An extension of the algebraic Aldous diffusion: or how to make money fast

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Journée-séminaire de combinatoire
Université Sorbonne Paris Nord
1 April 2025

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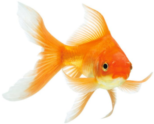

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$\mathfrak{T}_n$: set of trees with n labelled leaves and no degree 2 vertex.

Goal : study the limit of a Markov chain on $\mathfrak{T}_n$, as $n \to +\infty$.

How does the chain work ?

Remove a leaf uniformly at random, reattach it with a specific rule.

Why do we expect a limit ?

- The chain is parameterized by $\gamma \in (1, 2]$ and the limit was already shown when $\gamma = 2$.
- The invariant distributions of the chains converge to the same limit one obtains by only attaching leaves.

What about combinatorics ?

The chains converge to a limit process whose operators admit a spectral decomposition with a simple combinatorial description.

Plan of the talk

1. Definitions and context

1.1 Attachment rule, and the limit tree it produces.

1.2 Leaf-constant chain on $\mathfrak{T}_n$.

1.3 Theory of algebraic measure trees, and limit when $\gamma = 2$.

2. The limit process

2.1 Extension of the theory of algebraic measure trees.

2.2 Limit process in the case $\gamma \in (1, 2]$.

3. Spectral decomposition

3.1 Spectrum and eigenspaces of the limit generator.

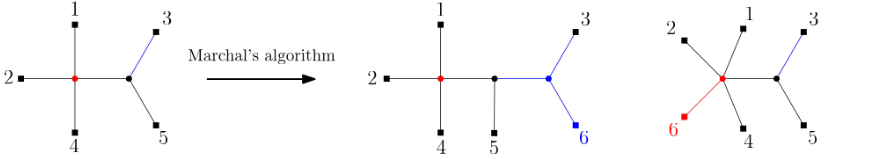
3.2 Consequences for the limit process.

3.3 (If time permits...) Sketch of proof.

The attachment rule

$\mathfrak{T}_n$: set of trees with n labelled leaves and no degree 2 vertex.

Marchal's algorithm (2008) : generates random trees $(T_n)_{n\geq 2}$
1. Given T_n (a.s. in $\mathfrak{T}_n$), put weight $\gamma - 1$ on each edge of T_n and $d - 1 - \gamma$ on each branch point with degree d of T_n.
2. Attach a leaf labelled $n+1$ to an edge or branch point of T_n sampled proportionally to the weights and define T_{n+1} as the newly created random tree in $\mathfrak{T}_{n+1}$.



Initial tree with 5 leaves. The blue edge is selected. The red branch point is selected.

$\mathfrak{M}_\gamma^n$: law of T_n on $\mathfrak{T}_n$

Special case $(\gamma = 2)$ **:** Rémy's algorithm (1980)
$\mathfrak{M}_2^n =$ uniform distribution on binary leaf-labelled trees

Real tree (T, d) **:** geodesic metric space with no subset homeomorphic to a circle.
$x, y \in T$: segment $[x, y]$ is the geodesic path from x to y
$x \in T$: $deg(x)$ = number of connected components of $T \backslash \{x\}$

Metric tree (T, d) **:** metric space that can be isometrically embedded into a real tree and contains all its branchpoints.

$\mathbb{M}$: space of (probability) measured compact metric spaces (up to measure preserving isometry).

d_{GHP} metric on $\mathbb{M}$ given for $\mathcal{X} = (X, d_X, \mu_X), \mathcal{Y} = (Y, d_Y, \mu_Y)$:

$$d_{GHP}(\mathcal{X}, \mathcal{Y}) = \inf_{\varphi_X, \varphi_Y} \left(d_{Hausdorff}(\varphi_X(X), \varphi_Y(Y)) + d_{Prokhorov}(\varphi_{X*}\mu_X, \varphi_{Y*}\mu_Y)\right)$$

where the infimum is over metric spaces Z and isometric injections $\varphi_X : X \to Z$, $\varphi_Y : Y \to Z$.

d_{GHP} makes $\mathbb{M}$ complete. [Abraham, Delmas, Hoscheit, 13']

The stable trees

Theorem (Curien & Haas, 2012)

$$n^{-\frac{\gamma-1}{\gamma}} T_n^{\gamma} \xrightarrow[n\to+\infty]{GHP} \mathcal{T}_{\gamma} \qquad a.s.$$

where $\mathcal{T}_\gamma$ is a random real tree called a γ-stable tree.

