Some problems on idomatic partitions and b-colorings of direct products of complete graphs*

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Abstract

In this note, we deal with the characterization of the idomatic partitions and b-colorings of direct products of complete graphs. We recall some known results on idomatic partitions of direct products of complete graphs and we present new results concerning the b-colorings of the direct product of two complete graphs. Finally, some open problems are given.

Keywords: direct product, complete graphs, idomatic partitions, b-colorings.

1 Introduction and preliminary results

Let $G = (V, E)$ be an undirected finite simple graph without loops (see reference [3] for classical concepts in graph theory). A set $S \subseteq V$ is called a *dominating set* if for every vertex $v \in V \setminus S$ there exists a vertex $u \in S$ such that u is adjacent to v. A set $S \subseteq V$ is called *independent* if no two vertices in S are adjacent. A set $S \subseteq V$ is called an *independent dominating set* of G if it is both independent and dominating. A partition of the vertex set V into independent dominating sets is called an *idomatic partition* of G [1, 2]. The maximum size of an idomatic partition of G is called the *idomatic number* $id(G)$. An idomatic partition of a graph G into k parts is called an *idomatic k-partition of G*. Notice that not every graph has an idomatic k-partition, for any k. For example, the cycle graph on five vertices C_5 has no idomatic k-partition for any k. A *proper coloring* (*coloring* for short) of

*This work is supported by the Math-AmSud Projet 10MATH-04 (France-Argentine-Brazil).

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G is an assignment of colors to the vertices of G such that adjacent vertices are assigned different colors. A k-coloring of G is a coloring using exactly k different colors. The smallest number k such that G admits a k-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. Given a k-coloring of G, a vertex v is said to be *dominant* if v is adjacent to at least one vertex receiving each of the $k-1$ colors not assigned to v. As remarked by Dunbar et al. in [4], an idomatic partition of a graph G represents a proper coloring of the vertices of G where all vertices are dominant. A *b-coloring* of G [7] is a coloring such that every color class admits at least one dominant vertex. So, b-colorings are relaxed versions of idomatic partitions. Notice that every coloring of G with $\chi(G)$ colors is a b-coloring. The *b-chromatic number* of G, denoted by $\chi_b(G)$, is the maximum number k such that G admits a b-coloring with k colors.

The *direct product* $G\times H$ of two graphs G and H is defined by $V(G\times H) = V(G)\times V(H)$, and where two vertices $(u_1,u_2),(v_1,v_2)$ are joined by an edge in $E(G\times H)$ if $\{u_1,v_1\}\in E(G)$ and $\{u_2,v_2\}\in E(H)$. This product is commutative and associative in a natural way (see reference [6] for a detailed description on product graphs).

Let n be a positive integer. We denote by $[n]$ the set $\{1,\dots,n\}$. The complete graph K_n will usually be on the vertex set $[n]$.

Idomatic partitions of graphs were studied in [4] as a special coloring problem on graphs defined as *fall colorings*. In that work, the authors show the following result.

Theorem 1 ([4]). *Let $n_1>1$ and $n_2>1$ be two integers. The direct product graph $K_{n_1}\times K_{n_2}$ admits an idomatic n_1-partition and an idomatic n_2-partition. Furthermore, if $t>1$ is an integer such that $t\notin\{n_1,n_2\}$, then $K_{n_1}\times K_{n_2}$ has no idomatic t-partition.*

Moreover, in [4] it is posed the question of characterizing the idomatic partitions of the direct product of three or more complete graphs. Recently, in [9] it is given a full characterization of the idomatic partitions of the direct product of three complete graphs. By following the same ideas given in [9], in [8] it is given a characterization of the idomatic sets of a direct product of four complete graphs.

The direct product of graphs $G_1,G_2,\dots,G_n$ will be denoted $\times_{i=1}^{n}G_i$. Let $G=\times_{i=1}^{k}K_{n_i}$ and let $u=(u_1,\dots,u_k)$ and $v=(v_1,\dots,v_k)$ be vertices of G. Then, let

$$e(u,v)=|\{i:u_i=v_i\}|$$

be the number of coordinates in which u and v coincide. With this notation we can state that u and v are adjacent in $G=\times_{i=1}^{k}K_{n_i}$ if and only if

$e(u,v)=0$. Therefore, $I \subseteq V(G)$ is independent if and only if $e(u,v)>0$ for any $u,v \in I$. Note also that $e(u,v) \leq k-1$ holds for any $u \neq v$.

Let $X \subset V(G)$, where $G = \times_{i=1}^{k} K_{n_i}$, and let

$$\{e(u,v) : u,v \in X \text{ and } u \neq v\} = \{j_1, \ldots, j_r\}.$$

Then, we say that X is a $T_{j_1,\ldots,j_r}$-set.

This note is organized as follows. Let $G = \times_{i=1}^{k} K_{n_i}$. In Section 2, we summarize the results concerning independent dominating sets and idomatic partitions of G and we pose two problems. Finally, Section 3 contains some new results concerning b-colorings of the direct product of two complete graphs and an interesting conjecture is posed.

