Lazard’s elimination in presented Lie algebras

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Knizhnik-Zamolodchikov equation

- Assume that $k$ is a commutative ring with unit.
- For $n \geq 2$, we denoted by $\mathcal{T}_n = \{ t_{i,j} \}_{1 \leq i < j \leq n}$ the set of noncommutative variables.
- The Knizhnik-Zamolodchikov equation (see for instance Drinfeld [1], Minh [5])

$$(KZ_n) \quad dF(z) = \Omega_n(z) F(z) \quad (1)$$

defined over the complex configuration space

$$\mathbb{C}_n^* = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j \},$$

where the system (so called the KZ connection)

$$\Omega_n(z) = \sum_{1 \leq i < j \leq n} \frac{t_{i,j}}{2i\pi} d \log(z_i - z_j), \quad (2)$$

where the logarithmic function is relative to some section of $\widetilde{\mathbb{C}}_n^*$, for example $\mathbb{C} \setminus [ -\infty, 0 ]$.
As a consequence of Arnold's theorem, the system (2) is completely integrable i.e. \(d\Omega_n - \Omega_n \wedge \Omega_n = 0\), it is equivalent to the fact that \(T_n = \{t_{i,j}\}_{1 \leq i < j \leq n}\) satisfy the infinitesimal pure braid relations

\[
R[n] = \begin{cases}
R_1[n] & [t_{i,j}, t_{i,k} + t_{j,k}] \\
R_2[n] & [t_{i,j} + t_{i,k}, t_{j,k}] \\
R_3[n] & [t_{i,j}, t_{k,l}] \\
\end{cases}
\]

for \(1 \leq i < j < k \leq n\), for \(1 \leq i < j < k \leq n\), and \(|\{i,j,k,l\}| = 4\).

The *Drinfeld-Kohno Lie algebra* \(\text{DK}_{k,n}\) is presented as

\[
\mathcal{L}_k(T_n) / \mathcal{J}_{R[n]}
\]

where \(\mathcal{J}_{R[n]}\) is the Lie ideal of \(\mathcal{L}_k(T_n)\) generated by \(R[n]\) (3).
By using the Knizhnik-Zamolodchikov equations, Kohno proved in [2] that $DK_{k,n}$ can be identified with $\mathfrak{gr}_k(\mathcal{PB}_n)$ the graded Lie algebra of the pure braid group $\mathcal{PB}_n$. Thus, Drinfeld-Kohno Lie algebra $DK_{k,n}$ is also called the Lie algebra of infinitesimal braids.

By some steps, we can construct a commutative diagram of $k$-modules with split short exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathfrak{gr}_k(F_n) & \longrightarrow & \mathfrak{gr}_k(\mathcal{PB}_{n+1}) & \longrightarrow & \mathfrak{gr}_k(\mathcal{PB}_n) & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathcal{L}_k(x_1,\ldots,x_n) & \longrightarrow & DK_{k,n+1} & \longrightarrow & DK_{k,n} & \longrightarrow & 0
\end{array}
$$

In particular, we obtain an isomorphism of $k$-modules

$$DK_{k,n+1} \cong \mathcal{L}_k(x_1,\ldots,x_n) \oplus DK_{k,n}. \quad (5)$$
A natural question is how to construct a Lie isomorphism from the Drinfeld-Kohno Lie algebra to a semidirect product of Lie algebras

\[ \text{DK}_{k,n+1} \rightarrow \mathcal{L}_k(x_1, \ldots, x_n) \rtimes \text{DK}_{k,n}. \]

We call the phenomenon by "the decomposition of Drinfeld-Kohno Lie algebra".

In this talk, we will give a proof for the existence of the decomposition of Drinfeld-Kohno Lie algebra as a corollary of our main theorem and Proposition 2.
Let us recall briefly Lazard’s elimination theorem in our setting.

**Lazard elimination theorem**

Let $X = B \sqcup Z$ be a set partitioned in two blocks. We have an isomorphism of split short exact sequences

\[
0 \to \mathcal{L}_k(B^*Z) \xrightarrow{j_{B|Z}} \mathcal{L}_k(X) \xrightarrow{p_{B|Z}} \mathcal{L}_k(B) \to 0
\]

with

- $u = b_1 \ldots b_k \in B^*$ and $z \in Z$ for

\[
rn(uz) = \left( \text{ad}_{b_1}^{\mathcal{L}_k(X)} \circ \ldots \circ \text{ad}_{b_k}^{\mathcal{L}_k(X)} \right)(z) =: \text{ad}_{(u)}^{\mathcal{L}_k(X)}(z)
\]
Lazard elimination theorem