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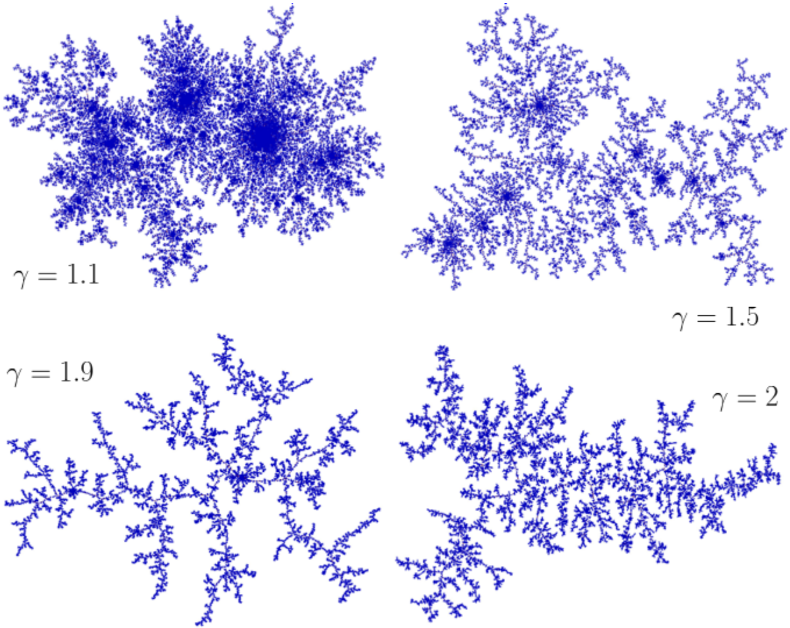
- Self-similarity (sample two leaves, the trees branching from the path between them are independent factored stable trees)

Illustrations (courtesy of I. Kortchemski)



The Markov chain on $\mathfrak{T}_n$

$X^n := (X^n_k)_{k\geq 0}$ Markov chain on $\mathfrak{T}_n$ that moves as follows :

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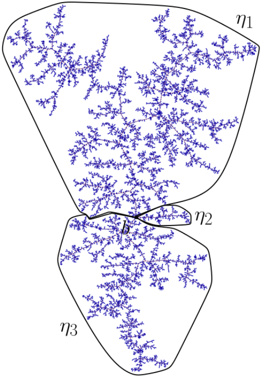
"Real world" motivation :
asymptotic study of phylogenetics MCMC-type algorithms.

Ω^γ_n : generator of continuous-time version with total rate $\sim \gamma n^2$.

Aldous's observation and conjecture ($\gamma = 2$)

Fix a branch point b and consider the proportions of leaves $\eta(b) := (\eta_1, \eta_2, \eta_3) \in \Delta_3$ branching from it.



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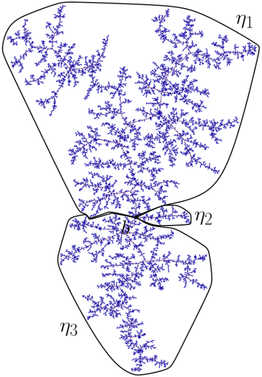
(1) : generator of a Wright-Fisher diffusion with **negative** mutation rate.



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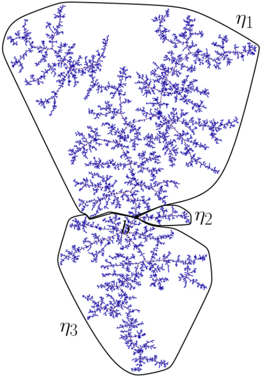

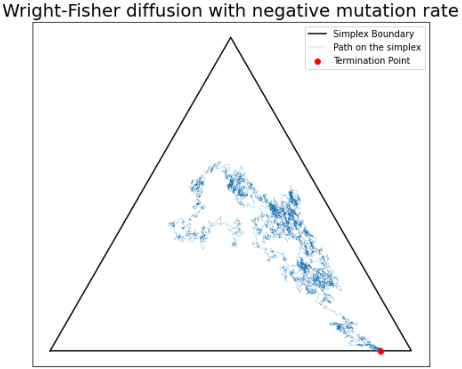

More generally :
Take k leaves and look at the mass distribution along the edges of the shape they induce : we obtain a similar diffusion on Δ_{2k-3}.

Question (Aldous, 1999) : Are we observing functionals of some limit tree diffusion (later nicknamed "Aldous diffusion") that is stationary w.r.t. the law of the Brownian tree ?

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1. [Forman, Pal, Rizzolo, Winkel : 2010-2023]
   - Obtained a continuous process on real trees that has the same properties as the conjectured one but :
   - The process has to be started at the invariant distribution.
   - Only the convergence in fdd's is proved.
   - Complicated construction that required a lot of papers.

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   - Only the convergence in fdd's is proved.
   - Complicated construction that required a lot of papers.
2. [Löhr, Mytnik, Winter : 2018-2023]
   - Decided to "forget" the metric and consider equivalence classes that they developed under the name of "algebraic measure trees". But :
   - Their process can be started at any "reasonable" tree.
   - Once the theory is developed, it reduces to a classical martingale problem.

Second process referred to as the **"algebraic" Aldous diffusion**.

Algebraic measure trees [L., W. 18']

Let T be any set and $c : T^3 \to T$ symmetric such that :

- For all $x, y \in T$, $c(x, x, y) = x$.
- For all $x, y, z \in T$, $c(x, y, c(x, y, z)) = c(x, y, z)$.
- For all $x, y, z, w \in T$: $c(x, y, z) \in \{c(x, y, w), c(x, w, z), c(w, y, z)\}$.

