Independence number of products of Kneser graphs

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November 19, 2018

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Abstract

We study the independence number of a product of Kneser graph $K(n,k)$ with itself, where we consider all four standard graph products. The cases of the direct, the lexicographic and the strong product of Kneser graphs are not difficult (the formula for $\alpha(K(n,k) \boxtimes K(n,k))$ is presented in this paper), while the case of the Cartesian product of Kneser graphs is much more involved. We establish a lower bound and an upper bound for the independence number of $K(n,2)\square K(n,2)$, which are asymptotically tending to $n^3/3$ and $3n^3/8$, respectively. The former is obtained by a construction, which differs from the standard diagonalization procedure, while for the upper bound the ℓ-independence number of Kneser graphs can be applied. We also establish some constructions in odd graphs $K(2k+1,k)$, which give a lower bound for the 2-independence number of these graphs, and prove that two such constructions give the same lower bound as a previously known one. Finally, we consider the s-stable Kneser graphs $K(ks+1,k)_{s-\mathrm{stab}}$, derive a formula for their ℓ-independence number, and give the exact value of the independence number of the Cartesian square of $K(ks+1,k)_{s-\mathrm{stab}}$.

Keywords: independence number, Kneser graph, graph product, Cartesian product, 2-independence number

AMS Subj. Class. (2010): 05C69, 05C76, 05D05

1 Introduction

The *Kneser graph*, $K(n,k)$, where n,k are positive integers such that $n > k$, has as the vertices the k-subsets of an n-set, and two k-subsets are adjacent in $K(n,k)$ if they are disjoint. By the famous Erdős-Ko-Rado theorem [5], the independence number $\alpha(K(n,k))$ of the Kneser graph $K(n,k)$, where $n \geq 2k$, equals $\binom{n-1}{k-1}$. The result was proven in the language of sets (or hypergraphs), showing that $\binom{n-1}{k-1}$ is the largest number of k-subsets of an n-set that are pairwise non-disjoint. This note is motivated by various related questions that can be posed in different products of two Kneser graphs, or, in other words, in the Cartesian product of families of k-subsets of an n-set.

Recall that for all of the standard graph products, the vertex set of the product of graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows. In the *lexicographic product* $G \circ H$ (also called *the composition* and denoted by $G[H]$), vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $g_1g_2 \in E(G)$ or ($g_1 = g_2$ and $h_1h_2 \in E(H)$). In the *strong product* $G \boxtimes H$ of graphs G and H vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever ($g_1g_2 \in E(G)$ and $h_1 = h_2$) or ($g_1 = g_2$ and $h_1h_2 \in E(H)$) or ($g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$). In the *direct product* $G \times H$ of graphs G and H two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. Finally, in the *Cartesian product* $G \square H$ of graphs G and H two vertices (g_1, h_1) and (g_2, h_2) are adjacent when ($g_1g_2 \in E(G)$ and $h_1 = h_2$) or ($g_1 = g_2$ and $h_1h_2 \in E(H)$). For a comprehensive survey on graph products cf. the monograph [10].

Let $\mathcal{K}_{n,k}$ represent the family of k-subsets of a given n-set. One can consider the following questions about the ordered pairs of sets in $X_{n,k}$ $(= \mathcal{K}_{n,k} \times \mathcal{K}_{n,k})$:

(a) What is the largest number of ordered pairs of sets $(A, B) \in X_{n,k}$ such that for any two pairs (A_1, B_1) and (A_2, B_2), we have either $A_1 = A_2$ and $B_1 \cap B_2 \neq \emptyset$, or $A_1 \neq A_2$ and $A_1 \cap A_2 \neq \emptyset$?

In other words, determine $\alpha(K(n, k) \circ K(n, k))$.

(b) What is the largest number of ordered pairs of sets $(A, B) \in X_{n,k}$ such that for any two distinct pairs (A_1, B_1) and (A_2, B_2), we have $A_1 \neq A_1 \cap A_2 \neq \emptyset$ or $B_1 \neq B_1 \cap B_2 \neq \emptyset$?

This question can be restated as determination of $\alpha(K(n, k) \boxtimes K(n, k))$. Indeed, two vertices (A_1, B_1) and (A_2, B_2) are non-adjacent in $K(n, k) \boxtimes K(n, k)$ if at least one of the following is true:

- $A_1 \neq A_2$ and $A_1A_2 \notin E(K(n, k))$, or
- $B_1 \neq B_2$ and $B_1B_2 \notin E(K(n, k))$

This is in turn equivalent to $A_1 \cap A_2 \neq \emptyset$ and $A_1 \neq A_2$, or $B_1 \cap B_2 \neq \emptyset$ and $B_1 \neq B_2$.

(c) What is the largest number of ordered pairs of sets $(A, B) \in X_{n,k}$ such that for any two distinct pairs (A_1, B_1) and (A_2, B_2), we have $A_1 \cap A_2 \neq \emptyset$ or $B_1 \cap B_2 \neq \emptyset$?

In other words, determine $\alpha(K(n, k) \times K(n, k))$.

(d) What is the largest number of ordered pairs of sets $(A, B) \in X_{n,k}$ such that for any two distinct pairs (A_1, B_1) and (A_2, B_2) at least one of the three conditions holds: $A_1 \neq A_1 \cap A_2 \neq \emptyset$, or $B_1 \neq B_1 \cap B_2 \neq \emptyset$ or ($A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$)?

This question can be restated as determination of $\alpha(K(n, k) \square K(n, k))$. Indeed, note that two vertices (A_1, B_1) and (A_2, B_2) are non-adjacent in $K(n, k) \square K(n, k)$ if at least one of the following is true:

- $A_1 \neq A_2$ and $A_1 \cap A_2 \neq \emptyset$, or
- $B_1 \neq B_2$ and $B_1 \cap B_2 \neq \emptyset$, or

- $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$.

The question (c) was resolved completely in [21], where it was shown that $\alpha(G \times H) = \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$. The question (a) can be easily answered even in the context of general graphs, notably $\alpha(G \circ H) = \alpha(G)\alpha(H)$ for any two graphs G, H, see e.g. [10]. In this paper we solve the problem (b), and shed some light on the problem (d). In particular, we show that $\alpha(K(n,k) \boxtimes K(n,k)) = \binom{n-1}{k-1}^2$ (see Section 3), while for the case of Cartesian products, we present some upper and lower bounds for $\alpha(K(n,2) \Box K(n,2))$ (see Section 4). For the lower bound, which is asymptotically $n^3/3$, we present a construction, which is different from standard constructions using diagonalization procedures, cf. Klavžar [11]. As it turns out, the problem of bounding the independence number of $K(n,k) \Box K(n,k)$ from above is closely related to determination of the so-called ℓ-independence number α_ℓ of $K(n,k)$, which is the size of a largest subset of vertices that induces an ℓ-colorable subgraph of $K(n,k)$. A formula for $\alpha_\ell(K(n,k))$ was established by Erdős [4] in the case when n is sufficiently large, and was later refined by Frankl and Füredi [6] and others, but the complete description is still open. Therefore in Section 5 we consider this number, and in the case of odd graphs present two constructions, which both give the same lower bounds for $\alpha_2(K(2k+1,k))$, which is in turn the same as the lower bound presented in the recent book of Godsil and Meagher [7] (it is non-trivial to establish the equality of these bounds, and we spend a good part of this section to do this). Finally, in Section 6 we consider the s-stable Kneser graphs $K(ks+1,k)_{s-\mathrm{stab}}$, establish that their ℓ-independence number equals ℓk, and prove that $\alpha(K(ks+1,k)_{s-\mathrm{stab}} \Box K(ks+1,k)_{s-\mathrm{stab}}) = k(ks+1)$.

2 Preliminaries and notation

Let G be a graph. The *order* of G will be denoted by $|V(G)|$, or simply $|G|$. A set S is *independent* (or *stable*) in G if any pair of vertices $u, v \in S$ are non-adjacent in G. Maximum independent sets in G will be also called *α-sets* in G. The *independence number* $\alpha(G)$ of a graph G is the cardinality of an α-set in G. Thus an independent set S in G is an α-set whenever $|S| = \alpha(G)$.

