords: Kneser graphs, Diameter of graphs.

1 Preliminaries

Let G be a connected graph. Given two vertices a, b in G, $dist(a, b)$, the distance between a and b, is defined as the length of the shortest path in G joining a to b. The diameter of G is defined as the maximum distance between any pair of vertices in G.

Let n, k be positive integers and let $[2n+k] = \{1, 2, \ldots, 2n+k\}$. Let $[2n+k]^n$ be the set of n-subsets of $[2n+k]$. The Kneser graph K_n^{2n+k} is the graph with vertex set $[2n+k]^n$ and where two n-subsets $A, B \in [2n+k]^n$ are joined by an edge if $A \cap B = \emptyset$. Note that K_2^5 is the well-known Petersen graph. It is easy to show that the Kneser graph K_n^{2n+k} is a connected regular graph having $\binom{2n+k}{n}$ vertices of degree $\binom{n+k}{n}$. Theoretical properties of Kneser graphs have been studied in past years (see for example [1, 2, 3] and ref.) In this note, we are interested in computing the diameter of these graphs. As far as we know this parameter has not been studied for such graphs.

Stahl shows in [3] the following result.

Lemma 1 *Let $A, B \in [2n+k]^n$ be two vertices in K_n^{2n+k} joined by a path of length $2p$ ($p \geq 0$). Then, $|A \cap B| \geq n - kp$.*

The following result is a consequence of Lemma 1.

Corollary 1 *Let $A, B \in [2n+k]^n$ be two vertices in K_n^{2n+k} joined by a path of length $2p+1$ ($p \geq 0$). Then, $|A \cap B| \leq kp$.*

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2 Main results

We start by showing the following result.

Proposition 1 *Let n, k be positive integers, with $n \geq 2$ and $k \geq n-1$. Then, the diameter of Kneser graph K_n^{2n+k} is equal to 2.*

Proof : By hypothesis, $k \geq n-1$, and so there are two vertices A and B non adjacent in K_n^{2n+k}. Therefore, the diameter of K_n^{2n+k} is at least equal to 2. Let A and B any two vertices non adjacent in K_n^{2n+k}, $1 \leq |A \cap B| \leq n-1$, and therefore $|[2n+k] \setminus (A \cup B)| \geq n$, which implies that there exists a vertex D of K_n^{2n+k} such that $A \cap D = \emptyset$ and $B \cap D = \emptyset$, and thus the diameter of K_n^{2n+k} is at most equal to 2. $\square$

Lemma 2 *Let $A, B \in [2n+k]^n$ be two different vertices of the Kneser graph K_n^{2n+k}, where $1 \leq k < n-1$. If $|A \cap B| = s$ then dist$(A, B) = \min\{2\lceil \frac{n-s}{k} \rceil, 2\lceil \frac{s}{k} \rceil + 1\}$.*

Proof : Let $C = A \cap B$, and let $s = |C|$. Let $D = [2n+k] \setminus (A \cup B)$, where $|D| = s + k$. Assume that $A = \{a_1, \ldots, a_{n-s}\} \cup C$, and $B = \{b_1, \ldots, b_{n-s}\} \cup C$. Let $t = 2\lceil \frac{n-s}{k} \rceil$. Consider the path $A = X_0, X_1, \ldots, X_t = B$ between A and B, where for $i < \frac{n-s}{k}$,

$$X_{2i-1} = \{a_1, \ldots, a_{(i-1)k}, b_{ik+1}, \ldots, b_{n-s}\} \cup D \text{ and } X_{2i} = \{b_1, \ldots, b_{ik}, a_{ik+1}, \ldots, a_{n-s}\} \cup C.$$

And $X_{t-1} = \{a_1, \ldots, a_{n-s-k}\} \cup D,\ X_t = B$.

Also let $D' \subseteq D$ such that $|D'| = s$. Consider the vertex $A' = (B \setminus C) \cup D'$. Note that $A \cap A' = \emptyset$, and $s' = |A' \cap B| = n - s$. Therefore, by the previous construction, there is a path between A' and B whose length is $2\lceil \frac{n-s'}{k} \rceil = 2\lceil \frac{s}{k} \rceil$. Thus, there is a path between A and B with length equal to $2\lceil \frac{s}{k} \rceil + 1$. So, $\text{dist}(A, B) \leq \min\{2\lceil \frac{n-s}{k} \rceil, 2\lceil \frac{s}{k} \rceil + 1\}$.

Conversely, let $d = \text{dist}(A, B)$. By Lemma 1, if $d = 2p$ then $s = |A \cap B| \geq n - kp$. Thus, $d = 2p \geq 2\lceil \frac{n-s}{k} \rceil$. Otherwise, if $d = 2p+1$ then, by Corollary 1, $s = |A \cap B| \leq kp$. Thus, $d = 2p + 1 \geq 2\lceil \frac{s}{k} \rceil + 1$. Therefore, $\text{dist}(A, B) \geq \min\{2\lceil \frac{n-s}{k} \rceil, 2\lceil \frac{s}{k} \rceil + 1\}$. $\square$

Theorem 1 *Let n and k be positive integers. Then, the diameter of the Kneser graph K_n^{2n+k} is equal to $\lceil \frac{n-1}{k} \rceil + 1$.*

Proof : Let D denote the diameter of the Kneser graph K_n^{2n+k}. Let $g(s) = 2\lceil \frac{n-s}{k} \rceil$, $h(s) = 2\lceil \frac{s}{k} \rceil + 1$ and $f(s) = \min\{g(s), h(s)\}$ be integer functions, where s takes values from the set $[n-1]$. By Lemma 2, we have that $D = \max_s f(s)$, and therefore it is enough to show

$$\max_s f(s) = \left\lceil \frac{n-1}{k} \right\rceil + 1.$$

Let $n-1 = 2qk + \varepsilon k + y$, where $q \geq 0$, $\varepsilon \in \{0, 1\}$ and $0 < y \leq k$. Thus, $\lceil \frac{n-1}{k} \rceil + 1 = 2q + \varepsilon + 2$. Now, let $s_0 = qk + \varepsilon k$. If $s \leq s_0$ then $f(s) \leq h(s) \leq h(s_0) = 2q + 2\varepsilon + 1 \leq 2q + \varepsilon + 2$, and if $s > s_0$ then $f(s) \leq g(s) \leq g(s_0 + 1) = 2q + 2 \leq 2q + \varepsilon + 2$. Therefore, $\max_s f(s) \leq \lceil \frac{n-1}{k} \rceil + 1$.

Conversely, we consider two cases:

- $\varepsilon = 0$. For this case, let $s_1 = qk + y$. Then, $g(s_1) = 2q + 2$ and $h(s_1) = 2q + 3$. So, $f(s_1) = g(s_1) = \lceil \frac{n-1}{k} \rceil + 1$.

- $\varepsilon = 1$. For this case, let $s_2 = qk + 1$. Then, $g(s_2) = 2q + 4$ and $h(s_2) = 2q + 3$. So, $f(s_2) = h(s_2) = \lceil \frac{n-1}{k} \rceil + 1$.

Therefore, $\max_s f(s) \geq \lceil \frac{n-1}{k} \rceil + 1$, which ends the proof of this theorem. $\square$

References

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