The packing chromatic number of hypercubes*

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k needed to proper color the vertices of G in such a way that the distance in G between any two vertices having color i be at least $i+1$. Goddard et al. [9] found an upper bound for the packing chromatic number of hypercubes Q_n. Moreover, they compute $\chi_\rho(Q_n)$ for $n \leq 5$ leaving as an open problem the remaining cases. In this paper, we obtain a better upper bound for $\chi_\rho(Q_n)$ and we improve the lower bounds for $\chi_\rho(Q_n)$ for $6 \leq n \leq 11$. In particular we compute the exact value of $\chi_\rho(Q_n)$ for $6 \leq n \leq 8$.

Keywords: Packing chromatic number, upper bound, hypercube graphs.

1 Introduction

We consider finite undirected graphs without loops or multiple edges. The concept of packing coloring comes from the area of frequency planning in wireless networks. This model emphasizes the fact that some frequencies might be used more sparely than the others. In graph terms, we ask for a partitioning of the vertex set of a graph G into disjoint classes $V_1, \ldots, V_k$ (representing frequency usage) according to the following constraints. Each color class V_i should be an *i-packing*, i.e. a set of vertices with the property that any distinct pair $u, v \in V_i$ satisfies $\mathrm{dist}(u, v) \geq i+1$. Here, $\mathrm{dist}(u, v)$ is the *distance* between u and v, i.e. the length of the shortest path in G from u to v. The greatest size of an i-packing of G is called the *i-packing number* of G and denoted by $\alpha_i(G)$.

Such partitioning into k classes is called a *packing k-coloring*, even though it is allowed that some sets V_i can be empty. The smallest integer k for which exists a packing k-coloring of G is called the *packing chromatic number* of G, and it is denoted by $\chi_\rho(G)$. The notion of the packing chromatic number was established by Goddard et al. [9] under the name *broadcast chromatic number*. The term packing chromatic number was introduced by Brešar et al. [4].

Much work has been devoted to the packing chromatic number of graphs [2–13, 15]. Fiala and Golovach [6] showed that determining the packing chromatic number is an NP-hard problem for trees. Goddard et al. [9] provided polynomial time algorithms for cographs and split graphs. Recently, Argiroffo et al. [2, 3] gave polynomial time algorithms for special subfamilies of trees, for partner limited graphs and for $(q, q-4)$ graphs. Lower and upper bounds for the packing chromatic

*Partially supported by MathAmSud Project 13MATH-07 (Argentina–Brazil–Chile–France) and by the French CNRS PEPS Project "Combinatoire et invariants des graphes topologiques et de leurs gnralisations" , 2011–2013.

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number have been obtained recently for some families of graphs [5,7,12,15] and some families of lattices [8,13]. Goddard and Xu [10,11] studied a generalized version of packing colorings of graphs.

In this paper, we are interested in bounding and, whenever possible, finding the packing chromatic number of hypercubes. For any $n \in \mathbb{Z}^+$, the *n-dimensional hypercube* (or *n-cube*), denoted Q_n, is the graph in which the vertices are all binary vectors of length n (i.e., the set $\{0,1\}^n$), and two vertices are adjacent if and only if they differ in exactly one position. Based on coding theory, Goddard et al. [9] gave an asymptotic result for the packing chromatic number of hypercubes. They proved that $\chi_\rho(Q_n) \sim (\frac{1}{2} - O(\frac{1}{n}))2^n$. More precisely, Goddard et al. [9] obtained that $\chi_\rho(Q_n) \leq 2 + (\frac{1}{2} - \frac{1}{4n})2^n$. In the same paper, they also computed $\chi_\rho(Q_n)$ for $1 \leq n \leq 5$, leaving as an open problem the remaining cases.

The *diameter*, $\mathrm{diam}(G)$, of a connected graph G is the maximum distance between two vertices of G. The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, vertices (g,h) and (g',h') being adjacent whenever $gg' \in E(G)$ and $h = h'$, or $hh' \in E(H)$ and $g = g'$. Brešar et al. [4] obtained that if G and H are connected graphs on at least two vertices, $\chi_\rho(G \Box H) \geq (\chi_\rho(G)+1)|H| - \mathrm{diam}(G \Box H)(|H|-1) - 1$. Moreover, if $H = K_n$, the complete graph on $n \geq 2$ vertices, then $\chi_\rho(G \Box K_n) \geq n\chi_\rho(G) - (n-1)\mathrm{diam}(G)$. It is well known that the binary hypercube Q_n is isomorphic to the graph $Q_{n-1} \Box K_2$. Then, we obtain directly the following lower bound for the packing chromatic number of Q_n.

Corollary 1.1 ([4]). *Let $n \geq 2$. Then, $\chi_\rho(Q_n) \geq 2\chi_\rho(Q_{n-1}) - (n-1)$.*

In this paper, we improve the upper bound found by Goddard et al. [9]. For this, we use elementary algebraic techniques in order to construct a packing coloring of Q_n, for $n \geq 4$. Furthermore, we obtain the exact values of $\chi_\rho(Q_n)$ for $n = 6, 7$ and 8.

2 Preliminaries

In this section we present definitions, notations and previous results. For each $n \in \mathbb{Z}^+$, let $[n] = \{1, \ldots, n\}$. All the graphs in this paper are finite, simple and connected. Given a graph G, $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively.

Let Γ be a group and C a subset of Γ closed under inverses and identity free. The Cayley graph $\mathrm{Cay}(\Gamma, C)$ is the graph with Γ as its vertex set, two vertices u and v being joining by an edge if and only if $u^{-1}v \in C$. The set C is then called the *connector set* of $\mathrm{Cay}(\Gamma, C)$. It is well known that Q_n is the Cayley graph of the Abelian group $\mathbb{Z}_2^n$ (the elements of $\mathbb{Z}_2^n$ are the binary n-vectors of the set $\{0,1\}^n$ and the group operation is the sum modulo two coordinatewise), where the connector set is the subset of n-vectors having exactly one component equal to 1.

Consider $v \in \mathbb{Z}_2^n$. For $i \in [n]$, we denote by v^i the i^{th} position of v, that is, $v = (v^1, \ldots, v^i, \ldots, v^n)$, where v^i is either 0 or either 1. The *weight* of v, denoted $wt(v)$, is the sum of its components, i.e. $wt(v) = \sum\limits_{i=1}^{n} v^i$ (the number of components in v equal to 1).

Clearly, $\mathrm{dist}(u,v) = wt(u+v)$ for any $u, v \in Q_n$. Moreover, for any $u, v \in \mathbb{Z}_2^n$, we can define $u \cap v$ as the n-vector $(u_1.v_1, u_2.v_2, \ldots, u_n.v_n)$, where $u_i.v_i = 1 \Leftrightarrow u_i = v_i = 1$. The following well known results are very easy to deduce (see [16] for example). Let $u, v \in Q_n$ then,

Observation 2.1. *$dist(u,v) = wt(u) + wt(v) - 2wt(u \cap v)$.*

Observation 2.2. *$wt(u)$ and $wt(v)$ are of the same parity if and only if $dist(u,v)$ is even.*

Let G be a graph and $|V(G)| = n$. For any $v \in V(G)$, $N(v)$ denotes the *open neighborhood of* v, i.e. $N(v) = \{u : uv \in E(G)\}$, and if $U \subset V(G)$ then, $N(U)$ is the set $\cup_{v \in U} N(v) \setminus U$.

Regarding i-packings, we define a more general concept. For $F \subset [n]$, we say that a subset of vertices X is an *F-packing of G* if $X = \cup_{i \in F} X_i$ where each X_i is an i-packing of G. The *F-packing number of G* is the maximum size of an F-packing of G and it is denoted by $\alpha_F(G)$, i.e. $\alpha_F(G) = \max\{|X| : X \text{ is an } F\text{-packing of } G\}$. By abuse of notation, if F is a singleton $\{i\}$, we will use $\alpha_i(G)$ instead of $\alpha_{\{i\}}(G)$. This concept allows us to give an alternative definition of the packing chromatic number of G, since $\chi_\rho(G) = \min\{k : \alpha_{[k]}(G) = n\}$. Argiroffo et al. [3] studied the packing-chromatic number of a graph G in terms of $[d-1]$-packings, $d \geq 2$ being the diameter of G. They obtain the following result.

Lemma 2.3 (Lemma 3 in [3])**.** *If G is a connected graph on n vertices and $d = diam(G) \geq 2$, then $\chi_\rho(G) \leq (d-1) + n - \alpha_{[d-1]}(G)$, with equality if $\alpha_{[d-1]}(G) < n$.*

From this last result we can obtain a lower bound for the packing chromatic number of connected graphs.

Lemma 2.4. *Let G be a connected graph where $|V(G)| = n$, $d = diam(G) \geq 2$ and $F \subseteq [d-1]$. If $\alpha_F(G) + \sum_{i \in [d-1] \setminus F} \alpha_i(G) < n$ then*

$$\chi_\rho(G) \geq d - 1 + n - \Big(\alpha_F(G) + \sum_{i \in [d-1] \setminus F} \alpha_i(G)\Big).$$

Proof. Observe that $\alpha_{[d-1]}(G) \leq \alpha_F(G) + \sum_{i \in [d-1]-F} \alpha_i(G) < n$.

From Lemma 2.3, we have that $\chi_\rho(G) = d - 1 + n - \alpha_{[d-1]}(G)$.

Therefore, $\chi_\rho(G) \geq d - 1 + n - \Big(\alpha_F(G) + \sum_{i \in [d-1]-F} \alpha_i(G)\Big)$. □

In Section 4.1 we will use the above results in order to obtain lower bounds for $\chi_\rho(Q_7)$ and $\chi_\rho(Q_8)$. For this, we will use the i-packing numbers of some hypercubes, that can be obtained from results on Coding Theory [1].