2 Independent dominating sets and idomatic partitions in $\times_{i=1}^{k} K_{n_i}$

In this section we give two open problems concerning the independent dominating sets and idomatic partitions in the graph $G = \times_{i=1}^{k} K_{n_i}$, with $k \geq 2$ and $n_i \geq 2$. We start by rephrasing known results in terms of $T_{j_1,\ldots,j_r}$-sets.

Notice first that if $G = \times_{i=1}^{2} K_{n_i}$, with $n_i \geq 2$, and if $I \subseteq V(G)$ is an independent dominating set, then I is a T_1-set (see [4]). For $k > 2$ we have:

Proposition 1 ([9]). *Let $G = \times_{i=1}^{3} K_{n_i}$, with $n_i \geq 2$, and let I be an independent dominating set of G. Then, I is either a T_1-set or a $T_{1,2}$-set.*

Moreover, in [9] it is characterized the structure of the independent dominating sets of $\times_{i=1}^{3} K_{n_i}$. Such results have been extended to the case $k = 4$ as follows :

Proposition 2 ([8]). *Let $G = \times_{i=1}^{4} K_{n_i}$, with $n_i \geq 2$, and let I be an independent dominating set of G. Then, I is either a T_1-set, or a $T_{1,2}$-set or a $T_{1,2,3}$-set*

Problem 1. *Let $G = \times_{i=1}^{k} K_{n_i}$, with $k, n_i \geq 2$. Then, for all $i \in [k-1]$, does there exist an independent dominating set of G which is a $T_{1,2,\ldots,i}$-set ?*

Notice that problem 1 holds for $k = 2, 3$ and 4 (see [4, 9, 8]). For $k > 4$ this is an open problem.

Let pr_i denote the projection homomorphism from G to the i^{th} factor K_{n_i}. It is not difficult to deduce that for each $i = 1, \ldots, k$, the sets

$pr_i^{-1}(1), \ldots, pr_i^{-1}(n_i)$ form an idomatic partition of G into $T_{1,2,\ldots,k-1}$-sets and such partitions are the only idomatic partitions of such type (see [9] for details).

For $k = 3$, it has been characterized in [9] the idomatic partitions into idomatic T_1-sets. Moreover, in [9] it is also proved that there exist idomatic partitions composed of T_1-sets and $T_{1,2}$-sets, and it is described how to construct such partitions. Therefore, from the total characterization of the idomatic partitions of the graph $G = \times_{i=1}^3 K_{n_i}$, the idomatic number of G can be easily deduced.

For $k = 4$, it has been characterized in [8] the idomatic partitions into idomatic T_1-sets and an example of idomatic partition into idomatic $T_{1,2}$-sets is given. However, it is not known whether there are idomatic partitions formed by various types of idomatic sets.

Problem 2. *For $k \geq 4$, a complete characterization of the idomatic partitions of the graph $\times_{i=1}^k K_{n_i}$ is still open.*

3 b-colorings of $\times_{i=1}^k K_{n_i}$

In this section we study the b-colorings of the graph $K_n \times K_m$, with $2 \leq n \leq m$. The main result of this section is the following theorem.

Theorem 2. *Let $G = K_n \times K_m$, with $2 \leq n \leq m$. Let Φ be a coloring of G. Thus, Φ is a b-coloring of G if and only if Φ induces an idomatic partition of G.*

As a consequence of Theorem 2, we have that G has only two b-colorings corresponding to the only two idomatic partitions of G, one with n colors and the other with m colors. Therefore, the b-chromatic number of G is equal to m.

It is clear that any idomatic partition of G is in fact a b-coloring of G. In order to prove the converse statement, we will prove the following lemmas.

Lemma 1. *Let $G = K_n \times K_m$ with $2 \leq n \leq m$. Assume that G is b-colored and that vertices (i, j) and (i, t) are dominant vertices for different colors a and b respectively, where $i \in [n]$ and $j, t \in [m]$ with $j \neq t$. Then, for any $k \in [n]$, with $k \neq i$, there exists no dominant vertex (k, s), with $s \in \{j, t\}$, for a color $c \notin \{a, b\}$.*

Proof. Let $a \neq b$ be the colors of vertices $x = (i, j)$ and $y = (i, t)$ respectively. By hypothesis, x and y are dominant vertices. So, by definition of direct product, there exist two vertices $x' = (i', j)$ and $y' = (i'', t)$, with $i', i'' \neq i$, such that vertex x' is colored with color a and vertex y' is colored with color b. Now, assume that there is a dominant vertex $z = (k, j)$, with $k \neq i, i'$, colored with a color c different from a and b. Clearly, this is impossible because vertex z has no neighbor colored with color a. Therefore, z can not be a dominant vertex for the color c. In an analogous way we can deduce that no vertex (k, t), with $k \neq i, i''$, can be a dominant vertex for a color different from a and b. □