Given a graph G and a positive integer ℓ let $\alpha_\ell(G)$ denote the *ℓ-independence number* of a graph G, i.e., the maximum order of an ℓ-colorable induced subgraph of G. Clearly, $\alpha_1(G) = \alpha(G)$, and $\alpha_\ell(G) = |V(G)|$ if and only if $\ell \geq \chi(G)$.

Given the Kneser graph $K(n,k)$ and $x \in [n] = \{1, \ldots, n\}$, we define $I(x)$ to be the *set with center* x as follows: $I(x) = \{A \in K(n,k) \,|\, x \in A\}$. It is clear that $I(x)$ is an independent set in $K(n,k)$, whose size is $\binom{n-1}{k-1}$, which equals $\alpha(K(n,k))$ by Erdős-Ko-Rado theorem. For $x, y \in [n]$, let $I(x_y) = \{A \in K(n,k) \mid x \in A, y \notin A\}$. For instance, in $K(6,2)$ the set $I(1_2)$ is the set $\{13, 14, 15, 16\}$, where the notation for sets in $K(n,k)$ is simplified so that we write elements of $K(n,k)$ without commas in between and with no brackets. Similarly, $I(xy_z)$ is defined as $I(xy_z) = \{A \in K(n,k) \mid x, y \in A, z \notin A\}$ for any $x, y, z \in [n]$. For example, in $K(7,3)$ we have $I(12_3) = \{124, 125, 126, 127\}$.

Let $G * H$ denote the product of graphs G and H, where $* \in \{\Box, \circ, \times, \boxtimes\}$. We denote by G^{m*} the *mth power of a graph G with respect to a given product* $*$. As all standard graph products are associative, this is well defined. For instance, by $G^{m\boxtimes}$ the m-power of G (with respect to the strong product of graphs) is defined as $\underbrace{G \boxtimes \cdots \boxtimes G}_{m \text{ factors}}$.

The *distance* between two vertices in a graph G is the number of edges on a shortest path connecting them. Let $\overline{G}$ denote the complement graph of the graph G, i.e. $\overline{G}$ has the same vertex set of G and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Let p be a positive integer. The *pth power* of a graph G, that we denote by $G^{(p)}$, is the graph having the same vertex set as G and where two vertices are adjacent in $G^{(p)}$ if the distance between them in G is at most p.

3 Strong product of Kneser graphs

The following observation follows from the fact that multiplying maximum independent sets of both factors gives an independent set in the strong product of two graphs.

Observation 1. $\alpha(K(n,k) \boxtimes K(n,k)) \geq (\alpha(K(n,k)))^2 = \binom{n-1}{k-1}^2$.

The *Shannon capacity* $\Theta(G)$ of a graph G is defined as

$$\Theta(G) = \sup_{m \in \mathbb{N}} \sqrt[m]{\alpha(G^{m\boxtimes})}.$$

As proven by Lóvasz in [13], $\Theta(K(n,k)) = \binom{n-1}{k-1}$, which, in particular, implies that

$$\sqrt{(\alpha(K(n,k) \boxtimes K(n,k))} \leq \binom{n-1}{k-1}.$$

Combined with Observation 1 this implies

Theorem 2. *For any n and any $k \leq n/2$ we have* $\alpha(K(n,k) \boxtimes K(n,k)) = \binom{n-1}{k-1}^2$.

In fact, by the same reasoning one can deduce that

$$\alpha((K(n,k)^{m\boxtimes}) = \binom{n-1}{k-1}^m.$$

4 Cartesian product of Kneser graphs

4.1 Lower bounds

The Cartesian product is the most intriguing in the context of independence number of the products of Kneser graphs. Several natural upper and lower bounds on the independence number of the Cartesian product of arbitrary graphs were presented by Klavžar [11].

The so-called *diagonalization procedure* is used to give lower bounds on the independence number of the Cartesian product of graphs [11]. Note that lower bounds in the maximization parameter such as $\alpha(G)$ often come from a construction of an independent set of appropriate size. It is clear that the product $A_1 \times B_1$ of two maximal independent sets A_1 and B_1, where A_1 is in G and B_1 in H, gives an independent set in the Cartesian product $G \,\Box\, H$. In addition, vertices in $A_1 \times B_1$ are not adjacent to any vertex in $(G - A_1) \times (H - B_1)$, with respect to $G \Box H$, hence one can continue with this procedure in the subgraphs $G - A_1$ and $H - B_1$; we denote these subgraphs by G_1 and H_1, respectively. In the ith step, let A_{i+1} be a maximal independent set in G_i and B_{i+1} be the maximal independent set in H_i. Then $A_{i+1} \times B_{i+1}$ is an independent set in $G \Box H$ (which is also independent with the previously determined independent sets in $G \Box H$), and we let $G_{i+1} = G_i - A_{i+1}$ and $H_{i+1} = H_i - B_{i+1}$. The procedure ends in the mth step, if G_{m+1} or H_{m+1} becomes empty. The resulting independent set D is

$$D = \bigcup_{i=1}^{m} A_i \times B_i \,.$$

The most natural and interesting case of a diagonalization procedure (although not always giving the largest independent set among all diagonalization procedures) is the case when in each step A_{i+1} and B_{i+1} present α-sets of graphs G_i and H_i, respectively. We call this a *greedy diagonalization procedure*. In the case of the Cartesian product of the Kneser graph $K(n,k)$ by itself, there is only one greedy diagonalization procedure (modulo permutations in $[n]$ and the corresponding changes of k-subsets in $K(n,k)$). This procedure gives the following lower bound on the independence number:

$$\alpha\big(K(n,k) \Box K(n,k)\big) \geq \binom{n-1}{k-1}^2 + \binom{n-2}{k-1}^2 + \cdots + \binom{2k-1}{k-1}^2 + \binom{2k-1}{k-1}^2 \qquad (1)$$

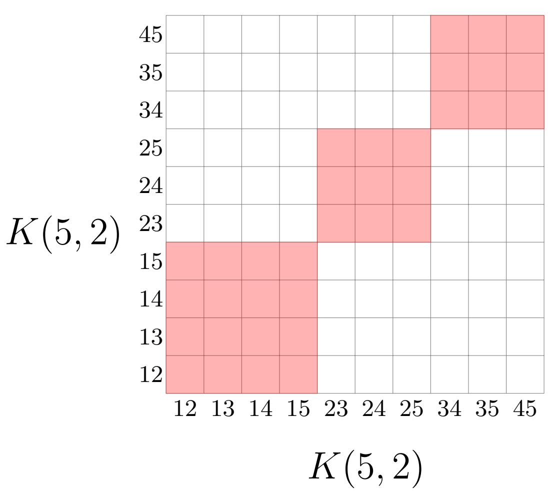

Figure 1: The independent set obtained by the greedy diagonalization procedure in $K(5,2) \Box K(5,2)$ represented by red squares. Sets $\{i,j\} \in V(K(5,2))$ are written in the short way as ij.

Indeed, noting that a maximum independent set I of $K(n,k)$ (for $n \geq 2k$) is obtained by taking one of the elements as the center (so $I = I(x)$, where $x \in [n]$), the resulting graph

G_1, defined as $K(n,k)-I$, is isomorphic to $K(n-1,k)$. By repeating the diagonalization procedure, we eventually arrive at the subgraph of $K(n,k)\Box K(n,k)$, which is isomorphic to $K(2k-1,k)\Box K(2k-1,k)$. Since the Kneser graph $K(2k-1,k)$ is isomorphic to the graph on $\binom{2k-1}{k}$ isolated vertices, we infer that $K(2k-1,k)\Box K(2k-1,k)$ is isomorphic to the graph on $\binom{2k-1}{k-1}^2$ isolated vertices, which explains the last item in the right-hand sum of (1).

In Fig. 1 one can see the construction of the (greedy) diagonalization procedure of the Cartesian product of the Petersen graph $K(5,2)$ by itself. As checked by computer the resulting independent set of size 34 is optimal, attaining $\alpha(K(5,2)\Box K(5,2))$.