An (n, d) *binary code* is a subset of $V(Q_n)$ such that the distance between any pair of vertices is at least d and $A(n, d)$ is the maximum size of an (n, d) binary code. Therefore, an (n, d) binary code is a $(d-1)$-packing of Q_n and $A(n, d) = \alpha_{d-1}(Q_n)$. In Table 1 we show some i-packing numbers of Q_n (see reference [1]) that we use in order to prove that every $F \subseteq [6]$ verifies the condition in Lemma 2.4 for Q_7. In fact,

$$\alpha_F(Q_7) + \sum_{i \in [6]-F} \alpha_i(Q_7) \leq \sum_{i=1}^{6} \alpha_i(Q_7) = 94 < 2^7.$$

For example, let $F = \{1, 2, 6\}$. In this case we have:

$$\chi_\rho(Q_7) \geq 6 + 2^7 - (\alpha_F(Q_7) + \sum_{i=3}^{5} \alpha_i(Q_7)) = 122 - \alpha_F(Q_7). \tag{1}$$

i	2	3	4	5	6	7	8	9	10
$\alpha_i(Q_6)$	8	4	2	2	1	1	1	1	1
$\alpha_i(Q_7)$	16	8	2	2	2	1	1	1	1
$\alpha_i(Q_8)$	20	16	4	2	2	2	1	1	1
$\alpha_i(Q_9)$	40	20	6	4	2	2	2	1	1
$\alpha_i(Q_{10})$	72	40	12	6	2	2	2	2	1
$\alpha_i(Q_{11})$	144	72	24	12	4	2	2	2	2

Table 1: Values of $\alpha_i(Q_n)$ from [1]

In Section 4.1 we will prove that $\alpha_{\{1,2,6\}}(Q_7) \leq 73$, which implies that $\chi_\rho(Q_7) \geq 49$ and from the upper bound in Table 2 (see Section 4), we will conclude that $\chi_\rho(Q_7) = 49$.

Analogously, $\sum\limits_{i=1}^{7} \alpha_i(Q_8) = 174 < 2^8$. If $F = \{1, 2, 4\}$, from Lemma 2.4 we have

$$\chi_\rho(Q_8) \geq 7 + 2^8 - (\alpha_F(Q_8) + 22) = 241 - \alpha_F(Q_8). \tag{2}$$

In Section 4.1 we will show that $\alpha_{\{1,2,4\}}(Q_8) \leq 146$ and therefore the lower and upper bounds coincide.

This paper is organized as follows. Next section is devoted to improve the upper bound for the packing chromatic number of Q_n given by Goddard et al. in [9]. In Section 4 we prove several properties on F-packings that allow us to obtain $\chi_\rho(Q_n)$ for $n = 6, 7, 8$. Using these results, in Section 5 we improve the bounds for the packing chromatic number of the n-cube for $n = 9, 10, 11$. Finally, we present a non-closed formula for an upper bound of Q_n for $n \geq 9$, that slightly improves the bound given in Section 3.

3 Packing colorings of Q_n: the upper bound

The principal result of this section is the following :

Theorem 3.1. *$\chi_\rho(Q_n) \leq 3 + 2^n(\frac{1}{2} - \frac{1}{2^{\lceil \log_2 n \rceil}}) - 2\lfloor \frac{n-4}{2} \rfloor$, for $n \geq 4$.*

In order to prove Theorem 3.1, we derive a packing coloring of Q_n with such a number of colors. For this, we use an algebraic approach to construct 2− and 3−packings for Q_n as follows.

As mentioned previously, Q_n is the Cayley graph of the Abelian group $\mathbb{Z}_2^n$ where the connector set is the subset of n-vectors having exactly one component equal to 1. By using the greedy algorithm that computes iteratively a maximum i-packing I_i in the subgraph of Q_n induced by the vertices $V(Q_n) \setminus \bigcup_{j<i} I_j$, for $i \geq 1$ and $n \leq 13$, we have observed that the 2− and 3−packings of Q_n obtained via this method had a very nice structure that we define as follows:

Definition 3.2. *Let $n \geq 4$. For $m \in \{4, \ldots, n\} \setminus \{2^k + 1 : k \in \mathbb{Z}^+\}$, consider the n-vectors of $\mathbb{Z}_2^n$ defined as follows :*

1. *Type I: If m is even,*

$$v_{m,n}^i = \begin{cases} 1, & if\ i \in \{1, 2, m-1, m\} \\ 0, & otherwise \end{cases}$$

2. *Type II: If m is odd and $m-1$ is not a power of 2,*

$$v^i_{m,n} = \begin{cases} 1, & \text{if } i \in \{1, m-2^{\lfloor \log_2 m \rfloor}, 1+2^{\lfloor \log_2 m \rfloor}, m\} \\ 0, & \text{otherwise} \end{cases}$$

We define the subset $W_n \subset \mathbb{Z}_2^n$ as the set containing the vectors $v_{m,n}$ defined previously.

Let see an example of W_n for $n = 13$:

$v_{4,13}$	1	1	1	1	0	0	0	0	0	0	0	0	0
$v_{6,13}$	1	1	0	0	1	1	0	0	0	0	0	0	0
$v_{7,13}$	1	0	1	0	1	0	1	0	0	0	0	0	0
$v_{8,13}$	1	1	0	0	0	0	1	1	0	0	0	0	0
$v_{10,13}$	1	1	0	0	0	0	0	0	1	1	0	0	0
$v_{11,13}$	1	0	1	0	0	0	0	0	1	0	1	0	0
$v_{12,13}$	1	1	0	0	0	0	0	0	0	0	1	1	0
$v_{13,13}$	1	0	0	0	1	0	0	0	1	0	0	0	1

Notice that every vector in W_n has weight four. Moreover, by construction, we have that:

Observation 3.3. *Let $n \geq 4$. Then, $|W_n| + 1 = |W_{n+1}|$ if and only if n is not a power of two, otherwise $|W_n| = |W_{n+1}|$. Moreover, if n is even (resp. odd), the new vector in W_{n+1} is of Type II (resp. I).*

By Observation 3.3 and by induction on n, we can deduce the following result.

Lemma 3.4. *Let $n \geq 4$. Then, $|W_n| = n - 1 - \lceil \log_2 n \rceil$.*

Definition 3.5. *Let $n \geq 4$. We denote by $\mathcal{G}_n$ the subgroup of the Abelian group $\mathbb{Z}_2^n$ generated by W_n.*

Theorem 3.6. *Let $n \geq 4$. The elements of the subgroup $\mathcal{G}_n$ form a 3-packing of Q_n.*

The proof of Theorem 3.6 follows from next Lemmas 3.7, 3.8, 3.9, and 3.10.

Lemma 3.7. *Let $n \geq 4$. The elements in $\mathcal{G}_n$ satisfy the following properties:*

1. *For all $u \in \mathcal{G}_n$, u has an even number (0 included) of components equal to 1, that is, wt(u) is even.*

2. *For all $u, v \in \mathcal{G}_n$, we have that dist(u, v) is even.*

Proof. We know that $\text{dist}(u, v) = \text{wt}(u + v)$ for all $u, v \in \mathbb{Z}_2^n$. Moreover, by construction, we know that $\text{wt}(u) = \text{wt}(v) = 4$ for all $u, v \in W_n$, and by Observation 2.1, we know that $\text{dist}(u, v) = \text{wt}(u + v)$ is even. Now, each $u \in \mathcal{G}_n$ is equal to the sum of some vectors in W_n. By applying appropriately Observation 2.1 and Observation 2.2, we can deduce that $\text{wt}(u)$ is even, which proves (1). Now, by (1) we have that the weight of both u and v is even. Therefore, by Observation 2.2, we have that $\text{dist}(u, v)$ is even, which proves (2). $\square$

Lemma 3.8. *Let $n \geq 4$ and let M be a nonempty set of even integers such that for each $m \in M$ we have that $v_{m,n} \in W_n$ is of Type I. Let $y = \sum_{m \in M} v_{m,n}$. Then, wt$(y) \geq 4$.*

Proof. By Definition 3.2, if $v_{m,n}$ is a vector of Type I then, it has four 1's in positions $1, 2, m-1, m$. Thus, if $|M| = 1$ then, the result trivially holds. Now, if $|M| \geq 2$ then, for any pair of different n-vectors $v_{m_1,n}$ and $v_{m_2,n}$ in W_n of Type I, with $m_1, m_2 \in M$, we have that $v_{m_1,n} + v_{m_2,n}$ has at least four 1's in positions $m_1 - 1, m_1, m_2 - 1, m_2$ because $m_1 \neq m_2$. Therefore, it is easy to deduce that $\mathrm{wt}(y) \geq 4$ in all the cases. □

Lemma 3.9. *Let $n \geq 4$ and let M be a nonempty set of odd integers such that for each $m \in M$ we have that $v_{m,n} \in W_n$ is of Type II. Let $y = \sum_{m \in M} v_{m,n}$. Then, $wt(y) \geq 4$.*

Proof. Let $k = |M|$. We will proceed by induction on k. If $k = 1$ then, by Definition 3.2, the results trivially holds. We assume that $k > 1$ and that the lemma holds for any $t \leq k$ and we will prove that if y is the sum of $k+1$ different vectors of Type II in W_n then, $\mathrm{wt}(y) \geq 4$. Thus, we have that $y = \sum_{m \in M} v_{m,n}$ with $|M| = k+1$. Let $m' = \max\{m : m \in M\}$ and let $M' = \{m \in M : \lfloor \log_2 m \rfloor = \lfloor \log_2 m' \rfloor\}$. Clearly, $1 \leq |M'| \leq k+1$. Let $y' = \sum_{m \in M'} v_{m,n}$. We consider the following two cases:

- *Case $M = M'$.* If $|M| \geq 2$ then, y' has 1's at least in positions $m \in M$ and in positions $m - \lfloor \log_2 m' \rfloor$. Therefore, $\mathrm{wt}(y') = \mathrm{wt}(y) \geq 4$.

- *Case $M \neq M'$.* Let $y'' = \sum_{m \in M \setminus M'} v_{m,n}$. Notice that if $m \in M \setminus M'$ then, $v^i_{m,n} = 0$ for all $i \geq 1 + 2^{\lfloor \log_2 m' \rfloor}$ (otherwise, there is a value $p \in M \setminus M'$ such that $p \geq 1 + 2^{\lfloor \log_2 m' \rfloor}$ and so, $\lfloor \log_2 p \rfloor \geq \lfloor \log_2 m' \rfloor$ and thus p must be in M'). By induction hypothesis, we have that $\mathrm{wt}(y') \geq 4$ and $\mathrm{wt}(y'') \geq 4$. Moreover, $y = y' + y''$ and y'' has 0's in all positions $i \geq 1 + 2^{\lfloor \log_2 m' \rfloor}$. As we have mentioned before, for each $m \in M$, we have that $m - 2^{\lfloor \log_2 m \rfloor} < 1 + 2^{\lfloor \log_2 m \rfloor} < m$. Suppose that $M' = \{v_{m',n}\}$. Then, $v_{m',n}$ has 1's in positions $1, m' - 2^{\lfloor \log_2 m' \rfloor}, 1 + 2^{\lfloor \log_2 m' \rfloor}$, and m'. By induction hypothesis, $\mathrm{wt}(y'') \geq 4$ and y'' has 0's in all positions $i \geq 1 + 2^{\lfloor \log_2 m' \rfloor}$. Therefore, y' and y'' differ in positions $1 + 2^{\lfloor \log_2 m' \rfloor}$ and m'. Moreover, y' has a 1 in position $j \in \{2, \ldots, 2^{\lfloor \log_2 m' \rfloor}\}$ and y'' has at least three 1's in positions $j_1, j_2, j_3 \in \{2, \ldots, 2^{\lfloor \log_2 m' \rfloor}\}$. Therefore, in this case, y' and y'' differ in at least two positions in the interval $\{2, \ldots, 2^{\lfloor \log_2 m' \rfloor}\}$. Thus, $\mathrm{wt}(y) \geq 4$.