- bracketing and $\overline{rn}$ is the restriction of $rn$ to its image as in the diagram.
- if $j_B : \mathcal{L}_k(B) \to \mathcal{L}_k(X)$ is the subalphabet embedding, (so that the restriction to its image is the isomorphism $\overline{j_B}$) then $\overline{j_B} \circ p_B|_Z$ is the projector on

$$\mathcal{L}_k(X)_B = \bigoplus_{\alpha \in \mathcal{N}(X)} \mathcal{L}_k(X)_\alpha$$

The kernel of $p_B|_Z$ is

$$\mathcal{L}_k(X)_{BZ} = \bigoplus_{\alpha \in \mathcal{N}(X)} \mathcal{L}_k(X)_\alpha$$

- The above diagram is a split SES, its section is given by $j_B$. 

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Main results: Quotients of Lazard’s eliminations

- **Observation and ideas**: Put $\mathcal{T}_{n+1} = \mathcal{T}_n \sqcup \mathcal{T}_{n+1}$ a set partitioned in two blocks, then the infinitesimal pure braid relator $R[n+1] \subset L_k(\mathcal{T}_{n+1})$ is compatible with the alphabet partition (see Example 1). Thus we deal with a special kind of relators i.e. relators being compatible with an elimination scheme.

- In general, let $X = B \sqcup Z$ be a set partitioned in two blocks. We suppose given a relator $r = \{r_j\}_{j \in J} \subset L_k(X)$ which is compatible with the alphabet partition i.e. there exists a partition of the set of indices $J = J_Z \sqcup J_B$ such that $r_B = \{r_j\}_{j \in J_B} = r \cap L_k(X)_B$ and $r_Z = \{r_j\}_{j \in J_Z} = r \cap L_k(X)_{BZ}$. The notations being as above, we construct the following ideals:
  1. $\mathcal{I}_B$ is the Lie ideal of $L_k(X)_B$ generated by $\{r_j\}_{j \in J_B}$
  2. $\mathcal{I}, \mathcal{I}_Z$ and $\mathcal{I}_{BZ}$ are the Lie ideals of $L_k(X)$ generated respectively by $r, r_Z$ and $r_{BZ} := \{\text{ad}_Q z\}_{Q \in \mathcal{I}_B, z \in Z}$. 
Example 1.

A typical example is for the partitioned $X := \mathcal{T}_{n+1} = \mathcal{T}_n \sqcup \mathcal{T}_{n+1} := B \sqcup Z$ and the infinitesimal pure braid relator $r := R[n+1] \subset \mathcal{L}_k(\mathcal{T}_{n+1})$. In this case, we observe that the relator $r_{\mathcal{T}_n} = R[n+1] \cap \mathcal{L}_k(\mathcal{T}_{n+1}) \mathcal{T}_n = R[n]$ and the relator $r_{\mathcal{T}_{n+1}} = R[n+1] \cap \mathcal{L}_k(\mathcal{T}_{n+1}) \mathcal{T}_n \mathcal{T}_{n+1} =$

\[
\begin{cases}
R_1^{-}[n+1] & [t_i,j, t_i,n+1 + t_j,n+1] & \text{for } 1 \leq i < j \leq n, \\
R_2^{-}[n+1] & [t_i,j + t_i,n+1, t_j,n+1] & \text{for } 1 \leq i < j \leq n, \\
R_3^{-}[n+1] & \pm[t_i,j, t_k,n+1] & \text{for } 1 \leq i < j \leq n, 1 \leq k \leq n, \text{ and } |\{i,j,k\}| = 3.
\end{cases}
\]

Then we can construct the following Lie ideals

- $\mathcal{J}_{\mathcal{T}_n} = \mathcal{J}_{R[n]}$ is the Lie ideal of $\mathcal{L}_k(\mathcal{T}_n)$ generated by the infinitesimal pure braid relator $r_{\mathcal{T}_n} = R[n]$.

- $\mathcal{J}_{\mathcal{T}_{n+1}}$ (resp. $\mathcal{J}_{\mathcal{T}_n \mathcal{T}_{n+1}}$) is the Lie ideal of $\mathcal{L}_k(\mathcal{T}_{n+1})$ generated by the relator $r_{\mathcal{T}_{n+1}}$ (resp. $r_{\mathcal{T}_n \mathcal{T}_{n+1}} = \{\text{ad}Q \ z\} Q \in \mathcal{J}_{R[n]}, z \in \mathcal{T}_{n+1}$).