Lemma 2. *Let $G = K_n \times K_m$ with $2 \leq n \leq m$. Assume that G is b-colored and that vertices (i, j) and (i, t) are dominant vertices for different colors a and b respectively, where $i \in [n]$ and $j, t \in [m]$, with $j \neq t$. For any $k \notin \{j, t\}$ and $i', i'' \in [n] \setminus \{i\}$, with $i' \neq i''$, there exist no dominant vertices (i', k) and (i'', k) for different colors c and d respectively, with c and d different from a and b.*

Proof. By Lemma 1, we have that $k \neq j, t$ and $i', i'' \neq i$. Without loss of generality, assume that $i < i' < i''$. As in the proof of Lemma 1, we can deduce that there are vertices $(s_1, j), (s_2, t), (i', p_1)$ and (i'', p_2) colored with colors a, b, c and d resp. with $s_1, s_2 \neq i$ and $p_1, p_2 \neq k$. Now, consider vertex (i, k). Such a vertex can not be colored with any color in $\{a, b, c, d\}$. Thus, let e be the color of (i, k). Now, as (i, j) is a dominant vertex for color a, it must have a neighbor (u, v) colored with color e. By construction, $u \neq i$ and $v = k$. Moreover, as (i', k) is a dominant vertex for color c, it must have a neighbor (u', v') colored with color e, with $v' \neq k$, which is not possible. An analogous contradiction is obtained for vertices $(i, j), (i, t)$ and (i'', t). □

By the commutativity of the direct product, a direct consequence of the previous lemma is the following corollary.

Corollary 1. *Let $G = K_n \times K_m$, with $2 \leq n \leq m$. Then, the b-chromatic number of G is equal to m.*

Lemma 3. *Let $G = K_n \times K_m$, with $2 \leq n \leq m$. Then, G has only b-colorings with n and m colors. Moreover, such b-colorings are always idomatic partitions of G.*

Proof. Assume that G has a b-coloring with k colors. By Corollary 1, we know that $k \leq m$. Moreover, it is well known that the chromatic number of G is equal to $\min\{n, m\} = n$, and thus $k \geq n$. So, assume first that $n < k < m$. Let s be the minimum number of columns containing at least one dominant vertex for each one of the k color classes. Reordering the columns representing G, we can assume w.l.o.g. that these are the first s columns. Moreover, by Lemmas 1 and 2, we can reorder the rows in such a way that the first p ones contain at least one dominant vertex for each one of the k colors classes, where p is the smallest positive integer for which this property holds. Moreover, we can assume that the first t rows of these p ones contain at least two dominant vertices of different colors. We consider the following cases :

- *Case* $t = 0$. In this case, by applying Lemma 1 to the columns of G, we have that k is at most equal to n, which is a contradiction to the definition of k.

- *Case* $t > 0$. By Lemmas 1 and 2, we can assume that dominant vertices for different colors in row i occur in consecutive positions, with $i \in \{1, \ldots, t\}$. Suppose that $t = 2$ and assume w.l.o.g. that the dominant vertices for different colors in rows 1 and 2 are $(1, 1), \ldots, (1, u_0)$ and $(2, u_0 + 1), \ldots, (2, u_0 + u_1)$, with $u_0, u_1 > 1$ respectively. Now, by using arguments as in the proof of Lemma 1, we know that for each vertex $(1, h)$, with $1 \leq h \leq u_0$, there exists at least one vertex (r, h), with $r \neq 1$, having the same color as $(1, h)$. Therefore, the color of vertex $(1, u_0 + w)$, with $1 \leq w \leq u_1$, must be equal to the color assigned to vertex $(2, u_0 + w)$. Otherwise, there is a contradiction to the assumption that dominant vertices in row 1 occur in consecutive positions. Moreover, each vertex $(1, u_0 + w)$ is a dominant vertex as $(2, u_0 + w)$ which is a contradiction to the minimality of p. Therefore, we have $t = 1$. By repeating the previous reasoning, we can deduce that p must be equal to 1. Now, we claim that $s = m$. Suppose that $s < m$ and consider vertex $(1, s + 1)$. By construction, such a vertex has at least one neighbor colored with each color in $\{1, \ldots, k\}$, and so, it must be assigned a color not in $\{1, \ldots, k\}$, which is a contradiction to the definition of k. So, the only possibility that remains is that $s = m$, and so in all cases there is a contradiction to the definition of k. Therefore, k must be equal to n or m. Finally, note that each vertex $(1, i)$ is a dominant vertex for the color class i, for $i = 1, \ldots, m$. So, each column i is colored with color i, which is an idomatic partition of G into m idomatic sets.

The same arguments can be used for the rows in order to prove that any b-coloring of G with n colors is an idomatic partition of G into n idomatic sets, where each row i is colored with color i. □

We present the main open question of this section as a conjecture.

Conjecture 1. *Let $G = \times_{i=1}^{k} K_{n_i}$, with $k, n_i \geq 2$. Then, any b-coloring of G is an idomatic partititon of G.*

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