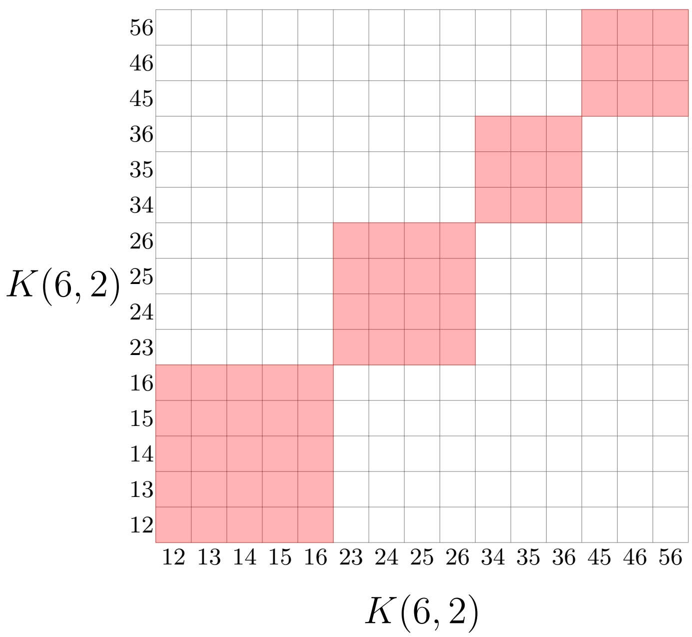

Figure 2: The independent set obtained by the greedy diagonalization procedure in $K(6,2)\Box K(6,2)$ represented by red squares.

The greedy diagonalization procedure yielding the independent set of $K(6,2)\Box K(6,2)$ of size 59 is presented in Fig. 2. The actual independence number of $K(6,2)\Box K(6,2)$, as checked by computer, is equal to 60. The optimal construction is not obtained by a diagonalization procedure, let alone by the greedy diagonalization procedure, see Fig. 3. The colors in the figure represent the sets, obtained as follows. By the notation regarding sets with center, the set in blue/dark color in Fig. 3 is $I(1_2)\times I(2_3)$, the set in green/dotted color is $I(2_3)\times I(1_2)$, and the set in yellow/light color is $I(3_1)\times I(3_1)$. The additional three vertices in red/lined are $(12,12)$, $(13,23)$ and $(23,13)$, and there is the red/lined 3×3 independent set, which is the same as in the diagonalization procedure.

The idea can be generalized to arbitrary $K(n,2)$ for $n\geq 6$. Note that the greedy diagonalization procedure gives the bound:

$$\alpha\big(K(n,2)\Box K(n,2)\big)\geq (n-1)^2+(n-2)^2+\cdots+3^2+3^2. \tag{2}$$

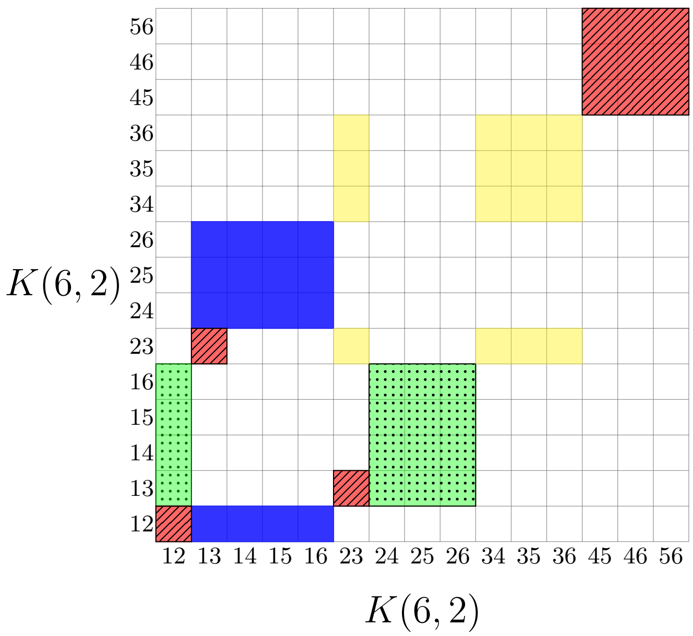

Figure 3: The independent set obtained by an alternative diagonalization procedure in $K(6,2)\Box K(6,2)$ represented by squares in colors/shapes.

In our case, we use the alternative construction in the subgraph of $K(n,2)$ induced by $I(1)\cup I(2)\cup I(3)$, by taking the independent set

$$(I(1_2)\times I(2_3))\cup(I(2_3)\times I(1_2))\cup(I(3_1)\times I(3_1))\cup\{(12,12),(13,23),(23,13)\},$$

which has $3(n-2)^2+3$ vertices. Note that, when $n\geq 6$, the term $3(n-2)^2+3$ is greater by exactly 1 than the sum of the first three terms given by the greedy diagonalization procedure, i.e. $(n-1)^2+(n-2)^2+(n-3)^2$. As the first three terms of the diagonalization procedure correspond to the subgraph of $K(n,2)$ that is induced by $I(1)\cup I(2)\cup I(3)$, we get that in both constructions the problem, after dealing with the subgraph of $K(n,2)\Box K(n,2)$ induced by $(I(1)\cup I(2)\cup I(3))\times(I(1)\cup I(2)\cup I(3))$, reduces to the independence number of $K(n-3,2)\Box K(n-3,2)$ (as soon as $n\geq 6$). By using a simple induction with respect to three different bases, $\alpha(K(3,2)^{2\Box})=9, \alpha(K(4,2)^{2\Box})=18$, and $\alpha(K(5,2)^{2\Box})=34$, we obtain the following result.

Proposition 3. *Let $n\geq 6$, and $n\equiv t\,(mod\;3)$. Then*

$$\alpha(K(n,2)\Box K(n,2))\geq 3(n-2)^2+3(n-5)^2+\cdots+3(t+4)^2+(n-t-3)+A_t,$$

where

$$A_t=\begin{cases}9 & ;t=0\\ 18 & ;t=1\\ 34 & ;t=2.\end{cases}$$

The bound is bigger by $(n-t-1)/3$ than the greedy diagonalization bound in (2).

The bound of Proposition 3 can be transformed from the sum to a function dependent on n as follows.

Corollary 4. *If $n \geq 3$, then*

$$\alpha(K(n,2)\Box K(n,2)) \geq \begin{cases} \frac{2n^3-3n^2+3n+18}{6} & ; \quad n \equiv 0\,(mod\;3) \\ \frac{2n^3-3n^2+3n+16}{6} & ; \quad n \equiv 1\,(mod\;3) \\ \frac{2n^3-3n^2+3n+14}{6} & ; \quad n \equiv 2\,(mod\;3). \end{cases} \tag{3}$$

A natural question is, if one can improve the bound of the greedy diagonalization procedure to find a larger independent set in $K(n,k)\Box K(n,k)$ also for $k > 2$.[1] In case $k = 2$, the question is if one can improve the lower bound from Corollary 4. We only know that the answer is negative in the case of $K(6,2)\Box K(6,2)$, for which we know that the independence number attains the bound from Proposition 3.

4.2 Upper bounds for $\alpha(K(n,2)^{\Box 2})$

Here we consider upper bounds on $\alpha(K(n,k)\Box K(n,k))$ with an emphasis on the case $k = 2$. One of the most general bounds was presented by Klavžar [11]; it generalizes several previously known bounds, including the classical one by Vizing [23]. In the bound we assume that a certain clique vertex cover of one factor, say H, is given. Notably, this is a partition of $V(H)$ into subsets A_i, for $i \in \{1,\ldots,m\}$, such that each A_i induces a clique, and let $\ell_i = |A_i|$. Then

$$\alpha(G\Box H) \leq \sum_{i=1}^{m} \alpha_{\ell_i}(G), \tag{4}$$

where $\alpha_{\ell_i}(G)$ denotes the ℓ_i-independence number of a graph G. Indeed, it is easy to see that for each A_i there are at most $\alpha_{\ell_i}(G)$ vertices that form an independent set in the subgraph of $G\Box H$ induced by $V(G) \times A_i$.