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$\mathbb{T}$:= set of (equivalence classes of) AMTs
$\mathbb{T}^{[n]}$:= AMTs with n leaves and uniform distribution on the leaves
$\tilde{\mathbb{T}}$:= AMTs with no atom outside leaves, $\tilde{\mathbb{T}}_2$:= binary AMTs in $\tilde{\mathbb{T}}$
$\mathbb{T}^c$:= AMTs with diffuse measure

Strategy in the case $\gamma = 2$ [L., M., W. 20']

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The limit process is Feller, continuous, ergodic and symmetric for the law of an "algebraic" Brownian tree.

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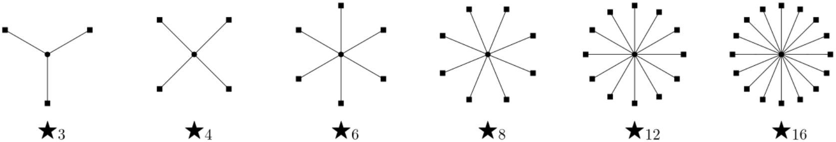
Aldous's observation extends to the case $\gamma < 2$, but

Problem : the sample shape topology can be extended to $\tilde{\mathbb{T}}$ but is no longer compact (star trees admit no convergent subsequence).



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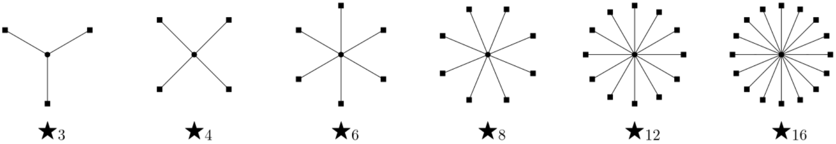

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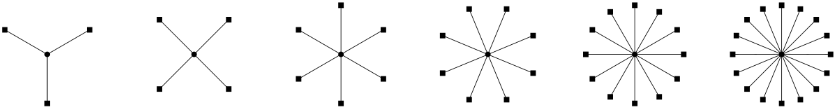


$\bigstar_3$ $\bigstar_4$ $\bigstar_6$ $\bigstar_8$ $\bigstar_{12}$ $\bigstar_{16}$

Solution : define a topology that makes star trees converge.

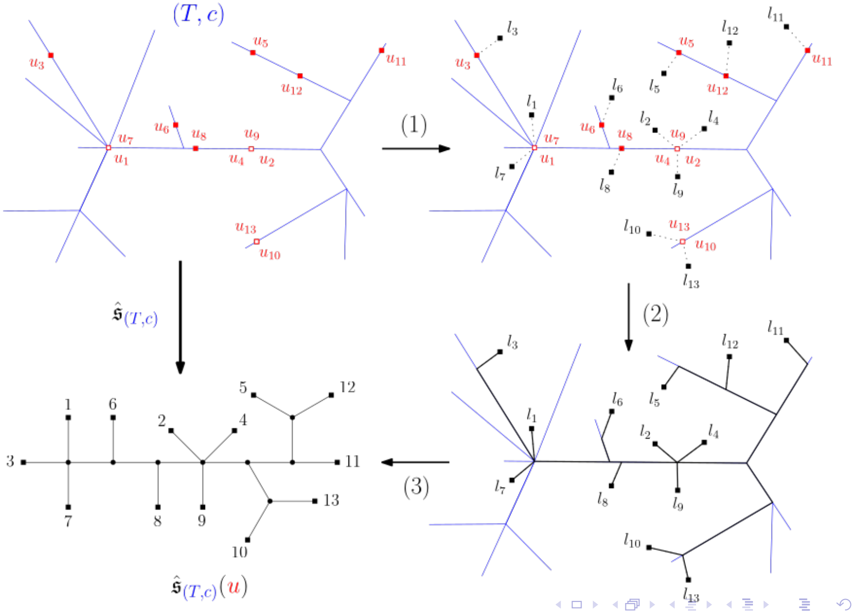
Topology based on sampling subtrees called "hierarchies" :
[Forman, Haulk, Pitman 18' ; Forman 20']

1. Take m points $u := (u_1, \dots, u_m) \in T^m$ in a tree (T, c).

2. Attach a leaf l_k on each of the points u_k $(1 \leq k \leq m)$.

3. Define the hierarchy $\hat{\mathfrak{s}}_{(T,c)}(u)$ induced by u in (T, c) as the labelled tree in $\mathfrak{T}_m$ induced by the leaves $(l_1, \dots, l_m)$ in (T, c).

Illustration of a sampled hierarchy



Sample hierarchy topology

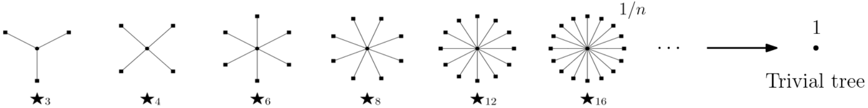
For $m \geq 3$, $\mathfrak{t} \in \mathfrak{T}_m$ and $[T, c, \mu] \in \mathbb{T}$,

$$\Psi^{m,\mathfrak{t}}([T, c, \mu]) := \int_{T^m} \hat{\mathfrak{s}}_{(T,c)}(u)\mu^{\otimes m}(\mathrm{d}u).$$

$\Psi^{m,\mathfrak{t}}([T, c, \mu])$: probablity of obtaining the hierarchy $\mathfrak{t} \in \mathfrak{T}_m$ if we sample m points from $[T, c, \mu]$.