  If $|M'| \geq 3$ then, there are at least three different values m_1, m_2 and m' in M', with $m_1 < m_2 < m'$, and thus, y' has 1's at least in positions m_1, m_2 and m'. Moreover, y'' is such that it has 0's in all positions $i \geq 1 + 2^{\lfloor \log_2 m' \rfloor}$. As $m_1 > 1 + 2^{\lfloor \log_2 m' \rfloor}$ then, $\mathrm{wt}(y' + y'') \geq 3$. Moreover, by Lemma 3.7(1), $\mathrm{wt}(y)$ is even, and thus, $\mathrm{wt}(y) \geq 4$.

  Finally, let $|M'| = 2$. Let $m, m' \in M'$ with $m < m'$. By induction hypothesis, $\mathrm{wt}(y'') \geq 4$ and y'' has 0's in all positions $i \geq 1 + 2^{\lfloor \log_2 m' \rfloor}$. Therefore, y' and y'' differ in positions m, m', and in at least one position $j \in \{2, \ldots, 2^{\lfloor \log_2 m' \rfloor}\} \setminus \{m, m'\}$, which implies that $\mathrm{wt}(y) \geq 3$. Again , by Lemma 3.7(1), $\mathrm{wt}(y)$ is even and thus, we conclude that $\mathrm{wt}(y) \geq 4$.

□

Now, we are ready to prove that the set $\mathcal{G}_n$ is a 3-packing of Q_n.

Lemma 3.10. *Let $n \geq 4$. For any $u, v \in Q_n$, $u \neq v$, such that $u, v \in \mathcal{G}_n$, we have that the distance between u and v in Q_n satisfies : $dist(u, v) \geq 4$.*

Proof. Let $x \in \mathcal{G}_n$. Therefore, $x = \sum_{w \in H_x} w$ for some $H_x \subseteq W_n$. Then, we define H'_x and H''_x as the sets of n-vectors in H_x of type I and II, respectively. For any pair $A, B \subseteq W_n$, denote $A \triangle B$ the set $(A \cup B) \setminus (A \cap B)$.

Let $u, v \in Q_n$, $u \neq v$. Consider $H_u \subseteq W_n$ and $H_v \subseteq W_n$ such that $u = \sum_{w \in H_u} w$ and $v = \sum_{w \in H_v} w$.

Notice that $u + v = \sum_{w \in H'_u \cup H''_u} w + \sum_{w \in H'_v \cup H''_v} w = \sum_{w \in H'_u \triangle H'_v} w + \sum_{w \in H''_u \triangle H''_v} w$.

Let $x = \sum_{w \in H'_u \triangle H'_v} w$ and $y = \sum_{w \in H''_u \triangle H''_v} w$. We have that $u + v = x + y$.

Moreover, we know that $\mathrm{dist}(u, v) = wt(u + v) = \sum_{i=1}^{n} (u + v)^i$, hence $\mathrm{dist}(x, y) = \sum_{i=1}^{n} (x + y)^i = \mathrm{dist}(u, v)$. We will prove that the distance between x and y is at least 4.

By Lemma 3.8 (resp. Lemma 3.9), if $H'_u \triangle H'_v \neq \emptyset$ (resp. $H''_u \triangle H''_v \neq \emptyset$) then, $\mathrm{wt}(x) \geq 4$ (resp. $\mathrm{wt}(y) \geq 4$). Therefore, if either $H'_u \triangle H'_v = \emptyset$ or $H''_u \triangle H''_v = \emptyset$ then, $\mathrm{dist}(x, y) = \mathrm{wt}(x + y) \geq 4$. Therefore, assume that $H'_u \triangle H'_v \neq \emptyset$ and $H''_u \triangle H''_v \neq \emptyset$. By Lemma 3.8 (resp. Lemma 3.9), $\mathrm{wt}(x) \geq 4$ (resp. $\mathrm{wt}(y) \geq 4$). Moreover, all the 1's in y occur in odd positions. Besides, x has at least one 1 in an even position greater than 2 and at least one 1 in an odd position greater than 1. The other possible 1's in x are in position 1 and 2. Therefore, x and y differ in at least three positions and thus, $\mathrm{wt}(x + y) \geq 3$. However, by Lemma 3.7(1), $\mathrm{wt}(x + y)$ is even and so, $\mathrm{wt}(x + y) = \mathrm{dist}(x, y) \geq 4$. $\square$

As a simple consequence of Definition 3.2, we can deduce easily the following observation:

Observation 3.11. *Let $n \geq 4$. Then, for any element $u \in W_n$ and for any nonempty set $M \subset W_n \setminus \{u\}$, we have that $u \neq \sum_{v \in M} v$.*

In fact, let $u = v_{i,n}$, $w = \sum_{v \in M} v$ and $m = \max\{j : v_{j,n} \in M\}$. It is easy to see that $j \neq i$ since $u \notin M$. If $j < i$, $w^i = 0$ and then $w \neq u$. Analogously, if $j > i$, $w^j = 1$ and then, $w \neq u$ since $u^j = 0$.

Since Observation 3.11 states that the elements of W_n are linearly independent, the following lemma is immediate.

Lemma 3.12. *Let $n \geq 4$, and let $\mathcal{G}_n$ be the subgroup of the Abelian group $\mathbb{Z}_2^n$ generated by W_n. Then, the order of $\mathcal{G}_n$ is equal to $2^{|W_n|}$.*

Definition 3.13. *Let $n \geq 4$. For each j, with $1 \leq j \leq \lfloor \frac{n-4}{2} \rfloor$, let A_j and B_j be 2-subsets of $\mathbb{Z}_2^n$ constructed as follows : A_j is formed by two n-vectors a_{j_1} and a_{j_2} where a_{j_1} has only one 1 in position $j+2$ and 0 otherwise, and a_{j_2} has 1 in positions $t \in [2j+4] \setminus \{j+2\}$, and 0 otherwise. The 2-set B_j is formed by two n-vectors b_{j_1} and b_{j_2}, where b_{j_1} (resp.b_{j_2}) is equal to a_{j_1} (resp. a_{j_2}) but with the two first positions complemented.*

Lemma 3.14. *Let $n \geq 6$. Let $g_2 \in \mathbb{Z}_2^n$ (resp. $g_3 \in \mathbb{Z}_2^n$) be the element having only the first component (resp. the second component) equal to 1 and the remaining components equal to 0. For each j, with $1 \leq j \leq \lfloor \frac{n-4}{2} \rfloor$, the 2-sets A_j and B_j in Definition 3.13 satisfy the following properties :*

1. *$A_j \cap (g_2 + \mathcal{G}_n) = A_j \cap (g_3 + \mathcal{G}_n) = B_j \cap (g_2 + \mathcal{G}_n) = B_j \cap (g_3 + \mathcal{G}_n) = \emptyset$.*

2. A_j (resp. B_j) is a $(2j+2)$-packing (resp. $(2j+3)$-packing) of Q_n.

Proof. First notice that, by construction, all the vectors in A_j and B_j have odd weight. By Lemma 3.7(1), we know that each vector in $\mathcal{G}_n$ has even weight. Therefore, each vector in $g_2 + \mathcal{G}_n$ (resp. $g_3 + \mathcal{G}_n$) has odd weight.

1. By Definition 3.13, $a_{j_1} \in A_j$ has only one 1 in position $j+2$. Moreover, g_2 (resp. g_3) belongs to the coset $g_2+\mathcal{G}_n$ (resp. $g_3+\mathcal{G}_n$) and it is easy to verify that $\text{dist}(a_{j_1}, g_2) = \text{dist}(a_{j_1}, g_3) = 2$. Thus, $a_{j_1} \notin g_2+\mathcal{G}_n$ (resp. $a_{j_1} \notin g_3+\mathcal{G}_n$). Now, by Definition 3.13, $a_{j_2} \in A_j$ has 1's in positions $t \in [2j+4] \setminus \{j+2\}$. Let v_A be the element in $\mathcal{G}_n$ generated by the sum of vectors $v_{2k+4} \in W_n$, for $0 \le k \le j$. Now, let $w_A = g_2 + v_A$. If j is even, w_A has 1's in positions $t \in [2j+4] \setminus \{2\}$, otherwise, if j is odd then, w_A has 1's in positions $t \in [2j+4] \setminus \{1\}$. In both cases, it is easy to verify that $\text{dist}(a_{j_2}, w_A) = 2$. Similarly, if $w_A = g_3 + v_A$ then, $\text{dist}(a_{j_2}, w_A) = 2$, therefore, $a_{j_2} \notin g_2 + \mathcal{G}_n$ (resp. $a_{j_2} \notin g_3 + \mathcal{G}_n$) which implies that $A_j \cap (g_2 + \mathcal{G}_n) = A_j \cap (g_3 + \mathcal{G}_n) = \emptyset$. In a similar way, we can deduce that $B_j \cap (g_2 + \mathcal{G}_n) = B_j \cap (g_3 + \mathcal{G}_n) = \emptyset$.

2. By Definition 3.13, we have that $\text{dist}(a_{j_1}, a_{j_2}) = \text{dist}(b_{j_1}, b_{j_2}) = 2j+4$. Moreover, notice that by construction, we have that $A_j \cap B_j = \emptyset$ and for all $j \neq j'$, we have that $A_j \cap A_{j'} = A_j \cap B_{j'} = B_j \cap A_{j'} = B_j \cap B_{j'} = \emptyset$. It shows that A_j (resp. B_j) is a $(2j+2)$-packing (resp. $(2j+3)$-packing) of Q_n as desired. Actually, A_j is in fact a $(2j+3)$-packing of Q_n and so, it is also a $(2j+2)$-packing of Q_n.