As a consequence of Baranyai's theorem [2], the vertex set of the Kneser graph $K(n,k)$ can be partitioned into $\theta_{n,k} = \left\lceil \binom{n}{k}/\lfloor n/k \rfloor \right\rceil$ cliques of size $\omega_{n,k} = \lfloor n/k \rfloor$. Combining this with the bound in (4) we get the following upper bound on the independence number of

[1] We considered a possible generalization of the construction of Proposition 3 to $K(n,k)$ for $k > 2$, by using the following natural idea. As in the case $k = 2$, we start with the independent set

$$(I(1_2) \times I(2_3)) \cup (I(2_3) \times I(1_2)) \cup (I(3_1) \times I(3_1)),$$

which has $3\binom{n-2}{k-1}^2$ vertices. Now, to this set we can add three additional independent sets $I(12_3) \times I(12_3)$, $I(13_2) \times I(23_1)$ and $I(23_1) \times I(13_2)$. Note that $(I(1_2) \times I(2_3)) \cup (I(2_3) \times I(1_2)) \cup (I(3_1) \times I(3_1)) \cup (I(12_3) \times I(12_3)) \cup (I(13_2) \times I(23_1)) \cup (I(23_1) \times I(13_2))$ is an independent set in $K(n,k)\Box K(n,k)$, which lies in the subgraph induced by $(I(1) \cup I(2) \cup I(3)) \times (I(1) \cup I(2) \cup I(3))$. Its size is $3\binom{n-2}{k-1}^2 + 3\binom{n-3}{k-2}^2$. Let us compare this with the part of the independent set in the greedy diagonalization procedure, which lies in the subgraph induced by $(I(1) \cup I(2) \cup I(3)) \times (I(1) \cup I(2) \cup I(3))$. We recall by (1) that its size is $\binom{n-1}{k-1}^2 + \binom{n-2}{k-1}^2 + \binom{n-3}{k-1}^2$. While we could not find an appropriate calculation to compare these values for all k and n, we have checked by computer that the greedy diagonalization procedure gives a better bound for all $k \in \{3,\ldots,100\}$ and $n \in \{2k,\ldots,1000\}$.

the Cartesian product of $K(n,k)$ by itself:

$$\alpha(K(n,k)\Box K(n,k)) \leq \left\lceil \frac{\binom{n}{k}}{\lfloor \frac{n}{k} \rfloor} \right\rceil \alpha_{\omega_{n,k}}(K(n,k)). \tag{5}$$

Clearly, the above upper bound raises the problem of determining the $\omega_{n,k}$-independence number of the Kneser graph. More generally, it is of independent interest to determine $\alpha_\ell(K(n,k))$ for an arbitrary ℓ. Note that the case $\ell = 1$ is resolved by the Erdős-Ko-Rado theorem, because this is just the independence number of the Kneser graph. At the other extreme, if $\ell \geq \chi(K(n,k))$, then $\alpha_\ell(K(n,k)) = |V(K(n,k))| = \binom{n}{k}$. By the well-known fact, given by Lovász's proof of Kneser's conjecture [12] (see also Matoušek [14]), this happens exactly when $\ell \geq n - 2k + 2$.

As the first step we determine the ℓ-independence number in the case when $k = 2$, i.e. for $K(n,2)$, by using both theorems, and also the Hilton-Milner result about the largest independent set in Kneser graphs that do not have a center. (It is very likely that the result was known before, but we present the proof for self-containment reasons.)

Proposition 5. *The ℓ-independence number of the Kneser graph $K(n,2)$, $n \geq 5$, is*

$$\alpha_\ell(K(n,2)) = \begin{cases} (n-1) + \cdots + (n-\ell) & ; \quad \ell < n-2 \\ \binom{n}{2} & ; \quad \ell \geq n-2\,. \end{cases} \tag{6}$$

Proof. Consider the Kneser graph $K(n,2)$ for $n \geq 5$. If $\ell = 1$ or $\ell \geq n-2 = \chi(K(n,2))$ the result follows from the mentioned classical theorems. So let $\ell \in \{2,\ldots,n-3\}$. Clearly, the sets $I(1), I(2_1), \ldots, I(\ell_{12\ldots(\ell-1)})$ are independent and pairwise disjoint, which implies that $\alpha_\ell(K(n,2)) \geq (n-1) + \cdots + (n-\ell)$.

For the reversed inequality, let $(A_1,\ldots,A_\ell)$ be a partition of a subset of $V(K(n,2))$ into independent sets. We may assume that $A_1,\ldots,A_t$ are sets having a center, while $A_{t+1},\ldots,A_\ell$ do not have a center, and we allow $t = 0$ and $t = \ell$. For the first part it is clear that

$$|A_1 \cup \cdots \cup A_t| \leq (n-1) + \cdots + (n-t).$$

For the second part, we use Hilton-Milner theorem [9], which states that the maximum size of an independent set in $K(n,k)$ having no center is $1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$; in case $k = 2$ this equals 3. Hence, we infer

$$|A_{t+1} \cup \cdots \cup A_\ell| \leq 3(\ell - t) \leq (n-t-1) + \cdots + (n-\ell),$$

where the second inequality follows from the fact that $n - \ell \geq 3$. The result now follows from both inequalities. $\square$

Now, we can apply the bound for $\alpha_\ell(K(n,2))$ in Proposition 5, when $\ell = \omega_{n,2}$, and insert it in formula (5). We get

$$\alpha(K(n,2)\Box K(n,2)) \leq \left\lceil \frac{\binom{n}{2}}{\lfloor \frac{n}{2} \rfloor} \right\rceil ((n-1) + \cdots + (n-\omega_{n,2}))\,,$$

which can be transformed to the following bounds (respecting the parity of n, and noting that $\omega_{n,2} = \lfloor n/2 \rfloor$):

$$\alpha(K(n,2)\Box K(n,2)) \leq \begin{cases} \frac{n(n-1)(3n-2)}{8} & ; \quad n \text{ even} \\ \frac{n(n-1)(3n-1)}{8} & ; \quad n \text{ odd.} \end{cases} \tag{7}$$

These bounds were proven using a direct, but more technical approach in [3]. Combining both bounds we get the main result of this section.

Theorem 6. *If $n \geq 5$, then*

$$\left.\begin{array}{ll} \frac{2n^3-3n^2+3n+18}{6} & ; \quad n \equiv 0\,(mod\ 3) \\ \frac{2n^3-3n^2+3n+16}{6} & ; \quad n \equiv 1\,(mod\ 3) \\ \frac{2n^3-3n^2+3n+14}{6} & ; \quad n \equiv 2\,(mod\ 3). \end{array}\right\} \leq \alpha(K(n,2)\Box K(n,2)) \leq \begin{cases} \frac{n(n-1)(3n-2)}{8} & ; \quad n \text{ even} \\ \frac{n(n-1)(3n-1)}{8} & ; \quad n \text{ odd.} \end{cases} \tag{8}$$

While the upper bound on $\alpha(K(n,2)\Box K(n,2))$ asymptotically tends to $\frac{3}{8}n^3$, the lower bounds tends to $\frac{n^3}{3}$.

5 The ℓ-independence number of Kneser graphs

As mentioned in the previous section, it would be interesting to determine the ℓ-independence number of Kneser graphs $K(n,k)$ also for $k > 2$ to obtain upper bounds on $\alpha(K(n,k)^{2\Box})$. A natural question is if the formula from Proposition 5 extends to the general case for $k > 2$, i.e., is it true that

$$\alpha_\ell(K(n,k)) = \begin{cases} \binom{n-1}{k-1} + \cdots + \binom{n-\ell}{k-1} & ; \quad \ell < n-2k+2 \\ \binom{n}{k} & ; \quad \ell \geq n-2k+2 \end{cases} \quad ? \tag{9}$$

The question was partially resolved by Erdős, who proved that for sufficiently large n, the equality (9) holds. He also asked if the results holds when $n \geq 2k+t-1$, but this was disproved by Frankl and Füredi [6].

As the full answer to the above general question is still open, we focus on the case $\ell = 2$, and graphs $K(2k+1,k)$ (the so-called *odd graphs*). In Chapter 9 of Godsil and Royle's book [8] one can find the following observation.

Lemma 7. [8, Lemma 9.7.1] *We have $\alpha_2(K(2k+1,k)) \leq \binom{2k}{k} + \binom{2k}{k-2}$.*

Therefore, $\alpha_2(K(5,2)) \leq 7$ (which is the optimal value); $\alpha_2(K(7,3)) \leq 26$ (which is also the optimal value). For $K(9,4)$ this upper bound gives 98. On the other hand, Tardif (personal communication in [8]) has found an induced bipartite subgraph of $K(9,4)$ with 96 vertices. In this section, we address the problem of finding good lower bounds for $\alpha_2(K(2k+1,k))$, improving the bound $\binom{2k}{k-1} + \binom{2k-1}{k-1}$, which is derived by taking the cardinality of $I(1) \cup I(2_1)$.

