Flip Graphs and Matroids

Jean Cardinal, ULB
Outline

Flip Graphs

Problems

Matroids

Polymatroids

Hypergraphic polytopes

Graph associahedra

References
Table of Contents

Flip Graphs
Problems
Matroids
Polymatroids
Hypergraphic polytopes
Graph associahedra
References
Flip Graphs

Graph on a set of combinatorial objects, such that two adjacent objects differ by a single, reversible, exchange operation between elements composing the structure.
Flip Graphs

Graph on a set of combinatorial objects, such that two adjacent objects differ by a single, reversible, exchange operation between elements composing the structure.
Spanning trees
Permutations

(T. Piesk, Creative Commons)
Acyclic orientations

(D. Eppstein, Wikimedia commons)
Triangulations

(Fomin, Zelevinsky)
Perfect matchings
Table of Contents

Flip Graphs

Problems

Matroids

Polymatroids

Hypergraphic polytopes

Graph associahedra

References
Polytopal flip graphs

Many flip graphs are skeletons of polytopes:
Polytopal flip graphs

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- **Spanning trees**  Spannning tree polytopes  
  Edmonds 1971
Polytopal flip graphs

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**Spanning trees**  Spanning tree polytopes  
**Permutations**  Permutohedra  

Edmonds 1971

Schoute 1911, Guilbaud-Rosenstiehl 1963
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- **Permutations**  Permutohedra  Schoute 1911, Guilbaud-Rosenstiehl 1963
- **Acyclic orientations**  Graphical zonotopes  Greene 1977, Greene-Zaslavsky 1983

Polytopal flip graphs
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**Triangulations** Associahedra  Tamari 1951, Stasheff 1963, Loday 2004
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Many flip graphs are skeletons of polytopes:

- **Spanning trees** Spanning tree polytopes
  - Edmonds 1971
- **Permutations** Permutohedra
  - Schoute 1911, Guilbaud-Rosenstiehl 1963
- **Acyclic orientations** Graphical zonotopes
  - Greene 1977, Greene-Zaslavsky 1983
- **Triangulations** Associahedra
  - Tamari 1951, Stasheff 1963, Loday 2004
- **Perfect matchings** Perfect matching polytope
  - Chvátal 1972
Polymatroidal flip graphs

Flip graphs are skeletons of (poly)matroid polytopes:
Polymatroidal flip graphs

Flip graphs are skeletons of (poly)matroid polytopes:

| Spanning tree polytopes | Matroids |
Flip graphs are skeletons of (poly)matroid polytopes:

<table>
<thead>
<tr>
<th>Spanning tree polytopes</th>
<th>Matroids</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permutohedra</td>
<td>Polymatroids</td>
</tr>
<tr>
<td>Associahedra</td>
<td></td>
</tr>
<tr>
<td>Graphical zonotopes</td>
<td></td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
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<th>Matroids</th>
</tr>
</thead>
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<td>Permutohedra</td>
<td>Polymatroids</td>
</tr>
<tr>
<td>Associahedra</td>
<td></td>
</tr>
<tr>
<td>Graphical zonotopes</td>
<td></td>
</tr>
<tr>
<td>Perfect matching polytope</td>
<td>Matroid intersections</td>
</tr>
</tbody>
</table>
Flip distances

Given two vertices of the polytope, can we efficiently compute the shortest path between them, on the skeleton of the polytope?
Flip distances

Given two vertices of the polytope, can we efficiently compute the shortest path between them, on the skeleton of the polytope?

Given two combinatorial objects of the same size, can we efficiently compute the flip distance between them?
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Geodesics vs. Combinatorial reconfiguration formulation
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Geodesics vs. Combinatorial reconfiguration formulation

https://reconf.wikidot.com/
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Given two combinatorial objects of the same size, can we efficiently compute the flip distance between them?

Geodesics vs. Combinatorial reconfiguration formulation
https://reconf.wikidot.com/

What is the complexity of computing the rotation distance between two binary trees?
Diameter

What is the diameter of the polytope?
Diameter

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What is the largest flip distance between any two combinatorial objects of some size?
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Two questions:

**Combinatorial** What are the best upper and lower bounds?

**Computational** Can we compute the diameter efficiently?
Diameter

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**Hirsch conjecture**: The diameter of dimension $n$ polytopes with $f$ faces is at most $f - n$.

Santos 2012
Diameter

What is the diameter of the polytope?

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Two questions:

**Combinatorial** What are the best upper and lower bounds?

**Computational** Can we compute the diameter efficiently?

**Hirsch conjecture**: The diameter of dimension $n$ polytopes with $f$ faces is at most $f/n$.

Santos 2012

**Polynomial Hirsch conjecture**: The diameter of dimension $n$ polytopes with $f$ faces is at most some polynomial in $n$ and $f$. 
Hamiltonicity

Is the skeleton of the polytope Hamiltonian?