□

From the previous lemmas, we are able to prove Theorem 3.1.

Proof of Theorem 3.1

Let $n \ge 4$. Let $\mathcal{G}_n$ be the subgroup of the Abelian group $\mathbb{Z}_2^n$ generated by the set W_n (see Definition 3.2). Clearly, the elements of $\mathbb{Z}_2^n$ correspond to the vertices of the binary n-dimensional hypercube Q_n. Let I_1^n, I_2^n and I_3^n be subsets of vertices of Q_n constructed as follows : I_1^n is the set of all vertices in Q_n having even weight (0 included). The sets I_2^n and I_3^n are the cosets $g_2+\mathcal{G}_n$ and $g_3+\mathcal{G}_n$, resp., where g_2 and g_3 are defined as in Lemma 3.14. By Lemma 3.10, it is clear that both I_2^n and I_3^n are disjoint of I_1^n, because all the elements in I_2^n and in I_3^n have odd weight. Moreover, I_2^n and I_3^n are disjoint by the Lagrange's Theorem. Therefore, I_i^n is an i-packing of Q_n, for $i = 1, 2, 3$. Now, observe that the family of sets in Definition 3.13 are pairwise disjoint. Furthermore, by Lemma 3.14, for $1 \le j \le \lfloor \frac{n-4}{2} \rfloor$, the sets A_j and B_j are $(2j+2)$-packing and $(2j+3)$-packing of Q_n respectively. By given a different color greater than $2\lfloor \frac{n-4}{2} \rfloor + 3$ to the remaining vertices in Q_n, we obtain the desired packing coloring of Q_n with $3 + 2^n(\frac{1}{2} - \frac{1}{2^{\lceil \log_2 n \rceil}}) - 2\lfloor \frac{n-4}{2} \rfloor$ colors. □

4 Lower bounds for $\chi_\rho(Q_n)$: the cases $n = 6$, 7 and 8.

As mentioned in the introduction, Goddard et al. [9] computed the packing chromatic numbers of the first five hypercubes and provided particular bounds for $\chi_\rho(Q_n)$ for $6 \le n \le 11$.

Moreover, by Corollary 1.1 and Theorem 3.1, actually we know that $25 \le \chi_\rho(Q_6) \le 25$, $44 \le \chi_\rho(Q_7) \le 49$ and $81 \le \chi_\rho(Q_8) \le 95$. Thus, we have the following direct result.

Corollary 4.1. $\chi_\rho(Q_6) = 25$.

n	1	2	3	4	5	6	7	8	9	10	11
$\chi_\rho(Q_n) =$	2	3	5	7	15	-	-	-	-	-	-
$\chi_\rho(Q_n) \geq$	-	-	-	-	-	15	28	63	132	285	610
$\chi_\rho(Q_n) \leq$	-	-	-	-	-	25	49	95	219	441	881
Gap	-	-	-	-	-	10	21	32	87	156	271

Table 2: Previous results

The main result in this section is the computation of a tight lower bound for the packing chromatic number of hypercubes of dimension 7 and 8. In order to obtain these lower bounds, we combine the results obtained by Agrell et al. [1] concerning the i-packing numbers of Q_7 and Q_8 (see Table 1), for $1 \leq i < 8$, with the ones concerning the maximum size of balanced independent (and dominating) sets on hypercubes obtained by Ramras [14].

4.1 The packing chromatic number of Q_7 and Q_8

In this section, we will obtain the packing chromatic number of Q_7 and Q_8. As we have proved in Section 2, from Lemma 2.4, we can compute $\chi_\rho(Q_7)$ and $\chi_\rho(Q_8)$ by proving that $\alpha_{\{1,2,6\}}(Q_7) \leq 73$ and $\alpha_{\{1,2,4\}}(Q_8) \leq 146$, respectively. Therefore, we will show that $\alpha_{\{1,2,6\}}(Q_7) = 73$ and $\alpha_{\{1,2,4\}}(Q_8) = 146$.

Previously, we present some definitions and technical results on hypercubes.

We denote by $V_e(Q_n)$ and $V_o(Q_n)$ the subset of vertices $v \in V(Q_n)$ with even weight (included 0) and odd weight, respectively.

Let $V^0(Q_n) = \{v \in V(Q_n) : v^1 = 0\}$ and $V^1(Q_n) = \{v \in V(Q_n) : v^1 = 1\}$.

For $i \in \{0,1\}$ and $j \in \{e,o\}$, $V^i_j(Q_n)$ denotes the set $V^i(Q_n) \cap V_j(Q_n)$.

Besides, we call Q^0_{n-1} and Q^1_{n-1} to the $(n-1)$-dimensional hypercube induced by $V^0(Q_n)$ and $V^1(Q_n)$ in Q_n, respectively. In Figure 1 we present a relation scheme.

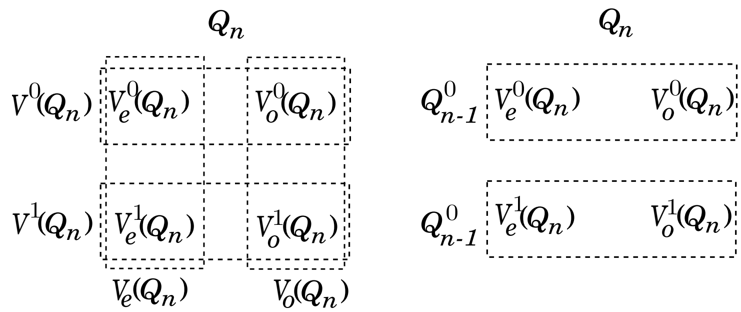

Figure 1: Relation scheme.

Remark 4.2. *It is not hard to see that the set of edges in the subgraph induced by $V^0_e(Q_n) \cup V^1_o(Q_n)$ is a matching of Q_n with size 2^{n-2}. Analogously, if we consider the set of edges in the subgraph induced by $V^1_e(Q_n) \cup V^0_o(Q_n)$, we have a matching of Q_n with size 2^{n-2}. Moreover, these matchings are disjoint and the union of them is a perfect matching of Q_n.*

Let us recall some results on 1-packings due to Ramras [14].

Lemma 4.3 (Proposition 4 in [14]). *If I is a 1-packing of Q_5 such that $I \cap V_e(Q_5) \neq \emptyset$ and $I \cap V_o(Q_5) \neq \emptyset$, then $|I| \leq 12$.*

Given a graph G, a subset of vertices U is a *dominating set* of G if $N[v] \cap U \neq \emptyset$ for all $v \in V$. A subset of vertices of a connected bipartite graph is called *balanced* if it has exactly half its elements in each of the partition sets.

Theorem 4.4 (Proposition 7 in [14]). *The maximum size of a balanced 1-packing of Q_7 is 44, and there is a balanced independent dominating set of this size.*

Following, we present some useful remarks that involves i-packings of hypercubes.

Let $v \in V(Q_n)$, we know that every pair of vertices in $N(v)$ are at distance 2. So, if X_i is an i-packing of Q_n with $i \geq 2$, then $|X_i \cap N(v)| \leq 1$. Moreover,

Remark 4.5. *Let X_i be an i-packing of Q_n with $i \geq 2$ and $I \subset V(Q_n)$. Then, $|X_i \cap N(I)| \leq |I|$.*

Now, consider X_i an i-packing of Q_n with i even. If $v, w \in X_i \cap V_e(Q_n)$, the distance between u and v is even and at least $i+1$. Therefore, $dist(u, v) \geq i+2$. This reasoning leads us to the following result.

Remark 4.6. *Let X_i be an i-packing of Q_n with i even. Then, $|X_i \cap V_e(Q_n)| \leq \alpha_{i+1}(Q_n)$ and $|X_i \cap V_o(Q_n)| \leq \alpha_{i+1}(Q_n)$.*

Next, we obtain a combinatorial result on hypercubes that we will apply to compute the packing chromatic numbers in the next section.

Lemma 4.7. *Let K be a subset of $V_e(Q_n)$ or $V_o(Q_n)$ such that $|K| = k$, for $k \leq 4$ and $k \leq n$. Then,*

$$|N(K)| \geq kn - \left(\frac{k(k+1)}{2} - 1\right),$$

with equality if $k = 1$.

Proof. If $K = \{v\}$, $N(K) = N(v)$ and the result is trivial.

For the remaining cases we proceed by induction on n. It is not hard to prove that the result holds if $n \leq 4$. Let $n \geq 5$ and suppose that the result holds if $K \subset V_e(Q_{n-1})$ or $K \subset V_o(Q_{n-1})$.

$k = 2$. Firstly, let us consider $K \subset V_e(Q_n)$. If $K \subset V_e^0(Q_n)$, we can apply the inductive hypothesis over Q_{n-1}^0 and we obtain $|N(K) \cap V_o^0(Q_n)| = |N(K) \cap V_o(Q_{n-1}^0)| \geq 2(n-1) - 2$. On the other hand, from Remark 4.2, $|N(K) \cap V_o^1(Q_n)| = 2$. Therefore, $|N(K)| \geq 2n - 2$. Analogously if $K \subset V_e^1(Q_n)$.

Now, consider $|K \cap V_e^0(Q_n)| = 1$ and let $K \cap V_e^0(Q_n) = \{x\}$ and $K \cap V_e^1(Q_n) = \{y\}$. Since Q_{n-1}^0 and Q_{n-1}^1 are $(n-1)$-cubes, $|N(x) \cap V_o^0(Q_n)| = |N(x) \cap V_o(Q_{n-1}^0)| = n-1$ and $|N(y) \cap V_o^1(Q_n)| = |N(y) \cap V_o(Q_{n-1}^1)| = n-1$. Thus $|N(K)| \geq 2n-2$.

$k = 3$. Suppose $K = \{x, y, z\} \subset V_e(Q_n)$. If $K \subset V_e^0(Q_n)$, by applying the inductive hypothesis over Q_{n-1}^0, we obtain that $|N(K) \cap V_o^0(Q_n)| \geq 3(n-1) - 5$ and from Remark 4.2, $|N(K) \cap V_o^1(Q_n)| = 3$. Then $|N(K)| \geq 3n - 5$. Analogously if $K \subset V_e^1(Q_n)$.