- $\mathcal{J} = \mathcal{J}_{R[n+1]}$ is the Lie ideal of $\mathcal{L}_k(\mathcal{T}_{n+1})$ generated by $R[n+1]$. 
Main Theorem.

With our constructions above, we get the following properties:

i) we have \(( \mathcal{J}_Z + \mathcal{J}_{BZ} ) \subset \mathcal{L}_k(X)_{BZ}\) (and then \(( \mathcal{J}_Z + \mathcal{J}_{BZ} ) \cap \mathcal{J}_B = \{0\}\)). Moreover, \(( \mathcal{J}_Z + \mathcal{J}_{BZ} )\) is a Lie ideal of \(\mathcal{L}_k(X)_{BZ}\) (and even, by definition, of \(\mathcal{L}_k(X)\)).

ii) the action of \(\mathcal{L}_k(X)_B\) on \(\text{Der}(\mathcal{L}_k(X)_{BZ})\) (by internal \(\text{ad}\)) passes to quotients as an action \(\alpha : \mathcal{L}_k(X)_B \to \text{Der}(\mathcal{L}_k(X)_{BZ} / (\mathcal{J}_Z + \mathcal{J}_{BZ}))\) such that \(r_B \subset \ker(\alpha)\) and then, we get an action

\[
\overline{\alpha} : \mathcal{L}_k(X)_B / \mathcal{J}_B \to \text{Der}(\mathcal{L}_k(X)_{BZ} / (\mathcal{J}_Z + \mathcal{J}_{BZ})) \tag{7}
\]

iii) we can construct an isomorphism from presented Lie algebra \(\mathcal{L}_k(X) / \mathcal{J}\) by the set \(r = \{r_j\}_{j \in J}\) of relators onto the semidirect product of Lie algebras \(\mathcal{L}_k(X)_{BZ} / (\mathcal{J}_Z + \mathcal{J}_{BZ}) \rtimes \mathcal{L}_k(X)_B / \mathcal{J}_B\).
Main Theorem.

iii) which will be denoted by

\[ \Phi : \mathcal{L}_k(X) / \mathcal{J} \cong \mathcal{L}_k(X)_{BZ} / (\mathcal{J}_Z + \mathcal{J}_{BZ}) \rtimes \mathcal{L}_k(X)_B / \mathcal{J}_B. \] (8)

iv) In fact, one has a commutative diagram of Lie algebras with split short exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_k(X)_{BZ} & \xrightarrow{j} & \mathcal{L}_k(X) & \xrightarrow{p} & \mathcal{L}_k(X)_B & \longrightarrow & 0 \\
& & \downarrow^{s\mathcal{J}_Z + \mathcal{J}_{BZ}} & & \downarrow^{s\mathcal{J}} & & \downarrow^{s\mathcal{J}_B} & \\
0 & \longrightarrow & \mathcal{L}_k(X)_{BZ} / (\mathcal{J}_Z + \mathcal{J}_{BZ}) & \longrightarrow & \mathcal{L}_k(X) / \mathcal{J} & \longrightarrow & \mathcal{L}_k(X)_B / \mathcal{J}_B & \longrightarrow & 0
\end{array}
\]
Elimination of the subalphabet $Z$

- In certain cases (which is that of the Lie algebras $DK_{k,n}$), it can happen that the left factor of the semidirect product (8) be isomorphic to $\mathcal{L}_k(Z)$. We start from the previous commutative diagram with an additional arrow

$$
\begin{array}{cccccc}
\mathcal{L}_k(Z) \\
j_Z \downarrow & & & & & \\
0 & \longrightarrow & \mathcal{L}_k(X)_{BZ} & \overset{j}{\longrightarrow} & \mathcal{L}_k(X) & \overset{p}{\longrightarrow} & \mathcal{L}_k(X)_B & \longrightarrow & 0 \\
\downarrow \scriptstyle{sJ_Z+J_{BZ}} & & & & & \downarrow \scriptstyle{sJ} & & \downarrow \scriptstyle{sJ_B} \\
0 & \longrightarrow & \mathcal{L}_k(X)_{BZ} \big/ (J_Z+J_{BZ}) & \longrightarrow & \mathcal{L}_k(X) \big/ J & \longrightarrow & \mathcal{L}_k(X)_B \big/ J_B & \longrightarrow & 0
\end{array}
$$

where $j_Z$ is the subalphabet embedding such that

$$
\text{Im}(j_Z) = \mathcal{L}_k(X)_Z = \bigoplus_{\alpha \in \mathbb{N}(X)} \mathcal{L}_k(X)_{\alpha}.
$$