Hamilton 1856
Hamiltonicity

Is the skeleton of the polytope Hamiltonian?  

Is there a **Gray code** for the combinatorial objects?

Hamilton 1856
Hamiltonicity

Is the skeleton of the polytope Hamiltonian? Hamilton 1856

Is there a Gray code for the combinatorial objects?

Again, two versions:

Combinatorial  Does there always exist a Hamiltonian cycle?
Computational  Can we compute it efficiently, say with bounded delay?
Table of Contents

Flip Graphs

Problems

Matroids

Polymatroids

Hypergraphic polytopes

Graph associahedra

References
Matroids

A matroid can also be defined as $M = (E, \mathcal{B})$, where $\mathcal{B}$ is a set of bases, satisfying the basis exchange axiom:
Matroids

A matroid can also be defined as $M = (E, B)$, where $B$ is a set of bases, satisfying the basis exchange axiom:

If $A$ and $B$ are two distinct bases, then for any element $a \in A \setminus B$, there exists an element $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in B$.

Whitney 1935, Nakasawa 1935-38, McLane 1936, Rado 1940s, Tutte 1950s
The bases of $M$ are its maximal independent sets.
Bases

- \( \{a, b, c\} \) with \( 11100 \)
- \( \{a, b, d\} \) with \( 11010 \)

\[ \begin{align*}
\text{Base} & \quad \text{Label} & \text{Index} \\
\{a, b, c\} & \quad abc & \quad 11100 \\
\{a, b, d\} & \quad abd & \quad 11010
\end{align*} \]
Matroid polytopes

The polytope of $M$ is the convex hull of the indicator vectors of the bases of $M$:

$$P_M = \text{conv}\{e_B : B \in \mathcal{B}\}$$
Matroid polytopes

The polytope of $M$ is the convex hull of the indicator vectors of the bases of $M$:

$$P_M = \text{conv}\{e_B : B \in \mathcal{B}\}$$

**Theorem**

A 0/1 polytope $P$ is the polytope of a matroid if and only if:

- every edge of $P$ is a translate of $e_i - e_j$, for some $i, j$,
- there exists a submodular rank function $r : 2^E \rightarrow \mathbb{N}$ s.t.:

$$P = P_r := \{x \in \mathbb{R}^E : \sum_{i \in U} x_i \leq r(U) \forall U \subset E \land \sum_{i \in E} x_i = r(E)\}.$$  

Gel’fand, Goresky, MacPherson, Serganova 1987
Distances and Hamiltonicity

- From the basis exchange axiom, the distance between two bases $A$ and $B$ is exactly $|AΔB|/2$. 

• Can be computed in polynomial time using the Matroid Union theorem and Edmonds' Matroid partition algorithm.

• It is known that any 0/1 polytope is Hamilton-connected.

Edmonds 1965

Naddef-Pulleyblank 1984

Efficient Gray codes using linear optimization as a black box.

Merino-Mütze 2023
Distances and Hamiltonicity

- From the basis exchange axiom, the distance between two bases $A$ and $B$ is exactly $|A \Delta B|/2$.
- The diameter $\delta(P_M)$ is therefore (half) the maximum symmetric difference between two bases.

References

- Edmonds 1965
- Naddef-Pulleyblank 1984
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Merino-Mütze 2023
Table of Contents

Flip Graphs
Problems
Matroids
Polymatroids
Hypergraphic polytopes
Graph associahedra
References
Theorem

A polytope $P$ is a polymatroid if and only if:

- every edge of $P$ is parallel to $e_i - e_j$, for some $i, j$,
- there exists a submodular function $f : 2^E \mapsto \mathbb{R}$ s.t.:

$$P = P_f := \{ x \in \mathbb{R}^E : \sum_{i \in U} x_i \leq f(U) \ \forall U \subset E \land \sum_{i \in E} x_i = f(E) \}.$$
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\]

- Greedy optimization algorithm
- Aka generalized permutahedra, or submodular polyhedra
Table of Contents

Flip Graphs

Problems

Matroids

Polymatroids

Hypergraphic polytopes

Graph associahedra

References
Acyclic orientations and graphical zonotopes

Given a simple, connected graph $G = ([n], E)$, let $f : 2^n \to \mathbb{N}$,

$$f(U) = |\{e \in E : e \cap U \neq \emptyset\}|.$$
Acyclic orientations and graphical zonotopes

Given a simple, connected graph $G = ([n], E)$, let $f : 2^{[n]} \to \mathbb{N}$,

$$f(U) = |\{ e \in E : e \cap U \neq \emptyset \}|.$$

- $P_f$ is the **Graphical zonotope** of $G$.

