Barrier resilience problems and crossing numbers of graphs

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Outline

Two independent parts

- Barrier resilience
- Crossing numbers of graphs

In common:

- Geometry
- Simple to state, interesting for general audience
- I have worked on them, I have something to explain
Original barrier resilience problem

[Kumar, Lai, Arora 2005], [Bereg, Kirkpatrick 2009]
Original barrier resilience problem

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rectangle

\[ s \]

\[ t \]

rectangle
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Polynomial time

rectangle

Unknown!!

annulus

• s

• t
Barrier resilience - Rectangular domain in $P$

[Kumar, Lai, Arora 2005] via Menger’s theorem

\[ \text{OPT} \geq \text{max nb } \ell - r \text{ vertex-disjoint paths} = \text{min size } \ell - r \text{ vertex cut} \geq \text{OPT} \]
Barrier resilience - Rectangular domain in $P$

[Kumar, Lai, Arora 2005] via Menger’s theorem

$$\text{OPT} \geq \max \text{nb } \ell-r \text{ vertex-disjoint paths} = \min \text{size } \ell-r \text{ vertex cut} \geq \text{OPT}$$
Barrier resilience - Rectangular domain in P

[Kumar, Lai, Arora 2005] via Menger’s theorem

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Barrier resilience - Rectangular domain in $\mathbb{P}$

[Kumar, Lai, Arora 2005] via Menger’s theorem

\[
\text{OPT} \geq \max_{\ell-r} \text{vertex-disjoint paths} = \min \text{size}\ \ell-r \text{vertex cut} \geq \text{OPT}
\]

Nice application of Menger’s theorem
No Menger-like theorem for the annular case
Related problems

If you cannot solve a problem, perturb it:
Related problems

If you cannot solve a problem, perturb it:

- other shapes instead of disks
  - the rectangular case can be solved in polynomial time for any reasonable shape; same argument
  - the annular case is NP-hard for (unit) segments or crossing rectangles

- the annular case is FPT wrt OPT for any connected shape

[Alt, C., Giannopoulos, Knauer 2017]
[Korman, Löffler, Silveira, Strash 2018]
[Tseng and Kirkpatrick 2011]

[Eiben and Lokshtanov 2020]
Related problems

If you cannot solve a problem, perturb it:

- other shapes instead of disks
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  - the annular case is NP-hard for (unit) segments or crossing rectangles

- the annular case is FPT wrt OPT for any connected shape

- change the criteria
  - shrink the disks, instead of deleting them

[Alt, C., Giannopoulos, Knauer 2017]
[Korman, Löffler, Silveira, Strash 2018]
[Tseng and Kirkpatrick 2011]

[Eiben and Lokshtanov 2020]
Minimum shrinking problem

\[
\min \sum \text{shrinking}
\]

[C., Jain, Lubiw, Mondal 2018]
Minimum shrinking problem

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\min \sum \text{shrinking}
\]

[C., Jain, Lubiw, Mondal 2018]

rectangle

annulus
Minimum shrinking problem

\[
\min \sum \text{ shrinking}
\]

[C., Jain, Lubiw, Mondal 2018]
[C., Colin de Verdière 2020]

Unknown!!
NP-hard

rectangle
annulus
## State of the art

<table>
<thead>
<tr>
<th></th>
<th>rectangular domain</th>
<th>annular domain</th>
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<tr>
<td>barrier problem</td>
<td>polynomial, $\tilde{O}(n^{3/2})$</td>
<td>unknown complexity</td>
</tr>
<tr>
<td>total failure</td>
<td>Menger’s theorem</td>
<td>FPT</td>
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<td></td>
<td>max flow</td>
<td>PTAS in some cases</td>
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<tr>
<td>shrinking barrier</td>
<td>unknown complexity</td>
<td>(weakly!!) NP-hard</td>
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<tr>
<td>shrinking barrier</td>
<td>$(1 + \epsilon)$-approx</td>
<td>different radii</td>
</tr>
<tr>
<td></td>
<td>in $O(n^5/\epsilon^{2.5})$ time</td>
<td></td>
</tr>
</tbody>
</table>

unit disks vs. disks vs. pseudodisks

unknown complexity

FPT

PTAS in some cases

(weakly!!) NP-hard

different radii

Menger's theorem

max flow

barrier problem

total failure
Outline

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barrier
A short interlude

How to start a paper; two examples
A short interlude

How to start a paper; two examples

The Magical Number Seven, Plus or Minus Two…
George A. Miller (1956), Harvard University
Psychological Review, 63, 81-97