5.1 Decomposing $K(2k+1,k)$

Let $k \geq 2$. It is well known that the diameter of $K(2k+1,k)$ is equal to k (see [20]), and we consider a partition of $V(K(2k+1,k))$ into $k+1$ spheres around an arbitrary vertex as follows. Let $x_0 = [k]$. For $0 \leq i \leq k$, let X_i be the set of vertices of $K(2k+1,k)$ defined as: $X_0 = \{x_0\}$ and $X_i = \{y : y \in V(K(2k+1,k)), |x_0 \cap y| = i-1\}$. Note that $\bigcup_{i=0}^{k} X_i$ forms a partition of the vertex set of $K(2k+1,k)$.

Lemma 8. *Let G_i be the subgraph of $K(2k+1,k)$ induced by the vertex set X_i, for $0 \leq i \leq k$. Then,*

1. *For $0 \leq i \leq k$, with $i \neq \lfloor \frac{k}{2} \rfloor + 1$, the subgraph G_i induces an independent set.*
2. *Let $G_A = \bigcup_{1 \leq i \leq \lfloor \frac{k}{2} \rfloor} G_i$ and let $G_B = \bigcup_{\lfloor \frac{k}{2} \rfloor + 2 \leq i \leq k} G_i$. Then, the subgraph G_A (resp. G_B) induces an independent set.*
3. *Let $i = \lfloor \frac{k}{2} \rfloor + 1$. The subgraph G_i is isomorphic to $\frac{1}{2}\binom{k}{k/2}$ disjoint copies of the direct product graph $K_2 \times K(k+1, k/2)$ if k is even, and G_i is isomorphic to $\frac{1}{2}\binom{k+1}{\lceil k/2 \rceil}$ disjoint copies of the direct product graph $K(k, \lfloor k/2 \rfloor) \times K_2$ if k is odd.*

Proof.

1. Clearly, the statement holds for $i = 0$. Let a, b be two vertices in G_i and assume that a is adjacent to b, that means, $|a \cap b| = 0$. This implies that $a \cup b$ is formed by $2(i-1)$ elements in $x_0 = \{1, \ldots, k\}$ and $2(k-i+1)$ elements in $z = \{k+1, \ldots, 2k+1\}$. Therefore, $2(i-1) \leq k$ and $2(k-i+1) \leq k+1$. However, if $1 \leq i \leq \lfloor k/2 \rfloor$ then, $2(k-i+1) \geq k+2$, which is a contradiction; and if $\lfloor k/2 \rfloor + 2 \leq i \leq k$ then $2(i-1) \geq k+1$, which is again a contradiction.

2. Assume that there are two adjacent vertices a and b with $|a \cap x_0| = i-1$ and $|b \cap x_0| = j-1$. By the previous case, $i \neq j$ and by construction, we must have that $i+j-2 \leq k$ and $2k - (i+j-2) \leq k+1$. However, if $a, b \in G_A$, then $1 \leq i \neq j \leq \lfloor k/2 \rfloor$ and $2k - (i+j-2) \geq 2k - 2\lfloor k/2 \rfloor + 3 \geq k+3$, which is a contradiction. Similarly, if $a, b \in G_B$, then $\lfloor k/2 \rfloor + 2 \leq i \neq j \leq k$ and $i+j-2 \geq 2\lfloor k/2 \rfloor + 3 \geq k+2$, which is again a contradiction.

3. Let $i = \lfloor \frac{k}{2} \rfloor + 1$. For a vertex u of G_i let $A_u = u \cap x$ and $B_u = u \cap z$. We define the map $\phi : G_i \to K(k, i-1) \times K(k+1, k-i+1)$ by letting $\phi(u) = (A_u, B_u)$. By definition of the Kneser graph and the direct product of graphs, it is trivial to see that ϕ is a graph isomorphism. Now, if k is even, then G_i is isomorphic to the graph $K(k, k/2) \times K(k+1, k/2)$, but $K(k, k/2)$ is isomorphic to a perfect matching of size $\frac{1}{2}\binom{k}{k/2}$. Moreover, if k is odd, then G_i is isomorphic to the graph $K(k, \lfloor k/2 \rfloor) \times K(k+1, \lceil k/2 \rceil)$, but $K(k+1, \lceil k/2 \rceil)$ is isomorphic to a perfect matching of size $\frac{1}{2}\binom{k+1}{\lceil k/2 \rceil}$.

□

5.2 Lower bound for $\alpha_2(K(2k+1,k))$

In this section, we obtain the following lower bound for $\alpha_2(K(2k+1,k))$:

Proposition 9. *For $k \geq 2$,*

$$\alpha_2(K(2k+1,k)) \geq \begin{cases} \binom{2k+1}{k} - \frac{(k+1)}{(k+2)}\binom{k}{k/2}^2 & \text{, if } k \text{ even} \\ \\ \binom{2k+1}{k} - \binom{k}{\lfloor k/2 \rfloor}^2 & \text{, if } k \text{ odd} \end{cases}$$

Proof. Let G_A and G_B the subgraphs of $K(2k+1,k)$ as defined in the previous section. Let W_1 and W_2 be two (initially empty) sets, which will form a bipartition of an induced bipartite subgraph of $K(2k+1,k)$. Let G_C be the subgraph of $K(2k+1,k)$ induced by the vertices in the set $X_{\lfloor k/2 \rfloor+1}$. We consider two cases depending on the parity of k:

(a) k even. Let $Y_1 = \{y : y \in G_C \text{ and } 1 \in y\}$, that is, Y_1 is the subset of all vertices in G_C containing the integer 1, i.e., $Y_1 = G_C \cap I(1)$. Set $W_1 = \{x_0\} \cup \{y : y \in G_B\} \cup Y_1$ and $W_2 = \{y : y \in G_A\}$. Notice that the size of Y_1 is equal to $\binom{k-1}{k/2-1}\binom{k+1}{k/2} = \frac{1}{2}\binom{k+1}{k/2}\binom{k}{k/2}$. Clearly, x_0 is neither adjacent to any vertex in G_B nor in Y_1. It remains to show that for any two vertices $a \in Y_1$ and $b \in G_B$, these vertices are not adjacent. Assume that there exist $a \in Y_1$ and $b \in G_B$ such that a is adjacent to b. By construction, $|a \cap x_0| = k/2$ and $|b \cap x_0| = j$, with $k/2+1 \leq j \leq k-1$, and we must have that $k/2 + j \leq k$, which is impossible. Let $p = |X_{k/2+1}| = \binom{k}{k/2}\binom{k+1}{k/2}$ be the number of vertices in G_C. It is not difficult to prove that $p - |Y_1| = \frac{k+1}{k+2}\binom{k}{k/2}^2$ which ends the proof of this part.