If $K \cap V_e^0(Q_n) = \{x, y\}$ and $K \cap V_e^1(Q_n) = \{z\}$, from previous result when $k = 2$, we have that $|N(\{x, y\}) \cap V_o^0(Q_n)| \geq 2(n-2)$. Therefore $|N(K)| \geq 3n - 5$, since $|N(z) \cap V_o^1(Q_n)| = n - 1$.

The case when $|K \cap V_e^0(Q_n)| = 1$ and $|K \cap V_e^1(Q_n)| = 2$ can be obtained in an analogous way.

$k=4$. First, consider $K=\{w,x,y,z\}\subset V_e(Q_n)$.

If $K\subset V_e^0(Q_n)$, by applying the inductive hypothesis over Q_{n-1}^0, we have that $|N(K)\cap V_o^0(Q_n)|\geq 4(n-1)-9$ and from Remark 4.2, $|N(K)\cap V_o^1(Q_n)|=4$. Therefore, $|N(K)|\geq 4n-9$. Analogously if $K\subset V_e^1(Q_n)$.

If $K\cap V_e^0(Q_n)=\{w,x,y\}$ and $K\cap V_e^1(Q_n)=\{z\}$, using previous results when $k=3$ over Q_{n-1}^0, we can conclude that $|N(\{w,x,y\})\cap V_o^0(Q_n)|\geq 3(n-1)-5$. Besides, we know that $|N(z)\cap V_o^1(Q_n)|=n-1$. Then, $|N(K)|\geq 3(n-1)-5+(n-1)=4n-9$. Similarly, the result holds if $|K\cap V_e^0(Q_n)|=1$ and $|K\cap V_e^1(Q_n)|=3$.

Finally, suppose that $K\cap V_e^0(Q_n)=\{w,x\}$ and $K\cap V_e^1(Q_n)=\{y,z\}$. From previous results when $k=2$, we obtain that $|N(\{w,x\})\cap V_o^0(Q_n)|\geq 2(n-2)$ and $|N(\{w,x\})\cap V_o^1(Q_n)|\geq 2(n-2)$. Therefore, $|N(K)|\geq 4(n-2)>4n-9$.

Applying similar reasoning as above for the case when $K\subset V_o(Q_n)$, we obtain the result. □

As we have mentioned before, our goal is to obtain the packing chromatic number of Q_7 and Q_8. To this end, it is enough to prove that $\alpha_{\{1,2,6\}}(Q_7)\leq 73$ and $\alpha_{\{1,2,4\}}(Q_8)\leq 146$, respectively.

Computing $\chi_\rho(Q_7)$

Along this section we assume that $F=\{1,2,6\}$. Firstly, we compute the size of an F-packing of Q_7 for a particular case.

Lemma 4.8. *Let X_i be an i-packing of Q_7 for $i\in F$. Consider $X=\cup_{i\in F}X_i$. If $|X_1\cap V_o(Q_7)|\leq 1$ or $|X_1\cap V_e(Q_7)|\leq 1$ then, $|X|\leq 73$.*

Proof. Without lost of generality we can suppose that X_1, X_2 and X_6 are pairwise disjoint. Then, $|X|=|X_1|+|X_2|+|X_6|$.

Consider $|X_1\cap V_o(Q_7)|\leq 1$ and let $I=X_1\cap V_o(Q_7)$.

From the facts that $X=(X\cap V_o(Q_7))\cup(X\cap V_e(Q_7))$ and $X_1\cap N(I)=\emptyset$, we can conclude that $X\subseteq I\cup(X_2\cap V_o(Q_7))\cup(X_6\cap V_o(Q_7))\cup(V_e(Q_7)\setminus N(I))\cup(X_2\cap N(I))\cup(X_6\cap N(I))$.

Therefore,

$|X|\leq|I|+|X_2\cap V_o(Q_7)|+|X_6\cap V_o(Q_7)|+|V_e(Q_7)|-|N(I)|+|X_2\cap N(I)|+|X_6\cap N(I)|$.

From Remark 4.6, we know that $|X_2\cap V_o(Q_7)|\leq\alpha_3(Q_7)=8$ and $|X_6\cap V_o(Q_7)|\leq\alpha_7(Q_7)=1$. Besides, applying Remark 4.5, $|X_2\cap N(I)|\leq|I|$ and $|X_6\cap N(I)|\leq\min\{|I|,|X_6\cap V_e(Q_7)|\}$.

Observe that, if I is the empty set, $|X_6\cap N(I)|=0$ and $|X_6\cap N(I)|\leq 1$ otherwise.

Then,

$$|X|\leq 73+2|I|-|N(I)|+\min\{|I|,|X_6\cap V_e(Q_7)|\}.$$

Therefore, if $I=\emptyset$, $|X|\leq 73$ and if $|I|=1$, $|N(I)|=7$ and $|X|\leq 69$.

If $|X_1\cap V_e(Q_7)|\leq 1$ the result follows similarly. □

Now, let $X=\cup_{i\in F}X_i$ be an F-packing of Q_7. To prove that $\alpha_F(Q_7)\leq 73$, from the previous lemma, it remains to analyze the case $|X_1\cap V_o(Q_7)|\geq 2$ and $|X_1\cap V_e(Q_7)|\geq 2$. Notice that $|X_2\cup X_6|\leq\alpha_2(Q_7)+\alpha_6(Q_7)=18$. Therefore, it is enough to show that $|X_1|\leq 55$ in this case. To do this, if $I=X_1\cap V_o(Q_7)$, we proceed by case analysis on $|I|$.

Lemma 4.9. *Let X_1 be a 1-packing of Q_7 and $I = X_1 \cap V_o(Q_7)$. If $2 \le |I| \le 7$ then, $|X_1| \le 55$.*

Proof. If $|I| = 2$, from Lemma 4.7, $|N(I)| \ge 12$ and then $|X_1| \le |V_e(Q_7) \setminus N(I)| + |I| = |V_e(Q_7)| - |N(I)| + |I| \le 64 - 12 + 2 = 54$. Analogously, if $3 \le |I| \le 7$, $|N(I)| \ge 16$. Then $|X_1| \le 55$. $\square$

Next, for the cases $8 \le |I| \le 22$ we will use a different strategy than before. Previously, we need to define some subsets of vertices in Q_7.

Definition 4.10. *Consider $i, j \in \{0, 1\}$ and let $V^{ij} = \{v \in V(Q_7) : v^1 v^2 = ij\}$. For $s \in \{e, o\}$, we denote V_s^{ij} to the set $V^{ij} \cap V_s(Q_7)$ (see Figure 2).*

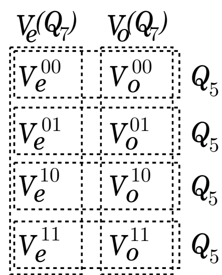

Figure 2: A relation scheme of sets V_s^{ij}, Q_5 and Q_7.

Let us observe that the subgraph induced by V^{ij} is isomorphic to Q_5 for all $i, j \in \{0, 1\}$. Notice that, from definition, $V_e(Q_7) = \cup_{i,j \in \{0,1\}} V_e^{ij}$ and $V_o(Q_7) = \cup_{i,j \in \{0,1\}} V_o^{ij}$.

Besides, it is not hard to see that a similar result that in Remark 4.2 can be apply to these sets. For instance, the edges of the subgraph induced by $V_e^{00} \cup V_o^{01}$ is a matching. Analogous result follows if we consider the subgraphs induced by $V_e^{00} \cup V_o^{10}$, $V_e^{01} \cup V_o^{00}$, $V_e^{01} \cup V_o^{11}$, $V_e^{10} \cup V_o^{00}$, $V_e^{10} \cup V_o^{11}$, $V_e^{11} \cup V_o^{01}$ or $V_e^{11} \cup V_o^{10}$.

Lemma 4.11. *Let X_1 be a 1-packing of Q_7 and $I = X_1 \cap V_o(Q_7)$. If $8 \le |I| \le 22$ then, $|X_1| \le 54$.*

Proof. Let $S_{ij} = V^{ij} \cap X_1$ for all $i, j \in \{0, 1\}$. We consider three different cases.

Case 1. Suppose $|I \cap V_o^{00}| \ge 12$. From Lemma 4.3 we have that $X_1 \cap V_e^{00} = \emptyset$. Besides, from Remark 4.2, we obtain that $|N(I) \cap V_e^{01}| \ge 12$. Thus, $|X_1 \cap V_e^{01}| \le 4$. Similarly, $|X_1 \cap V_e^{10}| \le 4$. Therefore, $|X_1 \cap V_e(Q_7)| = |X_1 \cap V_e^{00}| + |X_1 \cap V_e^{01}| + |X_1 \cap V_e^{10}| + |X_1 \cap V_e^{11}| \le 0 + 4 + 4 + 16 \le 24$ and $|X_1| \le 24 + |I| \le 46$.

Analogously, the same result follows if $|I \cap V_o^{01}| \ge 12$, $|I \cap V_o^{10}| \ge 12$, or $|I \cap V_o^{11}| \ge 12$.

Case 2. Consider $8 \le |I| \le 11$.

If $I \subseteq V_o^{00}$, from Lemma 4.3 we obtain that $|S_{00}| \le 12$. Besides, $|S_{01}| = |X_1 \cap V_e^{01}| \le 8$ since, from Remark 4.2, $|N(I) \cap V_e^{01}| \ge |I|$. Similarly, $|S_{10}| = |X_1 \cap V_e^{10}| \le 8$. Then, $|X_1| \le 12 + 8 + 8 + 16 = 44$.

We also obtain the result if $I \subseteq V_o^{01}$, $I \subseteq V_o^{10}$ or $I \subseteq V_o^{11}$.

Case 3. Suppose $I \cap V_o^{00} \neq \emptyset$, $I \cap V_o^{01} \neq \emptyset$ and $|I \cap V_o^{ij}| \leq 11$ for all $i, j \in \{0, 1\}$. From Lemma 4.3, we have that $|S_{00}| \leq 12$ and $|S_{01}| \leq 12$. Besides, applying Remark 4.2, $N(I) \cap V_e^{10} \neq \emptyset$ and $N(I) \cap V_e^{11} \neq \emptyset$. Therefore, $|S_{10}| \leq 15$ and $|S_{11}| \leq 15$. Then, $|X_1| \leq 54$.

By similar reasonings we can obtain the result for the cases when $|I \cap V_o^{ij}| \leq 11$ for all $i, j \in \{0, 1\}$ and I has no empty intersection with at least two of the sets $V_o^{00}, V_o^{01}, V_o^{10}, V_o^{11}$.