My problem is that I have been persecuted by an integer. For seven years this number has followed me around, has intruded in my most private data, and has assaulted me from the pages of our most public journals. This number assumes a variety of disguises, being sometimes a little larger and sometimes a little smaller than usual, but never changing so much as to be unrecognizable…
A short interlude

How to start a paper; two examples

On Hodge-Riemann Cohomology Classes
Julius Ross and Matei Toma
arXiv 2106.11285

Since the dawn of time, human beings have asked some fundamental questions: who are we? why are we here? is there life after death? Unable to answer any of these, in this paper we will consider cohomology classes on a compact projective manifold that have a property analogous to the Hard-Lefschetz Theorem and Hodge-Riemann bilinear relations.
Outline

Two independent parts

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Drawing of a graph

Drawing $D$ of a graph $G$ (in the plane)

- each vertex one point - injectively
- each edge one continuous, simple curve
- endpoints of edge $uv$ are points for $u$ and $v$
- the interior of an edge does not contain other vertices
- no common point in the interior of three edges

$\text{cr}(D)$: number of crossings in drawing $D$
$\text{cr}(G)$: minimum $\text{cr}(D)$ over all drawings $D$ of graph $G$
Crossing number 0

- \( \text{cr}(G) = 0 \) if and only if \( G \) planar
- Good understanding of planar graphs
- \( G \) planar \( \iff \) \( G \) does not contain a subdivision of \( K_5 \) or \( K_{3,3} \)
- Efficient algorithms to recognize planar graphs
Why crossing number

Purchase, Cohen, James 1995: "Increasing the number of arc crossings in a graph decreases the understability of the graph". Purchase, 1997: "reducing the number of edge crosses is by far the most important aesthetic".
Why crossing number

Purchase, Cohen, James 1995: "Increasing the number of arc crossings in a graph decreases the understandability of the graph". Purchase, 1997: "reducing the number of edge crosses is by far the most important aesthetic".

Huang, Eades, Hong, 2014: "The effect of crossing angles on human graph comprehension was validated."

Optimization, VLSI, Discrete Geometry
Main conjectures – Harary-Hill

The crossing number of complete graphs is

\[
\text{cr}(K_n) = \frac{1}{4} \prod_{k=0}^{n-3} \binom{n-k}{2} \approx \frac{n^4}{64} \approx \frac{|E(K_n)|}{8}
\]

True for \( n \leq 12 \)
Main conjectures – Zarankiewicz-Turán

The crossing number of complete bipartite graphs (here for balanced)

\[ cr(K_{n,n}) = \left\lfloor \frac{n}{2} \right\rfloor^2 \left\lfloor \frac{n-1}{2} \right\rfloor^2 \approx \frac{n^4}{16} \approx \left( \frac{|E(K_{n,n})|}{2} \right) / 8 \]

True for \( n \leq 8 \)
Planarity game

https://www.jasondavies.com/planarity/

Planarity

Can you untangle the graph? See if you can position the vertices so that no two lines cross.

Number of line crossings detected: 36.
Rectilinear drawings

Each edge drawn as a straight line segment
Rectilinear drawings

Each edge drawn as a straight line segment
\(\overline{cr}(G)\ldots\) rectilinear crossing number of \(G\)
Rectilinear drawings

Each edge drawn as a straight line segment
\( \overline{cr}(G) \) \ldots rectilinear crossing number of \( G \)

- \( G \) planar \( \iff \) \( cr(G) = 0 \iff \overline{cr}(G) = 0 \)

[Wagner 1936, Fáry 1948]
Rectilinear drawings

Each edge drawn as a straight line segment
\( \overline{cr}(G) \) \ldots rectilinear crossing number of \( G \)