(b) k odd. We consider the following vertex subsets of G_B: let $B_1 = \{y : y \in G_B,\ k+1 \in y, \text{ and } |y \cap x_0| = \lfloor k/2 \rfloor + 1\}$ and $B_2 = \{y : y \in G_B \text{ and } |y \cap x_0| \geq \lfloor k/2 \rfloor + 2\}$. Moreover, we partition the vertex set of G_C as follows: $C_1 = \{y : y \in G_C \text{ and } k+1 \in y\}$ and $C_2 = X_{\lfloor k/2 \rfloor+1} \setminus C_1$. Finally, set $W_1 = \{x_0\} \cup B_1 \cup B_2 \cup C_1$ and set $W_2 = \{y : y \in G_A\} \cup C_2$. Let a be any vertex in C_1 and let b be any vertex in B_2 and suppose that a is adjacent to b in $K(2k+1,k)$. By construction, $|a \cap x_0| = \lfloor k/2 \rfloor$ and $|b \cap x_0| = j$ with $\lfloor k/2 \rfloor + 2 \leq j \leq k-1$. However, by construction, $\lfloor k/2 \rfloor + j \leq k$, which is impossible. It shows that W_1 induces an independent set in $K(2k+1,k)$. Now, let a be any vertex in G_A and let b be any vertex in C_2 and suppose that a is adjacent to b in $K(2k+1,k)$. By construction, $|b \cap x_0| = \lfloor k/2 \rfloor$ and $|a \cap x_0| = j$ with $0 \leq j \leq \lfloor k/2 \rfloor - 1$. Again, by construction, $k - \lfloor k/2 \rfloor + k - j \leq k+1$, which is again impossible. Therefore, W_2 induces an independent set in $K(2k+1,k)$. Let $p = |X_{\lfloor k/2 \rfloor+2}| = \binom{k}{\lfloor k/2 \rfloor+1}\binom{k+1}{k-\lfloor k/2 \rfloor-1}$ and let $q = |B_1| = \binom{k}{\lfloor k/2 \rfloor+1}\binom{k}{k-\lfloor k/2 \rfloor-2}$. In order to end the proof of this part, it suffices to show that $p - q = \binom{k}{\lfloor k/2 \rfloor}^2$. It follows easy from the identities: $\binom{k}{\lfloor k/2 \rfloor+1} = \binom{k}{\lfloor k/2 \rfloor} = \frac{\lceil k/2 \rceil+1}{k+1}\binom{k+1}{\lfloor k/2 \rfloor}$, $\binom{k+1}{k-\lfloor k/2 \rfloor-1} = \binom{k+1}{\lfloor k/2 \rfloor}$, and $\binom{k}{k-\lfloor k/2 \rfloor-2} = \frac{\lfloor k/2 \rfloor}{k+1}\binom{k+1}{\lfloor k/2 \rfloor}$.

□

The bound in Proposition 9 can be obtained by an alternative construction. In fact, let G_A and G_B the subgraphs of $K(2k+1,k)$ defined in Lemma 8, and let G_C the subgraph of $K(2k+1,k)$ induced by the vertices in the set $X_{\lfloor \frac{k}{2} \rfloor+1}$ (see Section 5.1). It is not difficult to verify that if k is even (resp. odd) there are edges between some vertices in G_C and some vertices in the set $X_{k/2}$ (resp. $X_{\lfloor \frac{k}{2} \rfloor+1}$) and there are no edges between vertices of G_C and vertices in $X_{\frac{k}{2}+2}$ (resp. $X_{\lfloor \frac{k}{2} \rfloor}$). Therefore, let I be a maximum independent set in G_C. An alternative bipartition yielding the bound in Proposition 9 will be $(G_A, G_B \cup I)$ if k is even, and $(G_A \cup I, G_B)$ if k is odd. Let Z be a minimum vertex cover in G_C. It is clear that $|I| = |G_C| - |Z|$. Therefore, the size of the bipartite subgraph obtained by this construction will be $\binom{2k+1}{k} - |Z|$. Let k be even. By Lemma 8, G_C is is isomorphic to $\frac{1}{2}\binom{k}{k/2}$ disjoint copies of $K_2 \times K(k+1, k/2)$. Moreover, by [21], $\alpha(K_2 \times K(k+1, k/2)) = \max\{2\alpha(K(k+1, k/2)), \binom{k+1}{k/2}\} = \max\{2\binom{k-1}{k/2-1}, \binom{k+1}{k/2}\} = \binom{k+1}{k/2}$. So, a minimum vertex cover in $K_2 \times K(k+1, k/2)$ has size $2\binom{k+1}{k/2} - \binom{k+1}{k/2} = \binom{k+1}{k/2}$. Therefore, a minimum vertex cover Z in G_C has size $\frac{1}{2}\binom{k}{k/2}\binom{k+1}{k/2} = \frac{(k+1)}{(k+2)}\binom{k}{k/2}^2$. In the case when k is odd, we know by Lemma 8, that G_C is is isomorphic to $\frac{1}{2}\binom{k+1}{\lceil k/2 \rceil}$ disjoint copies of $K(k, \lfloor k/2 \rfloor) \times K_2$. So, $\alpha(K(k, \lfloor k/2 \rfloor) \times K_2) = \max\{2\binom{k-1}{\lfloor k/2 \rfloor -1}, \binom{k}{\lfloor k/2 \rfloor}\} = \binom{k}{\lfloor k/2 \rfloor}$. Thus, a minimum vertex cover in $K(k, \lfloor k/2 \rfloor) \times K_2$ has size $2\binom{k}{\lfloor k/2 \rfloor} - \binom{k}{\lfloor k/2 \rfloor} = \binom{k}{\lfloor k/2 \rfloor}$ and therefore, the size of a minimum vertex cover Z in G_C has size $\frac{1}{2}\binom{k+1}{\lceil k/2 \rceil}\binom{k}{\lfloor k/2 \rfloor} = \binom{k}{\lfloor k/2 \rfloor}^2$. This proves that the constructions give the same lower bounds.

In addition, we compared the bound in Proposition 9 with the bound stated in the recent book of Godsil and Meagher [7]: for $k \geq 2$,

$$\alpha_2(K(2k+1,k)) \geq \begin{cases} \sum_{i=k/2}^{k-1} \binom{k-1}{i}\binom{k+2}{k-i} + \sum_{i=k/2+1}^{k} \binom{k+1}{i}\binom{k}{k-i} & \text{, if } k \text{ even} \\ 2\sum_{i=(k+1)/2}^{k} \binom{k}{i}\binom{k+1}{k-i} & \text{, if } k \text{ odd} \end{cases}$$

The following lemma shows that the bound in Proposition 9 and the Godsil and Meagher bound are in fact the same.

Lemma 10. *Let $k \geq 2$ be an integer. Then the following equalities hold:*

(i) $\sum_{i=k/2}^{k-1} \binom{k-1}{i}\binom{k+2}{k-i} + \sum_{i=k/2+1}^{k} \binom{k+1}{i}\binom{k}{k-i} = \binom{2k+1}{k} - \frac{(k+1)}{(k+2)}\binom{k}{k/2}^2$, *if k is even;*

(ii) $2\sum_{i=(k+1)/2}^{k} \binom{k}{i}\binom{k+1}{k-i} = \binom{2k+1}{k} - \binom{k}{\lfloor k/2 \rfloor}^2$, *if k is odd.*

Proof. In order to prove this result, we need the well-known Vandermonde's identity:

$$\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}, \text{ for any non negative integers } n, m, r, \tag{10}$$

and some identities related to the sum of squares of binomial coefficients that have been obtained by Slavík in [18]. For any integer $n \geq 1$ and for $t \geq 0$, let $S_t(n) = \sum_{i=0}^{n-1} i^t \binom{2n}{i}^2$.

Then,

$$S_0(n) = \frac{1}{2}\left[\binom{4n}{2n} - \binom{2n}{n}^2\right]; \tag{11}$$

$$S_1(n) = n\left[\binom{4n-1}{2n-1} - 3\binom{2n-1}{n-1}^2\right]; \tag{12}$$

$$S_2(n) = 4n^2\left[\frac{1}{2}\binom{4n-2}{2n-1} - \binom{2n-1}{n-1}^2\right]; \tag{13}$$

$$S_3(n) = 2n^2\left[\binom{4n-2}{2n-1} - 2\binom{2n-1}{n-1}^2 + (2n-1)\left[\binom{4n-3}{2n-2} - 2\binom{2n-1}{n-1}\binom{2n-2}{n-2} - \binom{2n-2}{n-1}^2\right]\right]. \tag{14}$$

We distinguish two cases.

Case k *even*. Let $A = \sum_{i=k/2}^{k-1}\binom{k-1}{i}\binom{k+2}{k-i}$ and let $B = \sum_{i=k/2+1}^{k}\binom{k+1}{i}\binom{k}{k-1}$.

Claim 11. $A = \frac{1}{2}\binom{2k+1}{k} - \frac{k+1}{k+2}\binom{k}{k/2}^2 + \frac{1}{2}\binom{k}{k/2}^2$.