Sample hierarchy topology on $\mathbb{T}$: induced by $(\Psi^{m,\mathfrak{t}})_{m \geq 3, \mathfrak{t} \in \mathfrak{T}_m}$.

In this topology, the star trees converge to an atom of mass 1.



More generally : accumulation of mass results in creation of atoms.

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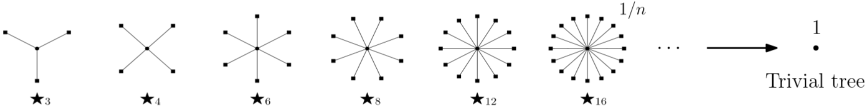

More generally : accumulation of mass results in creation of atoms.

Theorem

The sample hierarchy topology on $\mathbb{T}$ is compact.

Convergence of the chains

Hierarchy polynomials : linear combinations Ψ of $(\Psi^{m,\mathfrak{t}})_{m\geq 3,\mathfrak{t}\in\mathfrak{T}_m}$, dense subalgebra of cont. functions on $\mathbb{T}$ (Stone-Weierstrass).

Theorem (Convergence of generators)

$$\lim_{n\to+\infty} \sup_{\mathcal{T}:=[T,c,\mu]\in\mathbb{T}^{[n]}} |\Omega_n^\gamma \Psi(\mathcal{T}) - \Omega_\infty^\gamma \Psi(\mathcal{T})| = 0,$$

where

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The chains converge in law (for the sample hierarchy topology) to a limit process $X := (X_t)_{t\in\mathbb{R}_+}$ on $\mathbb{T}$ with (pre)generator Ω_∞^γ.

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The chains converge in law (for the sample hierarchy topology) to a limit process $X := (X_t)_{t\in\mathbb{R}_+}$ *on* $\mathbb{T}$ *with (pre)generator* Ω_∞^γ*.*

Properties in common with $\gamma = 2$: X is continuous, Feller, ergodic, reversible for the law $\mathcal{M}_\gamma$ of an algrebraic γ-stable tree.

Difference : X can be started at any point of $\mathbb{T}$.

Spectral decomposition of the limiting generator

Π_m : hierarchy polynomials of order $\leq m$ (for $m \geq 3$).
($=$ linear combinations from $\{\Psi^{k,\mathfrak{t}} : k \in [\![3, m]\!], \mathfrak{t} \in \mathfrak{T}_k\}$)

Observation : for $m \geq 3$, $\Psi^{m,\mathfrak{t}} = \sum_{\mathfrak{t}' \in \mathfrak{T}_{m+1} : \mathfrak{t} \nearrow \mathfrak{t}'} \Psi^{m,\mathfrak{t}'}$.

$\implies \Pi_m = \mathrm{Vect}(\{\Psi^{m,\mathfrak{t}} : \mathfrak{t} \in \mathfrak{T}_m\})$.

Set $\lambda_3 := 0$ and $\lambda_m := m((m-2)\gamma - 1)$ for $m \geq 4$.

$V_m \subset \Pi_m$: subspace of Π_m that can be described explicitely.

Theorem (Eigendecomposition of the generator in $L^2(\mathcal{M}_\gamma)$)

For $m \geq 3$, $-\lambda_m$ is an eigenvalue of Ω^γ_∞ with eigenspace V_m. Moreover, we have the orthogonal direct sum

$$L^2(\mathcal{M}_\gamma) = \bigoplus_{m \geq 3} V_m,$$

$\dim(V_3) = 1$, *and* $\dim(V_m) = \left|\mathbb{T}^{[m]}\right| - \left|\mathbb{T}^{[m-1]}\right|$ *for $m \geq 4$.*

Combinatorial description of the eigenspaces $(V_m)_{m\geq 4}$

Π_m : linear combinations of $\{\Psi^{m,\mathfrak{t}} : \mathfrak{t} \in \mathfrak{T}_m\}$
$\mathfrak{a} : \mathfrak{T}_m \to \mathbb{T}^{[m]}$: canonical map (forgets labels)

Observation : for all $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}_m$ s.t. $\mathfrak{a}(\mathfrak{t}) = \mathfrak{a}(\mathfrak{t}')$, $\Psi^{m,\mathfrak{t}} = \Psi^{m,\mathfrak{t}'}$.
$\implies$ For $\mathcal{T} \in \mathbb{T}^{[m]}$, define

$$\Psi^{m,\mathcal{T}} = \sum_{\mathfrak{t}\in\mathfrak{a}^{-1}(\{\mathcal{T}\})} \Psi^{m,\mathfrak{t}}.$$

$\Psi^{m,\mathcal{T}}$: probability of sampling the unlabelled shape $\mathcal{T}$.