Finally, observe that we cover every possible case for $8 \leq |I| \leq 22$. Therefore, $|X_1| \leq 54$. □

Notice that Lemmas 4.9 and 4.11 can be applied similarly when $I = X_1 \cap V_e(Q_7)$.

Lemma 4.12. *Let X_1 be a 1-packing of Q_7 such that $|X_1 \cap V_o(Q_7)| \geq 2$ and $|X_1 \cap V_e(Q_7)| \geq 2$. Then, $|X_1| \leq 55$.*

Proof. From Theorem 4.4 we have that $|X_1 \cap V_o(Q_7)| \leq 22$ or $|X_1 \cap V_e(Q_7)| \leq 22$. Since $|X_1 \cap V_o(Q_7)| \geq 2$ and $|X_1 \cap V_e(Q_7)| \geq 2$, the result follows from previous lemmas. □

It is clear from Lemmas 4.8 and 4.12 that if X is an F-packing of Q_7 then $|X| \leq 73$. Thus, $\alpha_F(Q_7) \leq 73$. Hence, we have the main result of this section.

Theorem 4.13. *$\chi_\rho(Q_7) = 49$.*

To conclude this section, notice that to compute $\chi_\rho(Q_7)$, it was enough to show that $\alpha_F(Q_7) \leq 73$. Actually, $\alpha_F(Q_7) = 73$. In fact, let $X_1 = V_e(Q_7)$,
$X_2 = \{(0,0,0,0,0,0,1), (0,0,0,1,1,1,0), (0,1,1,0,0,1,0), (0,1,1,1,1,0,1), (1,0,1,0,1,0,0), (1,0,1,1,0,1,1), (1,1,0,0,1,1,1), (1,1,0,1,0,0,0)\}$ and
$X_6 = \{(1,1,1,1,1,1,1)\}$.

It is not hard to see that $\{X_i\}_{i \in F}$ is a family of pairwise disjoint i-packing of Q_7. Thus, $X = \bigcup_{i \in F} X_i$ is an F-packing of Q_7 and $|X| = 73$. Therefore, $\alpha_F(Q_7) = 73$.

Computing $\chi_\rho(Q_8)$

This section is devoted to compute $\alpha_{\{1,2,4\}}(Q_8)$, which allows us to obtain $\chi_\rho(Q_8)$. For the rest of this section we consider $F = \{1, 2, 4\}$.

Before concluding that $\alpha_F(Q_8) \leq 146$, we are going to prove three lemmas, in which we compute an upper bound for an F-packing of Q_8.

Lemma 4.14. *Let X_i be an i-packing of Q_8 for $i \in F$. Consider $X = \cup_{i \in F} X_i$. If $|X_1 \cap V_e(Q_8)| \leq 10$ or $|X_1 \cap V_o(Q_8)| \leq 10$ then, $|X| \leq 146$.*

Proof. Without lost of generality we can suppose that X_1, X_2 and X_6 are pairwise disjoint. Let $I = X_1 \cap V_e(Q_8)$ and $i = |I|$.

If $I = \emptyset$, from Remark 4.6 we have that $|X \cap V_e(Q_8)| \leq \alpha_3(Q_8) + \alpha_5(Q_8) = 18$. Therefore, $|X| \leq |X \cap V_e(Q_8)| + |V_o(Q_8)| \leq 146$.

Consider $1 \leq i \leq 10$. It is clear that $N(I) \subset V_o(Q_8)$. Let $T = V_o(Q_8) \setminus N(I)$. Observe that,
$X = I \cup (X_2 \cap V_e(Q_8)) \cup (X_4 \cap V_e(Q_8)) \cup ((X_2 \cup X_4) \cap N(I)) \cup (X \cap T)$.

From Remark 4.6, $|X_2 \cap V_e(Q_8)| \leq \alpha_3(Q_8) = 16$, $|X_4 \cap V_e(Q_8)| \leq \alpha_5(Q_8) = 2$. Besides, from Remark 4.5, $|\,(X_2 \cup X_4) \cap N(I)| \leq |X_2 \cap N(I)| + |X_4 \cap N(I)| \leq i + 2$. Then, $|X| \leq |T| + 20 + 2i$.

If $i = 1$, $|T| = 120$ and $|X| \leq 142$. Applying Lemma 4.7 we have that

$$|T| \leq \begin{cases} 114 & \text{if } i = 2 \\ 109 & \text{if } i = 3 \\ 105 & \text{if } 4 \leq i \leq 10. \end{cases}$$

Therefore $|X| \leq 146$. Analogously, the result follows if $|X_1 \cap V_o(Q_8)| \leq 10$. $\square$

If $X = \cup_{i \in F} X_i$ is an F-packing of Q_8, to prove that $|X| \leq 146$, it remains to analyze the cases such that $|X_1 \cap V_e(Q_8)| \geq 11$ and $|X_1 \cap V_o(Q_8)| \geq 11$. In this regard, notice that $\alpha_2(Q_8) = 20$ and $\alpha_4(Q_8) = 4$. Then, it is enough to prove that, in these cases, $|X_1| \leq 122$. Moreover, in the following lemmas we prove that the size of X_1 is at most 110 for a bit more general cases.

Lemma 4.15. *Let X_1 be a 1-packing of Q_8 such that $|X_1 \cap V_s^i(Q_8)| \geq 2$ for all $i \in \{0,1\}$, $s \in \{e,o\}$. Then, $|X_1| \leq 110$.*

Proof. First, observe that $V_s^i(Q_8) = V_s(Q_7^i)$, for all $i \in \{0,1\}$, $s \in \{e,o\}$. Then, $|X_1 \cap V_s(Q_7^i)| \geq 2$ for all $i \in \{0,1\}$, $s \in \{e,o\}$. From Lemma 4.12, we obtain that $|X_1 \cap V^0(Q_8)| \leq 55$ and $|X_1 \cap V^1(Q_8)| \leq 55$. Then, $|X_1| \leq 110$. $\square$

Lemma 4.16. *Let X_1 be a 1-packing of Q_8 such that $|X_1 \cap V_e(Q_8)| \geq 11$ and $|X_1 \cap V_o(Q_8)| \geq 11$. If there exists $i \in \{0,1\}$ and $s \in \{e,o\}$ such that $|X_1 \cap V_s^i(Q_8)| \leq 1$, then $|X_1| \leq 110$.*

Proof. Without loss of generality, suppose $|X_1 \cap V_e^1(Q_8)| \leq 1$. From hypothesis, $|X_1 \cap V_e^0(Q_8)| \geq 10$, since $V_e^0(Q_8) \cup V_e^1(Q_8) = V_e(Q_8)$. Observe that, from Remark 4.2, the set of edges in the subgraph induced by $V_e^0(Q_8) \cup V_o^1(Q_8)$ is a matching. Then,

$$|N(X_1) \cap V_o^1(Q_8)| \geq |X_1 \cap V_e^0(Q_8)| \geq 10.$$

Therefore, $|X_1 \cap V_o^1(Q_8)| \leq 54$ and $|X_1 \cap V^1(Q_8)| = |X_1 \cap V_e^1(Q_8)| + |X_1 \cap V_o^1(Q_8)| \leq 55$.

To conclude, let us see that $|X_1 \cap V^0(Q_8)| \leq 55$.

If $|X_1 \cap V_o^0(Q_8)| \leq 1$, by a similar reasoning as before, we obtain $|X_1 \cap V^0(Q_8)| \leq 55$.

Finally, suppose that $|X_1 \cap V_o^0(Q_8)| \geq 2$. As we have seen, $|X_1 \cap V_e^0(Q_8)| \geq 10$, hence we can apply Lemma 4.12 to the 7-cube Q_7^0 and obtain that $|X_1 \cap V^0(Q_8)| \leq 55$. $\square$

From Lemmas 4.14, 4.15 and 4.16 we conclude that $\alpha_F(Q_8) \leq 146$ and, therefore, we have the principal result of this section.

Theorem 4.17. $\chi_\rho(Q_8) = 95$.

Finally, we obtain an F-packing of Q_8 with size 146, which proves that $\alpha_F(Q_8) \geq 146$.
To this end, consider $X = \cup_{i \in F} X_i$, where $X_1 = V_e(Q_8)$,
$X_2 = \{(0,0,0,0,0,0,0,1), (0,0,0,0,1,1,1,0), (0,0,1,1,0,0,1,0),$
$(0,0,1,1,1,1,0,1), (0,1,0,1,0,1,0,0), (0,1,0,1,1,0,1,1),$
$(0,1,1,0,0,1,1,1), (0,1,1,0,1,0,0,0), (1,0,0,1,0,1,1,1),$
$(1,0,0,1,1,0,0,0), (1,0,1,0,0,1,0,0), (1,0,1,0,1,0,1,1),$

$(1,1,0,0,1,1,0,1)$, $(1,1,0,0,0,0,1,0)$, $(1,1,1,1,0,0,0,1)$,
$(1,1,1,1,1,1,1,0)\}$ and
$X_4 = \{(0,0,0,0,0,1,1,1), (0,0,1,1,1,0,0,0)\}$.

It is not hard to see that $\{X_i\}_{i\in F}$ is a family of pairwise disjoint i-packing of Q_8. Thus, $X = \bigcup_{i\in F} X_i$ is an F-packing of Q_8 and $|X| = 146$. Therefore, $\alpha_F(Q_8) = 146$.

Remark 4.18. *$\alpha_F(Q_8) = 146$.*

5 Better bounds for $\chi_\rho(Q_n)$ for $n = 9, 10, 11$.

Our goal in this section is to improve the known previous lower bounds for the packing chromatic number of Q_9, Q_{10} and Q_{11} (Table 2).

In order to do that we use the following result on i-packing numbers of hypercubes.

Lemma 5.1. *Let $n \geq 1$. Then $\alpha_F(Q_{n+1}) \leq 2\alpha_F(Q_n)$ for every $F \subseteq [2^n]$.*

Proof. Observe that if X is an F-packing of Q_{n+1}, then $X \cap V^0(Q_{n+1})$ is an F-packing of the subgraph Q_n^0. Therefore, $|X \cap V^0(Q_{n+1})| \leq \alpha_F(Q_n)$.

Analogously, $|X \cap V^1(Q_{n+1})| \leq \alpha_F(Q_n)$. Then, $|X| = |X \cap V^0(Q_{n+1})| + |X \cap V^1(Q_{n+1})| \leq 2\alpha_F(Q_n)$. Thus, $\alpha_F(Q_{n+1}) \leq 2\alpha_F(Q_n)$. □

Now, we obtain the better lower bounds directly from Remark 4.18.