  Greene 1977, Greene-Zaslavsky 1983

- $P_f$ is also the **Minkowski sum** of segments

  \[ \text{conv}\{e_i, e_j\}, \ ij \in E. \]
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• The skeleton of $P_f$ is the **flip graph on acyclic orientations** of $G$. 

Acyclic orientations and graphical zonotopes

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- The skeleton of $P_f$ is the flip graph on acyclic orientations of $G$.

- Distances and diameter: Easy.

- Hamiltonicity: not always. When exactly is an open problem.
Example: Permutahedron

When $G$ is the complete graph, we obtain all permutations.
Example: Bilinski dodecahedron

When $G$ is a 4-cycle.
Hypergraphic polytopes

Given a hypergraph $H = (V, E)$, where $E \subseteq 2^V \setminus \{\emptyset\}$, let $f_H : 2^V \to \mathbb{N}$ be defined as

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- Minkowski sum of standard simplices
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- Minkowski sum of standard simplices
- Vertices $\leftrightarrow$ Acyclic orientations of hypergraphs, edges $\leftrightarrow$ flips

Benedetti, Bergeron, Machacek 2018, C., Hoang, Merino, Mička, Mütze 2023
Flip distances in hypergraphic polytopes

Theorem

*Computing the flip distance between two acyclic orientations of hypergraph $H$ is APX-hard even when the input hypergraph $H = (V, \mathcal{E})$ is known to have bounded maximum degree and be such that $|e| \leq 3$ for every $e \in \mathcal{E}$.* 

C., Steiner 2023
Associahedra are hypergraphic

Let $H = ([n], E)$ be the set of intervals in $[n]$: 

$$E := \{\{i, i + 1, \ldots, j\} : 1 \leq i < j \leq n\}.$$

Then the hypergraphic polytope of $H$ is Loday's associahedron.

Loday 2004
Associahedra are hypergraphic

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- Complexity of computing flip distances: wide open!
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- Diameter is exactly \( 2n - 6 \).

Sleator, Tarjan, Thurston 1988, Pournin 2014
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- Hamiltonicity: Yes.

Lucas 1987, Lucas, Roelants van Baronaigien, Ruskey 1993
## Table of Contents

- Flip Graphs
- Problems
- Matroids
- Polymatroids
- Hypergraphic polytopes
- Graph associahedra
- References
Graph associahedra and elimination trees

When \( H = (V, \mathcal{E}) \) is the graphical building set of a graph \( G = (V, E) \):

\[
\mathcal{E} := \{ S \subseteq V : G[S] \text{ is connected} \},
\]

then the hypergraphic polytope \( P_H \) of \( H \) is the graph associahedron of \( G \).
Graph associahedra and elimination trees

When $H = (V, \mathcal{E})$ is the graphical building set of a graph $G = (V, E)$:

$$\mathcal{E} := \{ S \subseteq V : G[S] \text{ is connected}\},$$

then the hypergraphic polytope $P_H$ of $H$ is the graph associahedron of $G$.

- Vertices of $P_H$ are one-to-one with elimination trees of $G$,
- and the skeleton of $P_H$ is the rotation graph on elimination trees of $G.$
Elimination trees

\[ a, b, c, d, e, f, g \]
Elimination trees
Elimination trees
Rotations in elimination trees
Graph Associahedra

associahedron

permutahedron

stellohedron
Distances and diameters of graph associahedra

- **Distances:** Computing rotation distances is NP-hard
  
  Ito, Kakimura, Kamiyama, Kobayashi, Maezawa, Nozaki, Okamoto 2023
Distances and diameters of graph associahedra

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  ... unless the graph is a star or a complete split graph.
  C., Pournin, Valencia-Pabon 2023

• **Diameter:**
  • **Tree** associahedra have worst-case diameter $\Theta(n \log n)$
    C., Langerman, Perez-Lantero 2018
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• **Diameter:**
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  • Tight bounds for complete split or complete bipartite graph associahedra.
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    C., Langerman, Perez-Lantero 2018
  • Tight bounds for complete split or complete bipartite graph
    associahedra.
    C., Pournin, Valencia-Pabon 2022

• **Hamiltonicity:** Always!
  Manneville-Pilaud 2015, C., Merino, Mütze 2023
Table of Contents

Flip Graphs
Problems
Matroids
Polymatroids
Hypergraphic polytopes
Graph associahedra

References
Associahedra

Polymatroids and generalized permutohedra

Acyclic orientations


Graph associahedra

Computational complexity