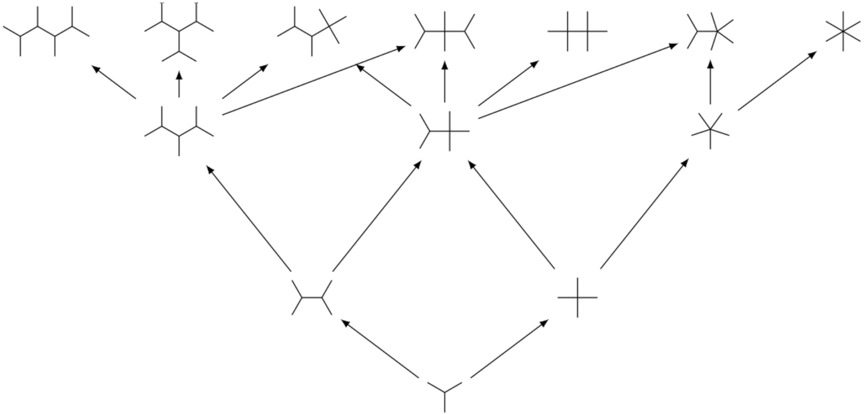
Proposition : The family $\{\Psi^{m,\mathcal{T}} : \mathcal{T} \in \mathbb{T}^{[m]}\}$ is a basis for Π_m.
$\implies \dim(\Pi_m) = \left|\mathbb{T}^{[m]}\right|$.

V_m : polynomials $\sum_{\mathcal{T}\in\mathbb{T}^{[m]}} \alpha_{\mathcal{T}} \Psi^{m,\mathcal{T}}$ such that, for all $\mathcal{S} \in \mathbb{T}^{[m-1]}$,

$$\sum_{\mathcal{T}\in\mathbb{T}^{[m]}} \pi_\gamma(\mathcal{S}, \mathcal{T})\alpha_{\mathcal{T}} = 0,$$

$\pi_\gamma(\mathcal{S}, \mathcal{T})$: probability of getting $\mathcal{T}$ from $\mathcal{S}$ with Marchal's algorithm.
$\implies \dim(V_m) = \left|\mathbb{T}^{[m]}\right| - \left|\mathbb{T}^{[m-1]}\right| \qquad \implies \Pi_m := \bigoplus_{k=3}^m V_m$

Illustration of the eigenspaces structure (V_3 to V_5)



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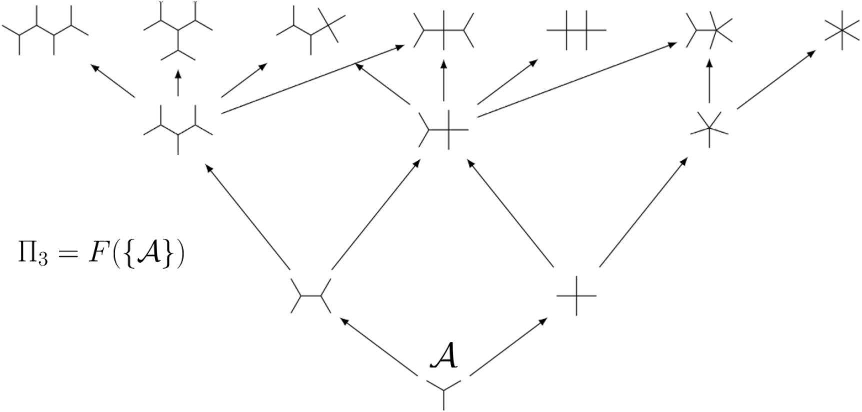

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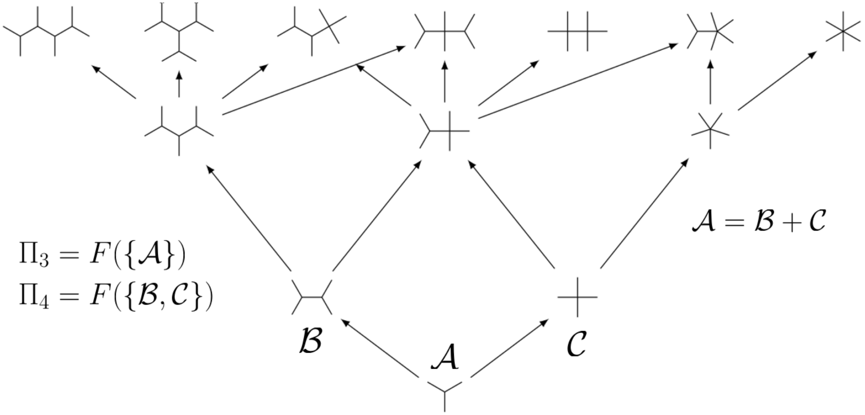