(9)
We are now in the position to state the following

**Proposition 2.**

With the notations as in Main Theorem, let us consider the composite map

\[ \beta = s\mathcal{J}_Z + \mathcal{J}_{BZ} \circ j_Z, \]

then

a. In order that \( \beta \) be injective, it is necessary and sufficient that

\[ (\mathcal{J}_Z + \mathcal{J}_{BZ}) \cap \mathcal{L}_k(X)_Z = \{0\}. \]

b. In order that \( \beta \) be surjective, it is necessary and sufficient that, for all \((b, z) \in B \times Z\), we had

\[ s\mathcal{J}_Z + \mathcal{J}_{BZ}([b, z]) \in s\mathcal{J}_Z + \mathcal{J}_{BZ}(\mathcal{L}_k(X)_Z). \]

(10)
Applications

The existence of the decomposition of Drinfeld-Kohno Lie algebra

Recall in Example 1, we denoted by $\mathcal{T}_{n+1} = \mathcal{T}_n \sqcup \mathcal{T}_{n+1}$ and the infinitesimal pure braid relator $R[n+1] \subset \mathcal{L}_k(\mathcal{T}_{n+1})$. In this case, the existence of the decomposition of Drinfeld-Kohno Lie algebra can be obtained as a consequence of our main theorem and by Proposition 2.

Corollary 3.

There is the decomposition of Drinfeld-Kohno Lie algebra i.e. in the category $k$-$\text{Lie}$,

$$DK_{k,n+1} \cong \mathcal{L}_k(X_n) \rtimes DK_{k,n}$$  \hfill (11)

where $X_n$ is any alphabet of cardinality $n$. 
About M.-P. Schützenberger’s questions on the Partially Commutative Free Lie algebra

- Let \( X \in \text{Set} \) be a set viewed as an alphabet. A commutation relation on \( X \) is a reflexive and symmetric graph \( \theta \subset X^2 \) (i.e. \( \theta = \theta^{-1} \) and \( \{(x, x)\}_{x \in X} \), the diagonal of \( X \), is a subset of \( \theta \)).

- Firstly, the free partially commutative monoid \( M(X, \theta) \) is the quotient of \( X^* \) by the congruence generated by the family \( (xy = yx)_{(x, y) \in \theta} \).

- We will consider the canonical surjection \( s_\theta : X^* \to M(X, \theta) \) as well as \( j_\theta : M(X, \theta) \to X^* \) an arbitrary set-theoretical section of it.

- The terminal alphabet \( \text{TAIph}(t) \) (where \( t \in M(X, \theta) \)) can be characterized as the set of last letters of preimages of \( t \) w.r.t. \( s_\theta \), it means that \( \text{TAIph}(t) = \{x \in X \mid t \in M(X, \theta).x\} \).

- Secondly, the free partially commutative Lie algebra \( \mathcal{L}_k(X, \theta) \) is the quotient of \( \mathcal{L}_k(X) \) by the ideal generated by the relator \( r_\theta = \{[x, y]\}_{(x, y) \in \theta} \).
Theorem 4.

Let \((X, \theta)\) be an alphabet with commutations. We consider a partition of \(X\), \(X = B \sqcup Z\) such that \(Z\) is totally non-commutative i.e. no two letters of \(Z\) commute between themselves \((\theta \cap Z^2 = \Delta_Z)\) and the code

\[
C_B(Z) = \{ s_\theta(uz) | u \in B^*, z \in Z, T\text{Alph}(s_\theta(uz)) = \{ z \} \} \quad (12)
\]

Let \(C = j_\theta(C_B(Z))\) and \(j_C\) be the subalphabet embedding, we have the diagram
Theorem 4.

Then, with the above hypotheses ($Z$ totally non-commutative and $C = j_\theta(C_B(Z))$, $sJ_Z + J_{BZ} \circ j_C$ is an isomorphism. In particular, the left factor of the semi-direct product (8), here $\mathcal{L}_k(X)_{BZ} \left/ (J_Z + J_{BZ}) \right. \rightarrow \mathcal{L}_k(X) \left/ J \right. \rightarrow \mathcal{L}_k(X)_B \left/ J_B \right. \rightarrow 0$ is a free Lie algebra.
It would be interesting to have alternative proofs for answers to Schützenberger’s questions about the Partially Commutative Free Lie algebra (cf. Duchamp and Krob [4], Thm. III.3) as a consequence of our main theorem.