Proof of Claim 11. By considering $\binom{m}{n} = 0$ for $m < n$, we have that $A = \sum_{i=0}^{k}\binom{k-1}{i}\binom{k+2}{k-i} - A_1$, where $A_1 = \sum_{i=0}^{k/2-1}\binom{k-1}{i}\binom{k+2}{k-i}$. By using Eq. (10), we have that $A = \binom{2k+1}{k} - A_1$.
Expressing $\binom{k-1}{i}$ and $\binom{k+2}{k-i}$ in terms of $\binom{k+2}{i+2}$, we get $A_1 = \sum_{i=0}^{k/2-1}\binom{k+2}{i+2}^2\left(\frac{(k-i)(i+1)(i+2)}{k(k+1)(k+2)}\right) = \sum_{i=0}^{k/2+1}\binom{k+2}{i}^2\left(\frac{(k-i+2)(i-1)i}{k(k+1)(k+2)}\right)$. By using the fact that $\binom{k+2}{k/2+1} = \frac{4(k+1)}{(k+2}\binom{k}{k/2}$, we can rewrite A_1 as $A_1 = A_2 + \frac{2(k+1)}{(k+2)}\binom{k}{k/2}^2$, where $A_2 = \sum_{i=0}^{k/2}\binom{k+2}{i}^2\left(\frac{(k-i+2)(i-1)i}{k(k+1)(k+2)}\right)$.
Finally, A_2 can be decomposed as follows: $A_2 = \frac{1}{k(k+1)(k+2)}\left[(k+3)A_3 - (k+2)A_4 - A_5\right]$, where $A_3 = \sum_{i=0}^{k/2} i^2\binom{k+2}{i}^2$, $A_4 = \sum_{i=0}^{k/2} i\binom{k+2}{i}^2$, and $A_5 = \sum_{i=0}^{k/2} i^3\binom{k+2}{i}^2$. Let $n = k/2+1$. Notice that A_3 corresponds to $S_2(k/2+1)$ (Eq. (13)), A_4 corresponds to $S_1(k/2+1)$ (Eq. (12)) and A_5 corresponds to $S_3(k/2+1)$ (Eq. (14)). Therefore, by expressing $\binom{2k+2}{k+1}$ and $\binom{2k+3}{k+1}$ in terms of $\binom{2k+1}{k}$, and by expressing $\binom{k+1}{k/2}$ and $\binom{k}{k/2-1}$ in terms of $\binom{k}{k/2}$, we have:
$A_3 = (k+2)^2\binom{2k+1}{k} - 4(k+1)^2\binom{k}{k/2}^2$; $A_4 = (2k+3)\binom{2k+1}{k} - 6\frac{(k+1)^2}{(k+2)}\binom{k}{k/2}^2$; and
$A_5 = \left[(k+2)^2 + \frac{(k+2)^2(k+1)}{2}\right]\binom{2k+1}{k} - \left[4(k+1)^2 + 2k(k+1)^2 + \frac{(k+2)^2(k+1)}{2}\right]\binom{k}{k/2}^2$.
Now, as A_2 is expressed in terms of A_3, A_4 and A_5, we can rewrite this as
$A_2 = \frac{1}{2}\binom{2k+1}{k} - \frac{(k+1)}{(k+2)}\binom{k}{k/2}^2 - \frac{1}{2}\binom{k}{k/2}^2$. So, as A_1 is expressed in terms of A_2, it can be rewriten as $A_1 = \frac{1}{2}\binom{2k+1}{k} + \frac{(k+1)}{(k+2)}\binom{k}{k/2}^2 - \frac{1}{2}\binom{k}{k/2}^2$. Finally, as $A = \binom{2k+1}{k} - A_1$, the claim holds. ($\square$)

Claim 12. $B = \frac{1}{2}\binom{2k+1}{k} - \frac{1}{2}\binom{k}{k/2}^2$.

Proof of Claim 12. Expression B can be expressed as $\sum_{i=0}^{k}\binom{k+1}{i}\binom{k}{k-i} - \sum_{i=0}^{k/2}\binom{k+1}{i}\binom{k}{k-i}$ and by Eq. (10), $B = \binom{2k+1}{k} - B_1$, where $B_1 = \sum_{i=0}^{k/2}\binom{k+1}{i}\binom{k}{k-i} = \sum_{i=0}^{k/2}\binom{k+1}{i}\binom{k}{i}$. In order to compute B_1, we express both $\binom{k+1}{i}$ and $\binom{k}{i}$ in terms of $\binom{k+2}{i}$ obtaining in this way that $B_1 = \frac{1}{(k+2)^2(k+1)}\left[((k+1)+2(k+2))B_4 + (k+2)^2(k+1)B_2 - ((k+2)^2 + 2(k+2)(k+1))B_3 - B_5\right]$, where $B_4 = \sum_{i=0}^{k/2} i^2\binom{k+2}{i}^2$; $B_2 = \sum_{i=0}^{k/2}\binom{k+2}{i}^2$; $B_3 = \sum_{i=0}^{k/2} i\binom{k+2}{i}^2$; and $B_5 = \sum_{i=0}^{k/2} i^3\binom{k+2}{i}^2$.

Notice that $B_3 = A_4 = S_1(k/2+1)$; $B_4 = A_3 = S_2(k/2+1)$; $B_5 = A_5 = S_3(k/2+1)$; and $B_2 = S_0(k/2+1)$. In order to compute B_2, we express $\binom{2k+4}{k+2}$ and $\binom{k+2}{k/2+1}$ in terms of $\binom{2k+1}{k}$ and $\binom{k}{k/2}$, respectively. Thus, $B_2 = \frac{2(2k+3)}{(k+2)}\binom{2k+1}{k} - \frac{8(k+1)^2}{(k+2)^2}\binom{k}{k/2}^2$. As B_1 is expressed in terms of B_2, B_3, B_4 and B_5, after regrouping and simplifying, we obtain that it can be rewritten as $B_1 = \frac{1}{2}\binom{2k+1}{k} + \frac{1}{2}\binom{k}{k/2}^2$. Finally, as $B = \binom{2k+1}{k} - B_1$, the claim follows. ($\square$)

Therefore, by Claims 11 and 12, the case (i) of the lemma holds.

Case k *odd*. Let $C = 2\sum_{i=\frac{k+1}{2}}^{k}\binom{k}{i}\binom{k+1}{k-i}$. Clearly, $C = 2\left[\sum_{i=0}^{k}\binom{k}{i}\binom{k+1}{k-i} - \sum_{i=0}^{\lfloor k \rfloor}\binom{k}{i}\binom{k+1}{k-i}\right]$. By Eq. (10), $C = 2\left[\binom{2k+1}{k} - C_1\right]$, where $C_1 = \sum_{i=0}^{\lfloor k \rfloor}\binom{k}{i}\binom{k+1}{k-i} = \sum_{i=0}^{\lfloor k \rfloor}\binom{k}{i}\binom{k+1}{i+1}$. By expressing $\binom{k}{i}$ in terms of $\binom{k+1}{i+1}$, we have that $C_1 = \frac{1}{k+1}\sum_{i=0}^{\lfloor k/2 \rfloor}(i+1)\binom{k+1}{i+1}^2 = \frac{1}{k+1}C_2 + \frac{1}{2}\binom{k+1}{\lceil k/2 \rceil}^2$, where $C_2 = \sum_{i=0}^{\lfloor k/2 \rfloor} i\binom{k+1}{i}^2$. Clearly, $C_2 = S_1(\lceil k/2 \rceil) = \frac{k+1}{2}\left[\binom{2k+1}{k} - 3\binom{k}{\lfloor k/2 \rfloor}^2\right]$. Now, by expressing $\binom{k+1}{\lceil k/2 \rceil}$ in terms of $\binom{k}{\lfloor k/2 \rfloor}$, we obtain that $C_1 = \frac{1}{2}\binom{2k+1}{k} + \frac{1}{2}\binom{k}{\lfloor k/2 \rfloor}^2$. Finally, $C = 2\left[\binom{2k+1}{k} - C_1\right] = \binom{2k+1}{k} - \binom{k}{\lfloor k/2 \rfloor}^2$ which proves the case (ii) of the lemma. $\square$