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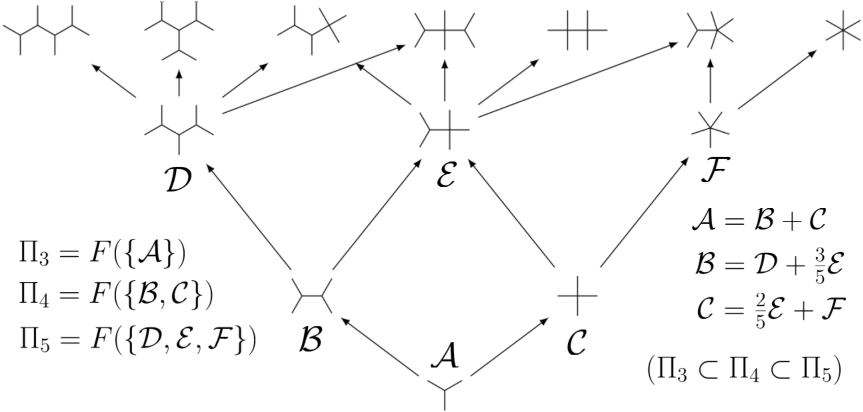
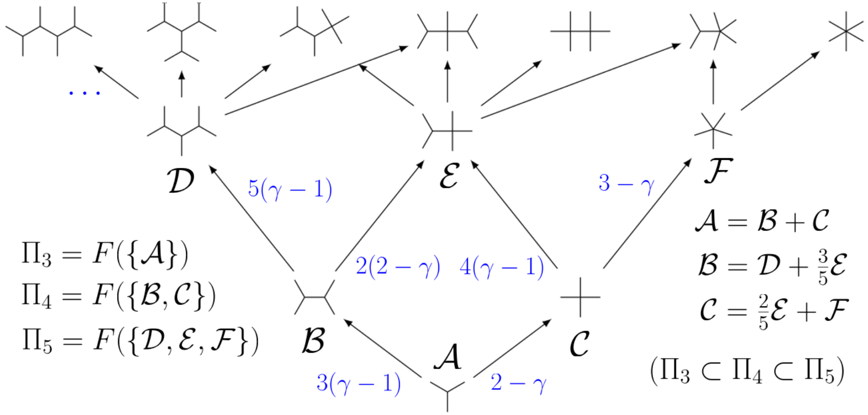


$$\Pi_3 = F(\{\mathcal{A}\})$$
$$\Pi_4 = F(\{\mathcal{B}, \mathcal{C}\})$$
$$\Pi_5 = F(\{\mathcal{D}, \mathcal{E}, \mathcal{F}\})$$

$$\mathcal{A} = \mathcal{B} + \mathcal{C}$$
$$\mathcal{B} = \mathcal{D} + \tfrac{3}{5}\mathcal{E}$$
$$\mathcal{C} = \tfrac{2}{5}\mathcal{E} + \mathcal{F}$$

$$(\Pi_3 \subset \Pi_4 \subset \Pi_5)$$



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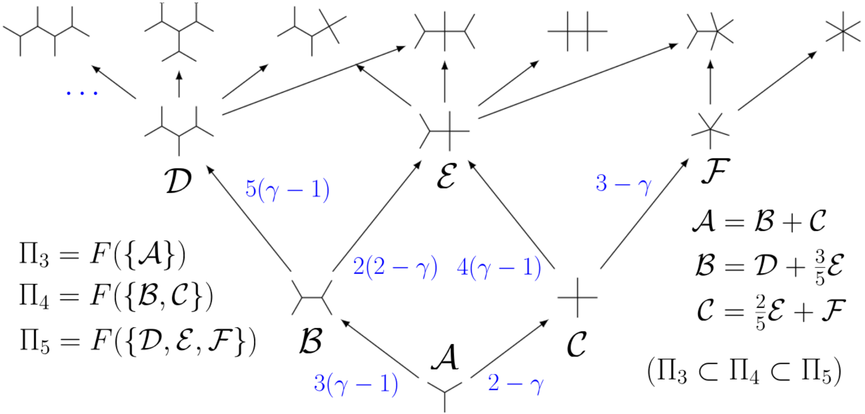

$V_3 = \Pi_3$, $V_4 = \{\alpha_{\mathcal{B}}\mathcal{B} + \alpha_{\mathcal{C}}\mathcal{C} \in \Pi_4 : 3(\gamma-1)\alpha_{\mathcal{B}} + (2-\gamma)\alpha_{\mathcal{C}} = 0\}$,

$V_5 =$ subspace of $\alpha_{\mathcal{D}}\mathcal{D} + \alpha_{\mathcal{E}}\mathcal{E} + \alpha_{\mathcal{F}}\mathcal{F} \in \Pi_5$ such that

$$5(\gamma-1)\alpha_{\mathcal{E}} + 2(2-\gamma)\alpha_{\mathcal{E}} = 0,$$
$$4(\gamma-1)\alpha_{\mathcal{E}} + (3-\gamma)\alpha_{\mathcal{F}} = 0.$$

Eigenstructure of the continuous-time Markov chain

For $m \geq 3$, let $s_m : \mathbb{R}^{\mathfrak{T}_m} \to \Pi_m$ be the linear surjection given by

$$s_m\left(\sum_{\mathfrak{t}\in\mathfrak{T}_m} \alpha_{\mathfrak{t}} \mathbb{1}_{\{\mathfrak{t}\}}\right) = \sum_{\mathfrak{t}\in\mathfrak{T}_m} \alpha_{\mathfrak{t}} \Psi^{m,\mathfrak{t}}$$

Corollary : Ω_m^{γ} has eigenvalues $\{-\lambda_k : k \in [\![3, m]\!]\}$ with respective eigenspaces $\{s_m^{-1}(V_k) : k \in [\![3, m]\!]\}$.