Theorem 5.2. *$\chi_\rho(Q_9) \geq 198$, $\chi_\rho(Q_{10}) \geq 395$ and $\chi_\rho(Q_{11}) \geq 794$.*

Proof. From Lemma 5.1 and recalling that $\alpha_{\{1,2,4\}}(Q_8) = 146$, we obtain that $\alpha_{\{1,2,4\}}(Q_9) \leq 292$, $\alpha_{\{1,2,4\}}(Q_{10}) \leq 584$ and $\alpha_{\{1,2,4\}}(Q_{11}) \leq 1168$.

Besides, from the values in Table 1, we have

$$\alpha_3(Q_9) + \sum_{i=5}^{8} \alpha_i(Q_9) = 30,\ \alpha_3(Q_{10}) + \sum_{i=5}^{9} \alpha_i(Q_{10}) = 54 \text{ and}$$

$$\alpha_3(Q_{11}) + \sum_{i=5}^{10} \alpha_i(Q_{11}) = 96.$$

Notice that we can apply Lemma 2.4, since $\alpha_{\{1,2,4\}}(Q_n) + \alpha_3(Q_n) + \sum_{i=5}^{n-1} \alpha_i(Q_n) < 2^n$, for $n = 9, 10, 11$.

Hence, $\chi_\rho(Q_9) \geq 198$, $\chi_\rho(Q_{10}) \geq 395$ and $\chi_\rho(Q_{11}) \geq 794$. □

6 Discussion

In this section we present a non-closed formula for an upper bound of the packing chromatic number of hypercubes that slightly improves the bounds in Theorem 3.1, for $n \geq 9$.

To do this, we will construct i-packings of the hypercubes Q_n, for $n \geq 9$. Recall the sets $W_{n'}$ defined in Section 3. For each $v_{m,n'} \in W_{n'}$, consider the vector obtained by replacing each component of $v_{m,n'}$ by a block of i components (see Figure 3).

$$\begin{aligned} v_{m,n'} &= (\underbrace{\ldots,0,\ldots}\qquad \ldots,1,\ldots) \\ {}_iv_{m,n} &= (\ldots,\underbrace{0,\ldots,0}_{i},\ldots,\underbrace{\overbrace{1,\ldots,1}}_{i},\ldots) \end{aligned}$$

Figure 3: Vectors $v_{m,n'}$ and ${}_iv_{m,n}$.

Formally, let $i \geq 2$, $n \geq 4i$ and $n' = \left\lfloor \frac{n}{i} \right\rfloor$. For $n \in \{in', \ldots, i(n'+1)-1\}$, we define $W_n^i = \{{}_iv_{m,n} \in \mathbb{Z}_2^n : v_{m,n'} \in W_{n'}\}$, where

$${}_iv_{m,n}^j = \begin{cases} v_{m,n'}^k & \text{if } j = i(k-1)+1, \ldots, ik; \\ 0 & \text{otherwise} \end{cases}$$

For example, let $i = 2$, $n = 15$ and $n' = 7$. We know that $W_7 = \{v_{4,7}, v_{6,7}, v_{7,7}\}$, where

$v_{4,7} = (1,1,1,1,0,0,0)$,
$v_{6,7} = (1,1,0,0,1,1,0)$,
$v_{7,7} = (1,0,1,0,1,0,1)$.

Then, $W_{15}^2 = \{{}_2v_{4,15}\ ,\ {}_2v_{6,15}\ ,\ {}_2v_{7,15}\}$, where

${}_2v_{4,15} = (1,1,1,1,1,1,1,1,0,0,0,0,0,0,0)$,
${}_2v_{6,15} = (1,1,1,1,0,0,0,0,1,1,1,1,0,0,0)$,
${}_2v_{7,15} = (1,1,0,0,1,1,0,0,1,1,0,0,1,1,0)$.

Next, we compute the size of W_n^i. First, notice that from previous definition, if $n+1$ is not a multiple of i then $|W_{n+1}^i| = |W_n^i|$. Otherwise, if $n+1$ is a multiple of i, then $|W_{n+1}^i| = |W_{\frac{n+1}{i}}|$ and $|W_n^i| = |W_{\frac{n+1}{i}-1}|$. So, from Observation 3.3, $|W_{n+1}^i| = |W_n^i| \Leftrightarrow \frac{n+1}{i} - 1$ is a power of 2. Then, it is easy to see the following property.

$$|W_{n+1}^i| = |W_n^i| \Leftrightarrow n+1 \text{ is not a multiple of } i \text{ or } \frac{n+1}{i} - 1 \text{ is a power of 2.}$$

Using this fact, it is not hard to show the following remark by induction on n in a similar way to the proof of Lemma 3.4.

Remark 6.1. *Let $i \geq 2$ and $n \geq 4i$. Then, $|W_n^i| = \left\lfloor \frac{n}{i} \right\rfloor - 1 - \left\lceil \log_2 \left\lfloor \frac{n}{i} \right\rfloor \right\rceil$.*

For $i \geq 2$ and $n \geq 4i$, we denote by $\mathcal{G}_n^i$ the subgroup of $\mathbb{Z}_2^n$ generated by W_n^i. From Lemma 3.10, it can be obtained a lower bound for the distance between two vectors in $\mathcal{G}_n^i$.

Lemma 6.2. *Let $i \geq 2$ and $n \geq 4i$. Then, $dist(v,w) \geq 4i$ for all $v, w \in \mathcal{G}_n^i$, $v \neq w$.*

Observe that this result allows us to define i-packings in the hypercubes. Firstly, we see a property concerning the sets $\mathcal{G}_n^i$ and $\mathcal{G}_n$.

We will show that $\mathcal{G}_n^i$ is a subgroup of $\mathcal{G}_n$ when i is an even positive integer. This fact plays an important role in the proof of the main theorem in this section. To this end, we show that every vector in W_n^i belongs to $\mathcal{G}_n$.

Let i be an even positive integer, $n \geq 4i$ and $n' = \left\lfloor \frac{n}{i} \right\rfloor$. Consider ${}_iv_{m,n} \in W_n^i$ the vector obtained from $v_{m,n'} \in W_{n'}$. Then, ${}_iv_{m,n}$ has four blocks of i consecutive components equal to 1 starting in positions 1, $q+1$, $s+1$ and $r+1$, with q, s, r even numbers such that $q \geq i$, $s \geq q+i$ and $r \geq s+i$. Then,

$$
{}_i v^j_{m,n} = \begin{cases} 1 & \text{if } j = 1, \ldots, i, q+1, \ldots, q+i, s+1, \ldots, s+i, r+1, \ldots, r+i \\ 0 & \text{otherwise} \end{cases}
$$

Consider the sets $V_1 = \{v_{m',n} \in W_n : 4 \le m' \le i \ \wedge m' \text{ even}\}$, $V_2 = \{v_{m',n} \in W_n : q+2 \le m' \le q+i \ \wedge \ m' \text{ even}\}$, $V_3 = \{v_{m',n} \in W_n : s+2 \le m' \le s+i \ \wedge m' \text{ even}\}$ and $V_4 = \{v_{m',n} \in W_n : r+2 \le m' \le r+i \ \wedge m' \text{ even}\}$.

Let $v_k = \sum_{v_{m',n} \in V_k} v_{m',n}$ for $k = 1, \ldots, 4$ and $v = v_1 + v_2 + v_3 + v_4$. It is trivial that $v \in \mathcal{G}_n$. We will prove that $v = {}_i v_{m,n}$.

First, observe that $|V_1| = \frac{i}{2} - 1$, $|V_k| = \frac{i}{2}$ for $k = 2, 3, 4$, and, if $U = \bigcup_{k=1}^{4} V_k$, $|U| = 2i - 1$.

Furthermore, notice that $v^1_{m',n} = v^2_{m',n} = 1$ for all $v_{m',n} \in U$. Then, $v^1 = v^2 = 1$ since $|U|$ is odd.

Now, let $j \ge 3$. From definition, we have that $v^j_1 = 1$ if and only if $j \in \{3, \ldots, i\}$. Analogously, $v^j_2 = 1$ if and only if $j \in \{q+1, \ldots, q+i\}$, $v^j_3 = 1$ if and only if $j \in \{s+1, \ldots, s+i\}$ and $v^j_4 = 1$ if and only if $j \in \{r+1, \ldots, r+i\}$. Therefore, $v = {}_i v_{m,n}$.

Lemma 6.3. *If i is an even positive integer, $\mathcal{G}^i_n$ is a subgroup of $\mathcal{G}_n$.*

Our goal now is to obtain k-packings of Q_n. To do this, we will construct cosets of $\mathbb{Z}_2^n$ by using previous subgroups.

Let $n \ge 9$. For $k \in \{2, \ldots, n\}$, let $g_k \in \mathbb{Z}_2^n$ be the vector having only the $(k-1)$-th component equal to 1 and the remaining components equal to 0. Consider the cosets $I^k_n \subset \mathbb{Z}_2^n$ such that I^1_n, I^2_n and I^3_n are defined as in the proof of Theorem 3.1, i.e. $I^1_n = V_e(Q_n)$, $I^2_n = g_2 + \mathcal{G}_n$ and $I^3_n = g_3 + \mathcal{G}_n$. Furthermore, let $I^k_n = g_k + \mathcal{G}^2_n$ for $k = 4, 5, 6, 7$ and if $n \ge 16$, for $k \ge 8$ let I^k_n defined as follows:

- Consider $t = \left\lfloor \frac{n}{8} \right\rfloor$. For each $j \in \{2, \ldots, t\}$, let $I^k_n = g_k + \mathcal{G}^{2j}_n$ for $k = 8(j-1), \ldots, 8j-1$.

As we have seen, I^k_n is a k-packing of Q_n for $k = 1, 2, 3$. Besides, from Lemma 6.2, we have that $\mathcal{G}^{2j}_n$ is an $(8j-1)$-packing of the n-cube for all $j \in [t]$. Then, from the fact that every i-packing is a j-packing for all $j \le i$, we conclude that I^k_n is a k-packing of Q_n, for $k = 1, \ldots, 8t-1$. Furthermore, from Lemma 6.3, we can see that $I^k_n \subset V_o(Q_n)$ for all $k \ge 2$. Moreover, these sets are pairwise disjoint in Q_n.