Observation.[2] Actually, the sums in Lemma 10 can be evaluated algorithmically by using the Zeilberger's algorithm [16] to produce recurrences which they satisfy, then the algorithm Hyper of Petkovšek [15] to solve these recurrences in closed form. For instance, in order to prove Claim 11 where k is even, we can write $k = 2n$ and using the Zeilberger's algorithm on the sum $S_1(n) = \sum_{i=n}^{2n-1}\binom{2n-1}{i}\binom{2n+2}{2n-i}$ yields the inhomogeneous first-order recurrence $(1+n)(3+2n)S_1(n+1) - 2(3+4n)(5+4n)S_1(n) = \frac{(1+n)(-6-4n+11n^2+8n^3)\binom{1+2n}{n}\binom{4+2n}{2+n}}{4(1+2n)^2(3+2n)}$ whose second-order homogeneous version is $0 = (1+n)(2+n)(3+n)(5+2n)(-6-4n+11n^2+8n^3)S_1(n+2) - 2(1+n)(2+n)(-540-621n+1377n^2+2624n^3+1440n^4+256n^5)S_1(n+1) + 8(1+2n)^2(3+4n)(5+4n)(9+42n+35n^2+8n^3)S_1(n)$. According to algorithm Hyper, this recurrence has two linearly independent hypergeometric solutions, $h_1(n) = \frac{n}{n+1}\binom{2n}{n}^2$ and $h_2(n) = \binom{4n+1}{2n}$, and so $S_1(n) = Ah_1(n) + Bh_2(n)$, where the values of A and B are determined from the initial values $S_1(1) = 4$, $S_1(2) = 51$ as $A = -\frac{1}{2}$, $B = \frac{1}{2}$. Hence $S_1(n) = \frac{1}{2}\left[\binom{4n+1}{2n} - \frac{n}{n+1}\binom{2n}{n}^2\right] = \frac{1}{2}\left[\binom{2k+1}{k} - \frac{k}{k+2}\binom{k}{k/2}^2\right]$. In the same way, by set $k = 2n$ and by applying Zeilberger's algorithm to the sum $S_2(n) = \sum_{i=n+1}^{2n}\binom{2n+1}{i}\binom{2n}{2n-i}$, we can obtain a second-order homogeneous recurrence which can be solved by using Hyper algorithm in order to obtain Claim 12. Finally, by setting $k = 2n+1$ and by applying the same process as before to the sum $S_3 = \sum_{i=n}^{2n-1}\binom{2n-1}{i}\binom{2n}{2n-i-1}$ we obtain the result in Lemma 10(ii).

By Lemma 10 and preceding discussion, we note that three different constructions of a (large) induced bipartite subgraph of $K(2k+1,k)$ give the same value. We wonder whether this is in fact an exact value of $\alpha_2(K(2k+1,k))$, and pose it as a problem.

[2]We are in debt with one of the anonymous referees for pointing out this important fact.

Problem 1. *Is it true that for any $k \geq 2$,*

$$\alpha_2(K(2k+1,k)) = \begin{cases} \binom{2k+1}{k} - \frac{(k+1)}{(k+2)}\binom{k}{k/2}^2 & \text{, if } k \text{ even} \\ \\ \binom{2k+1}{k} - \binom{k}{\lfloor k/2 \rfloor}^2 & \text{, if } k \text{ odd} \end{cases}$$

6 Stable Kneser graphs

The 2-*stable Kneser graph* (also known as the Schrijver graphs) [17] is the induced subgraph of the Kneser graph $K(n,k)$, obtained by restricting the vertex set to the k-subsets that are 2-*stable*, that is, that do not contain two consecutive elements of $[n]$ (where 1 and n are also considered to be consecutive).

In general, a subset $S \subseteq [n]$ is s-*stable* if any two of its elements are at least at distance s apart on the n-cycle, that is, if $s \leq |i-j| \leq n-s$ for distinct $i,j \in S$. For $s,k \geq 2$ and $n \geq ks$, the s-stable Kneser graph $K(n,k)_{s-\mathrm{stab}}$ is the subgraph of $K(n,k)$ obtained by restricting the vertex set of $K(n,k)$ to the s-stable k-subsets of $[n]$.

Let $n \geq 2k$ be positive integers. The Cayley graphs $\mathrm{Cay}(\mathbb{Z}_n, \{k, k+1, \ldots, n-k\})$, that we denote by $G(n,k)$, are known as *circular graphs*, where $\mathbb{Z}_n$ is the cyclic group of order n. Note that the graph $G(n,k)$ is isomorphic to the graph $\overline{C_n^{(k-1)}}$, i.e., the complement graph of the $(k-1)$st power of cycle C_n. Vince [22] has shown that $\chi(G(n,k)) = \lceil \frac{n}{k} \rceil$ and $\alpha(G(n,k)) = k$. In [19] the following proposition was proven.

Proposition 13. [19] *If $s,k \geq 2$, then $G(ks+1,k) \simeq K(ks+1,k)_{s-\mathrm{stab}}$.*

By using Proposition 13, the following result can be proved.

Lemma 14. [19] *If $k \geq s \geq 2$ and $G = K(ks+1,k)_{s-\mathrm{stab}}$, then G is a hom-idempotent graph, that is, there exists a homomorphism from $G \Box G$ to G.*

From Proposition 13 it follows that the graph $K(ks+1,k)_{s-\mathrm{stab}}$ is a Cayley graph and so a vertex transitive graph. Moreover, it is well known that the Cartesian product graph of two vertex transitive graphs is also vertex transitive. The following "No-Homomorphism Lemma" of Albertson and Collins [1] is very useful when there exists a graph homomorphism to a vertex transitive graph.

Lemma 15 (No-Homomorphism Lemma)**.** *Let X be a graph and let Y be a vertex transitive graph. If there is a graph homomorphism from X to Y then, $\alpha(Y)/|Y| \leq \alpha(X)/|X|$.*

Using the above results we can easily derive the independence number of the Cartesian product of $K(ks+1,k)_{s-\mathrm{stab}}$ with itself.

Proposition 16. *If $k \geq s \geq 2$ and $G = K(ks+1,k)_{s-\mathrm{stab}}$, then $\alpha(G \Box G) = k(ks+1)$.*

Proof. There is a natural homomorphism from G to $G \Box G$. Moreover, by Lemma 14, there is a homomorphism from $G \Box G$ to G. Therefore, by applying twice the No-Homomorphism Lemma (Lemma 15) we have $\frac{\alpha(G)|G \Box G|}{|G|} \leq \alpha(G \Box G) \leq \frac{\alpha(G)|G \Box G|}{|G|}$. Finally, by Proposition 13, we have that $|G| = ks+1$ and $\alpha(G) = k$, which ends the proof. $\Box$

In the rest of this section we focus on the ℓ-independence number of $K(ks+1,k)_{s-\mathrm{stab}}$.

Given a graph H, a trivial upper bound for the ℓ-independence number of H, denoted by $\alpha_\ell(H)$, is $\ell\alpha(H)$. The following result shows that the ℓ-independence number of the graph $K(ks+1,k)_{s-\mathrm{stab}}$ is equal to the trivial upper bound.

Proposition 17. *If $k \geq s \geq 2$ and $G = K(ks+1,k)_{s-\mathrm{stab}}$, then, for $1 \leq \ell \leq s$, $\alpha_\ell(G) = \ell k$.*

Proof. By Proposition 13, G is isomorphic to the circular graph $G(ks+1,k)$ which is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_{ks+1}, \{k, k+1, \ldots, n-k\})$. It is well known [22] that $\alpha(G(ks+1,k)) = k$ and $\chi(G(ks+1,k)) = \lceil \frac{ks+1}{k} \rceil = s+1$. Moreover, a simple observation shows that $G(ks+1,k)$ can be decomposed into s independent sets of size k plus one independent set of size one: $\{0,1,\ldots,k-1\}, \{k,k+1,\ldots,2k-1\}, \ldots, \{k(s-1),\ldots,ks-1\}$ and the singleton $\{ks\}$. Therefore, the result holds. $\square$

Acknowledgements

The authors are grateful to two anonymous reviewers for careful reading of the manuscript and giving a number of useful suggestions. They also thank Tim Kos for performing some computer experiments. B.B. acknowledges the financial support from the Slovenian Research Agency (project grant No. J1-9109 and research core funding No. P1-0297). He also thanks the Paris-Nord University for the financial support and hospitality. M. V-P. acknowledges the financial support from the Paris-Nord University. He also thanks University of Maribor for the financial support and hospitality.

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