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Sketch of proof : Recall that

$$\Omega_\infty^\gamma \Psi^{m,\mathfrak{t}}([T, c, \mu]) = \int_{T^m} \Omega_m \mathbb{1}_{\{\mathfrak{t}\}}(\hat{\mathfrak{s}}(u))\mu^{\otimes m}(\mathrm{d}u).$$

For all $f := \sum_{\mathfrak{t}\in\mathfrak{T}_m} \alpha_{\mathfrak{t}} \mathbb{1}_{\{\mathfrak{t}\}} \in \mathbb{R}^{\mathfrak{T}_m}$, $\lambda \in \mathbb{R}$ and $[T, c, \mu] \in \mathbb{T}$,

$$(\Omega_\infty^\gamma - \lambda)s_m f([T, c, \mu]) = 0 \iff \int_{T^m} (\Omega_m^\gamma - \lambda) f(\hat{\mathfrak{s}}(u))\mu^{\otimes m}(\mathrm{d}s) = 0.$$

By density, $(\Omega_\infty^\gamma - \lambda)s_m f \equiv 0 \iff (\Omega_m^\gamma - \lambda) f \equiv 0$.

But $(\Omega_\infty^\gamma - \lambda)s_m f \equiv 0$ iff $\exists k \in [\![3, m]\!]$ s.t. $\lambda = \lambda_k$ and $s_m f \in V_k$.

Eigenstructure of the discrete-time Markov chain

M_m^γ : transition matrix of the discrete-time Markov chain on $\mathfrak{T}_m$.

Corollary : M_m^γ has eigenvalues $\left\{1 - \frac{\lambda_k}{\lambda_m} : k \in [\![3, m]\!]\right\}$ with respective eigenspaces $\{s_m^{-1}(V_k) : k \in [\![3, m]\!]\}$.

Proof : Observe that $M_m^\gamma = I_m + \frac{1}{\lambda_m}\Omega_m^\gamma$.

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$$\tau_{\text{rel}}^m = \frac{\lambda_m - \lambda_4}{\lambda_m} = \frac{m((m-2)\gamma - 1))}{4(2\gamma - 1)} = \Theta(m^2)$$

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Open question : mixing time of order m^2 ?

Spectral decomposition of the L^2 semigroup

$(P_t)_{t \geq 0}$: Markov semigroup of the process X (with generator Ω_∞^γ)

Theorem

For $t > 0$, there exists $\varphi_t \in L^2(\mathcal{M}_\gamma^{\otimes 2})$ such that, for $f \in L^2(\mathcal{M}_\gamma)$,

$$P_t f(\cdot) = \int_{\mathbb{T}} \varphi_t(\cdot, \mathcal{T}) f(\mathcal{T}) \mathcal{M}_\gamma(\mathrm{d}\mathcal{T}) \in L^2(\mathcal{M}_\gamma).$$

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For all $f : \mathbb{T} \to \mathbb{R}$ continuous for the sample hierarchy topology,

$$\sup_{\mathcal{T}\in\mathbb{T}} |P_t f(\mathcal{T}) - \mathcal{M}_\gamma(f)| \underset{t\to+\infty}{\longrightarrow} 0.$$

Conjecture : continuity of the integral kernel

Recall : $(P_t)_{t\geq 0}$ semigroup of the limit process X, satisfying

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Sufficient condition :

Find orthonormal basis $(\phi_{m,k})_k$ of V_m with $\sup_k \|\phi_{m,k}\|_\infty \ll e^{m^2}$.

Why ? $\varphi_t(\cdot, \cdot) = \sum_{m\geq 3} e^{-\lambda_m t} \sum_{k=1}^{\dim(V_m)} \phi_{m,k}(\cdot)\phi_{m,k}(\cdot) \in L^2(\mathcal{M}_\gamma^{\otimes 2})$

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Consequences if the conjecture is true :

$\Longrightarrow$ Convergence in the stronger sample shape topology

$\Longrightarrow$ Strong Feller property

$\Longrightarrow$ The diffusion instantaneously enters $\mathbb{T}^c$ and stays within it ("big bang" phenomenon when started from the trivial tree)

Generator of the Markov chain

Ω_n^γ : generator of the continuous-time Markov chain on $\mathfrak{T}_n$.

$$\Omega_n^\gamma f(\mathfrak{t}) := \sum_{l=1}^{n} \sum_{h \in \mathcal{H}(\mathfrak{t}_{\wedge l})} \pi_{\mathfrak{t}_{\wedge l}}^\gamma(h) \left(f(\mathfrak{t}^{(l,h)}) - f(\mathfrak{t}) \right),$$

where

- $\mathfrak{t}_{\wedge l}$: labelled tree $\mathfrak{t}$ with the leaf labelled l removed,
- $\mathcal{H}(\mathfrak{t}_{\wedge l})$: set of edges and branch points of $\mathfrak{t}_{\wedge l}$,
- $\pi_{\mathfrak{t}_{\wedge l}}^\gamma(h)$: Marchal's weight associated to h in $\mathfrak{t}_{\wedge l}$, i.e.

$$\pi_{\mathfrak{t}_{\wedge l}}^\gamma(h) = \begin{cases} \gamma - 1, & \text{if } h \text{ is a leaf} \\ d - 1 - \gamma, & \text{if } h \text{ is a branch point of degree } d \geq 3 \end{cases}$$

- $\mathfrak{t}^{(l,h)}$: labelled tree $\mathfrak{t}$ after the leaf l has been moved to h.