Theorem 6.4. *Let $n \ge 9$ and $t = \left\lfloor \frac{n}{8} \right\rfloor$. Then, $\{I^k_n\}_{k=1}^{8t-1}$ is a family of pairwise disjoint k-packings of Q_n.*

Proof. We know that I^k_n is a k-packings of Q_n for $k = 1, \ldots, 8t-1$.

Then, we only need to prove that $\{I^k_n\}_{k=1}^{8t-1}$ is a family of pairwise disjoint subsets of $V(Q_n)$. First, observe that $I^1_n \cap I^k_n = \emptyset$, since $wt(v)$ is odd for all $v \in \bigcup_{k=2}^{8t-1} I^k_n$. Therefore, it is enough to show that $\{I^k_n\}_{k=2}^{8t-1}$ is a family of pairwise disjoint subsets of $V(Q_n)$.

Suppose that there are two different subsets $I^{k_1}_n$ and $I^{k_2}_n$ such that $I^{k_1}_n \cap I^{k_2}_n \ne \emptyset$. Let $v \in I^{k_1}_n \cap I^{k_2}_n$. From definition, $v + g_{k_1} \in \mathcal{G}^{2i}_n$ and $v + g_{k_2} \in \mathcal{G}^{2j}_n$ for some $i, j = 1, \ldots, t$ (consider $\mathcal{G}^1_n = \mathcal{G}_n$). Observe that $dist(v + g_{k_1}, v + g_{k_2}) = \sum_{i=1}^{n} (v + g_{k_1} + v + g_{k_2})^i = \sum_{i=1}^{n} (g_{k_1} + g_{k_2})^i = 2$.

On the other hand, from Lemma 6.3, $v + g_{k_1} \in \mathcal{G}^1_n$ and $v + g_{k_2} \in \mathcal{G}^1_n$. Then, from Lemma 3.10, $dist(v + g_{k_1}, v + g_{k_2}) \ge 4$, wich contradicts the previous fact.

Therefore $\{I^k_n\}_{k=1}^{8t-1}$ is a family of pairwise disjoint k-packings of Q_n. □

Finally, observe that $\bigcup_{k=1}^{8t-1} I_n^k$ is an $[8t-1]$-packing of the n-cube. Then, if we assign a different color greater than $8t-1$ to each vertex in $V(Q_n) \setminus \bigcup_{k=1}^{8t-1} I_n^k$, we obtain a packing h-coloring of Q_n with $h = 8t - 1 + 2^n - \sum_{i=1}^{8t-1} |I_n^i|$. Notice that $|I_n^i|$ can be obtained from Lemma 3.4 and Remark 6.1. Then, we have the following upper bound.

Corollary 6.5. *Let $n \geq 9$ and $t = \left\lfloor \frac{n}{8} \right\rfloor$. Then:*
If $n \leq 15$,

$$\chi_\rho(Q_n) \leq 7 + 2^n \left(\frac{1}{2} - 2^{-\lceil \log_2 n \rceil} - 2^{1 - \lceil \frac{n}{2} \rceil - \lceil \log_2 \lfloor \frac{n}{2} \rfloor \rceil} \right).$$

If $n \geq 16$,

$$\chi_\rho(Q_n) \leq 8t - 1 + 2^n \left(\frac{1}{2} - 2^{-\lceil \log_2 n \rceil} - 2^{1 - \lceil \frac{n}{2} \rceil - \lceil \log_2 \lfloor \frac{n}{2} \rfloor \rceil} - \sum_{j=2}^{t} 2^{2 - \lceil \frac{n}{2j} \rceil - \lceil \log_2 \lfloor \frac{n}{2j} \rfloor \rceil} \right).$$

7 Conclusion

Observe that the packing chromatic number of Q_n reaches the upper bounds given by Goddard et al. [9] for $n = 6, 7, 8$. This behavior leads to assess whether the same is true for $n \geq 9$. In that sense, we prove in the following that this statement is false for $n = 9, 10$ by giving a packing 211-coloring of Q_9 and a packing 421-coloring of Q_{10}. Finally, we show previous results in Table 3 and we summarize our results in Table 4.

n	6	7	8	9	10	11
$\chi_\rho(Q_n) =$						
$\chi_\rho(Q_n) \geq$	15	28	63	132	285	610
$\chi_\rho(Q_n) \leq$	25	49	95	219	441	881
Gap	10	21	32	87	156	271

Table 3: Previous results

n	6	7	8	9	10	11
$\chi_\rho(Q_n) =$	25	49	95			
$\chi_\rho(Q_n) \geq$	-	-	-	198	395	794
$\chi_\rho(Q_n) \leq$	-	-	-	211	421	881
Gap	0	0	0	13	26	87

Table 4: New bounds

A packing 211-coloring of Q_9

For $v \in V(Q_n)$, let $\hat{v}^1 = 1 - v^1$, $\hat{v}^2 = 1 - v^2$, and $\hat{v}^i = v^i$ for all $3 \leq i \leq n$.

Consider the following packing coloring of Q_9.

$X_1 = V_e(Q_9)$,

$X_2 = \{(1,1,1,0,0,0,0,0,0),\ (1,1,0,0,1,1,1,0,0),\ (1,1,1,1,1,1,0,1,0),$
$(1,1,0,1,1,0,0,0,1),\ (1,1,1,0,0,1,1,1,1),\ (0,0,1,1,1,0,0,0,0),$
$(0,0,0,0,0,1,0,0,0),\ (0,0,1,0,1,1,1,1,0),\ (0,0,0,1,1,1,1,0,1),$
$(0,0,1,0,0,0,0,1,1),\ (0,1,0,1,0,0,1,0,0),\ (0,1,1,1,0,1,0,0,1),$
$(0,1,1,0,1,0,1,0,1),\ (0,1,0,0,1,1,0,1,1),\ (1,0,1,1,0,1,1,0,0),$
$(1,0,0,0,1,0,0,1,0),\ (1,0,1,0,1,1,0,0,1),\ (1,0,0,0,0,0,1,0,1),$
$(1,0,0,1,0,1,0,1,1),\ (1,0,1,1,1,0,1,1,1)\},$
$X_3 = \{\hat{v} : v \in X_2\},$
$X_4 = \{(1,1,0,0,0,0,1,1,1),\ (1,1,0,1,1,1,0,0,0),$
$(0,0,1,0,0,0,0,0,0),\ (0,0,1,1,1,1,1,1,1)\},$
$X_5 = \{\hat{v} : v \in X_4\},$
$X_6 = \{(1,1,1,0,1,1,1,1,0),\ (0,0,0,1,0,0,0,0,0)\},$
$X_7 = \{\hat{v} : v \in X_6\}.$

Observe that the family $\{X_j\}_{j=1}^{7}$ is pairwise disjoint and $\left|\bigcup_{j=1}^{i-1} X_j\right| = 308$. Then, for $i = 8, \ldots, 211$ we assign a vertex in $V(Q_9) \setminus \bigcup_{j=1}^{i-1} X_j$ to each X_i. Therefore $\{X_j\}_{j=1}^{211}$ is a packing 211-coloring of Q_9.

A packing 421-coloring of Q_{10}

Similarly, the following i-packings X_i in Q_{10} are pairwise disjoint,
$X_1 = V_e(Q_{10}),$
$X_2 = \{(0,1,1,1,0,0,0,0,0,0),\ (0,1,1,1,1,1,0,1,1,0),$
$(0,1,1,0,0,0,1,1,1,0),\ (0,1,1,0,0,1,0,1,0,1),\ (0,1,1,1,1,0,1,1,0,1),$
$(0,1,1,0,1,1,1,0,1,1),\ (0,0,0,1,0,0,1,1,0,0),\ (0,0,0,1,1,1,1,0,1,0),$
$(0,0,0,0,0,1,0,1,1,0),\ (0,0,0,0,0,0,0,0,0,1),\ (0,0,0,1,1,1,0,1,0,1),$
$(0,0,0,0,1,0,1,1,1,1),\ (0,0,1,0,1,1,1,1,0,0),\ (0,0,1,0,1,0,0,0,1,0),$
$(0,0,1,1,0,1,1,0,0,1),\ (0,0,1,1,0,0,0,1,1,1),\ (0,1,0,0,0,1,1,0,0,0),$
$(0,1,0,0,1,0,0,1,0,0),\ (0,1,0,1,1,0,0,0,1,1),\ (0,1,0,1,0,1,1,1,1,1),$
$(1,0,1,0,0,0,0,1,0,0),\ (1,0,1,0,0,1,1,0,1,0),\ (1,0,1,1,1,0,0,0,0,1),$
$(1,0,1,1,1,1,1,1,1,1),\ (1,1,0,0,0,0,0,0,1,0),\ (1,1,0,0,1,1,1,1,1,0),$
$(1,1,0,1,1,1,1,0,0,1),\ (1,1,0,1,0,0,0,1,0,1),\ (1,1,1,0,1,1,0,0,0,0),$
$(1,1,1,1,0,1,1,1,0,0),\ (1,1,1,1,1,0,1,0,1,0),\ (1,1,1,0,0,0,1,0,0,1),$
$(1,1,1,1,0,1,0,0,1,1),\ (1,1,1,0,1,0,0,1,1,1),\ (1,0,0,1,0,1,0,0,0,0),$
$(1,0,0,0,1,0,1,0,0,0),\ (1,0,0,1,1,0,0,1,1,0),\ (1,0,0,0,0,1,1,1,0,1),$
$(1,0,0,0,1,1,0,0,1,1),\ (1,0,0,1,0,0,1,0,1,1)\},$
$X_3 = \{\hat{v} : v \in X_2\},$
$X_4 = \{(1,1,1,1,0,0,0,0,0,0),\ (1,0,0,1,1,1,0,1,,1,0),$
$(1,0,0,0,0,0,1,0,1,1),\ (1,0,1,0,1,0,1,1,0,0),\ (1,1,0,1,1,1,1,0,0,1),$
$(1,1,1,0,0,1,0,1,1,1)\},$
$X_5 = \{\hat{v} : v \in X_4\},$
$X_6 = \{(0,0,0,0,0,0,0,1,0,0),\ (1,1,1,1,1,1,1,0,1,1)\},$
$X_7 = \{\hat{v} : v \in X_6\},$
$X_8 = \{(0,0,0,0,0,1,0,0,0,0),\ (1,1,1,1,1,0,1,1,1,1)\},$

$X_9 = \{\hat{v} : v \in X_8\}$.

By applying analogous reasoning as before, we have a packing 421-coloring of Q_{10}.

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