- \( G \) planar \( \iff \) \( cr(G) = 0 \) \( \iff \) \( \overline{cr}(G) = 0 \)

- \( cr(G) = 1 \) \( \iff \) \( \overline{cr}(G) = 1 \)
- \( cr(G) = 2 \) \( \iff \) \( \overline{cr}(G) = 2 \)

Claims without proof that \( cr(G) = 3 \) \( \iff \) \( \overline{cr}(G) = 3 \)
For each \( k \geq 4 \) there is \( G \) with \( cr(G) = 4 \) and \( \overline{cr}(G) = k \)

[Wagner 1936, Fáry 1948]
[Bienstock, Dean 1993]
Near-planar graphs

Non-planar $H$ is near-planar if $H = G + xy$ for planar $G$

- weak relaxation of planarity
- near-planar $\subsetneq$ toroidal, apex
Near-planar – Riskin

- $G$ planar, 3-connected, and 3-regular
  - $cr(G + xy)$ attained by the following drawing:
    draw $G$ planarly (unique) and insert $xy$ minimizing crossings

[Riskin 1996]
Near-planar graphs are hard

Theorem
Computing $cr(G)$ for near-planar graphs is NP-hard.

[C., Mohar 2013]
Near-planar graphs are hard

Theorem
Computing \( cr(G) \) for near-planar graphs is NP-hard.

- adding one edge makes a big mess
- crossing number of toroidal graphs hard
- new reduction from SAT
  - previous reductions were from Linear Ordering
- new problem: red-blue anchored drawings

[C., Mohar 2013]
Red-blue anchored drawings?
Red-blue anchored drawings?
Crossing number of near-planar graphs

forcing

\(x_1 \lor x_2\)

\(x_2 \lor \neg x_3\)

\(\neg x_2 \lor \neg x_4\)

\(\neg x_1 \lor \neg x_3 \lor x_4\)
Crossing number of near-planar graphs

\[\neg x_1 \lor \neg x_3 \lor x_4 \]
\[\neg x_2 \lor \neg x_4 \]
\[x_2 \lor \neg x_3 \]
\[\neg x_1 \lor \neg x_3 \lor x_4 \]

\[\begin{array}{cccc}
T & F & T & F \\
+1 & +1 & +2 & +1 \\
+2 & +1 & +1 & +1 \\
-1 & -1 & -1 & -1 \\
\end{array} \]
Crossing number of near-planar graphs

\[
\begin{align*}
\text{forcing} & : x_1 \lor \neg x_3 \lor x_4 \\
& \lor \neg x_2 \lor \neg x_4 \\
& \lor x_2 \lor \neg x_3 \\
& \lor \neg x_1 \lor \neg x_3 \lor x_4
\end{align*}
\]
Crossing number of near-planar graphs
Crossing number of near-planar graphs

\( \neg x_1 \lor \neg x_3 \lor x_4 \)

forcing

\( \neg x_2 \lor \neg x_4 \)

\( x_2 \lor \neg x_3 \)

\( \neg x_2 \lor \neg x_4 \)

\( \neg x_1 \lor \neg x_3 \lor x_4 \)
Crossing number of near-planar graphs

Forcing

\[ \neg x_1 \lor \neg x_3 \lor x_4 \]
\[ \neg x_2 \lor \neg x_4 \]
\[ x_2 \lor \neg x_3 \]
\[ x_1 \lor x_2 \]
Crossing number of near-planar graphs

- adding one edge makes a big mess
- we need large degrees
  - three vertices of large degree suffice \cite{Hlineny2023}
- \(\lfloor \Delta / 2 \rfloor\)-approximation \cite{C., Mohar 2011}
  - number of edge-disjoint cycles separating \(x\) and \(y\)
  - number of vertex-disjoint cycles separating \(x\) and \(y\)
- Is it NP-hard for max degree 4?
- Research also on adding a vertex
- Similar proof for 1-planarity of near-planar graphs
Conclusions

barrier

crossing number

near-planar
Conclusions

THANKS for your time!!

barrier

crossing number

e

near-planar