**Corollary 5. (Lazard’s Partially Commutative Elimination)**

Let $X$ be a set equipped with a commutation relation $\theta$ and $B$ be a subset of $X$ such that $Z = X - B$ is totally non-commutative. Then there is an isomorphism from the free partially commutative Lie algebra $\mathcal{L}_k(X, \theta)$ to the semidirect of product of Lie algebras, namely

$$\mathcal{L}_k(X, \theta) \simeq_{k-Lie} \mathcal{L}_k(C) \rtimes \mathcal{L}_k(B, \theta_B).$$

(13)
Suppose that a commutative ring \( k \) of characteristic zero (hence \( \mathbb{Q} \hookrightarrow k \)) and \( X = B \sqcup Z \) is a set partitioned in two blocks, where \( B = \{b_1, \ldots, b_n\} \) and \( Z = \{z_1, z_2, z_3, \ldots\} \). Let us consider the polynomial algebra

\[
k\langle X \rangle = k\langle b_1, \ldots, b_n, z_1, z_2, z_3, \ldots \rangle.
\]

The collection (called by Magnus polynomials (cf. Nakamura [3]))

\[
u. \text{ad}(w_1) z_{i_1} \ldots \text{ad}(w_k) z_{i_k},
\]

where \( k \geq 0, w_1, \ldots, w_k \in B^*, i_1, \ldots, i_k \geq 1 \) and \( u \in B^* \), are \( k \)-linear basis of \( k\langle X \rangle \).
We introduce the *half-shuffle* in the polynomial algebra $k\langle X \rangle$ as the linear extension of the binary product on words given by

$$(x_1 \ldots x_p)\frac{\shuffle}{2} (x_{p+1} \ldots x_n) = x_1 (x_2 \ldots x_p \shuffle x_{p+1} \ldots x_n),$$

$$1x^* \frac{\shuffle}{2} (x_{p+1} \ldots x_n) = 0,$$

$$(x_1 \ldots x_p)\frac{\shuffle}{2} 1x^* = x_1 \ldots x_p$$

and then elements arising by the half-shuffle of $ZB^*$:

$$z_{i_1} w_1 \frac{\shuffle}{2} (z_{i_2} w_2 \frac{\shuffle}{2} (\ldots \frac{\shuffle}{2} (z_{i_{k-1}} w_{k-1} \frac{\shuffle}{2} z_i w_k) \ldots), (15)$$

where $k \geq 0, w_1, \ldots, w_k \in B^*, i_1, \ldots, i_k \geq 1$ (if $k = 0$ then (15) will be denoted by $1x^*$). Henceforth we write simply $z_{i_1} w_1 \frac{\shuffle}{2} \ldots \frac{\shuffle}{2} z_i w_k$ instead of (15).
The purpose of the following theorem is to describe the dual of Magnus basis under the standard pairing $\langle \bullet | \bullet \rangle : k\langle X \rangle^\vee \otimes k\langle X \rangle = k\langle\langle X \rangle\rangle \otimes k\langle X \rangle \to k$

with respect to the monomials of $k\langle X \rangle$ (here for all $S \in k\langle\langle X \rangle\rangle$ and $P \in k\langle X \rangle$ then the pairing $\langle S|P \rangle = \sum_{w \in X^*} \langle S|w \rangle \langle P|w \rangle$).

**Theorem 6.**

The collections

$$\{ u.(-1)^{|w_1|} \text{ad}_{(w_1)} z_{i_1} \ldots (-1)^{|w_k|} \text{ad}_{(w_k)} z_{i_k} \}_{i_1,\ldots,i_k \geq 1, u \in B^*}^{k \geq 0, w_1,\ldots,w_k \in B^*}$$

and

$$\{ u \sqcup \left( z_{i_1} \tilde{w}_1 \frac{w_1}{2} \ldots \frac{w_k}{2} z_{i_k} \tilde{w}_k \right) \}_{i_1,\ldots,i_k \geq 1, u \in B^*}^{k \geq 0, w_1,\ldots,w_k \in B^*}$$

are dual bases of, respectively $k\langle X \rangle$ and $k\langle X \rangle^\vee$, where $\tilde{w} = b_{i_k} b_{i_{k-1}} \ldots b_{i_1}$ reverses the order of letters in the word $w = b_{i_1} b_{i_2} \ldots b_{i_k} \in B^*$. 
Describe the dual basis in a suitable algebraic framework.

Applying the dual basis to provide finally solutions for Knizhnik-Zamolodchikov equations given in (1) with asymptotic conditions by dévissage.
Some references


Thank you very much for your attention!