Total Marchal's weight on a tree $\mathfrak{t}$ with n leaves :

$$\pi_{\mathfrak{t}}^\gamma(\mathcal{H}(\mathfrak{t})) := \sum_{h \in \mathcal{H}(\mathfrak{t})} \pi_{\mathfrak{t}}^\gamma(h) = (n-1)\gamma - 1.$$

Ω^γ_∞ : generator of the limiting process, acting on $\bigcup_{m\geq 3} \Pi_m$.

$$\Omega^\gamma_\infty \Psi^{m,\mathfrak{t}}(\mathcal{T}) := \int_{\mathbb{T}^m} \Omega^\gamma_m \mathbb{1}_{\{\mathfrak{t}\}}(\mathfrak{t}')\, \hat{\mathfrak{s}}_* \mu^{\otimes m}(\mathrm{d}\mathfrak{t}')$$

$$= \int_{\mathfrak{T}_m} \sum_{l=1}^{m} \sum_{h\in\mathcal{H}(\mathfrak{t}'_{\wedge l})} \pi^\gamma_{\mathfrak{t}'_{\wedge l}}(h) \left(\mathbb{1}_{\{\mathfrak{t}\}}(\mathfrak{t}'^{(l,h)}) - \mathbb{1}_{\{\mathfrak{t}\}}(\mathfrak{t}') \right) \hat{\mathfrak{s}}_* \mu^{\otimes m}(\mathrm{d}\mathfrak{t}')$$

But, $\quad \mathfrak{t}'^{(l,h)} = \mathfrak{t} \implies \mathfrak{t}_{\wedge l} = \mathfrak{t}'_{\wedge l} \quad \& \quad \exists! h_l \in \mathcal{H}(\mathfrak{t}_{\wedge l}),\ \mathfrak{t}^{(l,h_l)} = \mathfrak{t}.$

$$\implies \Omega^\gamma_\infty \Psi^{m,\mathfrak{t}} = \sum_{l=1}^{m} \left(\pi^\gamma_{\mathfrak{t}_{\wedge l}}(h_l) \Psi^{m,\mathfrak{t}_{\wedge l}} + \sum_{h\in\mathcal{H}(\mathfrak{t}_{\wedge l})} \pi^\gamma_{\mathfrak{t}_{\wedge l}}(h) \Psi^{m,\mathfrak{t}} \right)$$

$$\implies \left(\Omega^\gamma_\infty + m((m-2)\gamma - 1)\right) \Psi^{m,\mathfrak{t}} = \sum_{l=1}^{m} \pi^\gamma_{\mathfrak{t}_{\wedge l}}(h_l) \Psi^{m,\mathfrak{t}_{\wedge l}} \in \Pi_{m-1}$$

$\implies$ Ω^γ_∞ is triangular with eigenvalues $(-m((m-2)\gamma - 1))_{m\geq 3}$.

Recall :

$$(\Omega_\infty^\gamma + \lambda_m)\,\Psi^{m,\mathfrak{t}} = \sum_{l=1}^{m} \pi_{\mathfrak{t}_{\wedge l}}^{\gamma}(h_l)\Psi^{m-1,\mathfrak{t}_{\wedge l}},\ \ \Psi^{m,\mathcal{T}} := \sum_{\mathfrak{t}\in\mathfrak{a}^{-1}(\{\mathcal{T}\})} \Psi^{m,\mathfrak{t}}$$

V_m : subspace of $\Psi := \sum_{\mathcal{T}\in\mathbb{T}^{[m]}} \alpha_{\mathcal{T}}\Psi^{m,\mathcal{T}} = \sum_{\mathfrak{t}\in\mathfrak{T}_m} \alpha_{\mathfrak{a}(\mathfrak{t})}\Psi^{m,\mathfrak{t}}$ s.t.

$$(\Omega_\infty^\gamma + \lambda_m)\Psi = 0 \iff \sum_{\mathfrak{t}\in\mathfrak{T}_m} \alpha_{\mathfrak{a}(\mathfrak{t})} \sum_{l=1}^{m} \pi_{\mathfrak{t}_{\wedge l}}^{\gamma}(h_l)\Psi^{m-1,\mathfrak{t}_{\wedge l}} = 0$$

$$\iff \sum_{\mathfrak{s}\in\mathfrak{T}_{m-1}} \Psi^{m-1,\mathfrak{s}} \sum_{\mathfrak{t}\in\mathfrak{T}_m:\mathfrak{s}\nearrow\mathfrak{t}} \pi(\mathfrak{s},\mathfrak{t})\alpha_{\mathfrak{a}(\mathfrak{t})} = 0$$

$$\iff \sum_{\mathcal{S}\in\mathbb{T}^{[m]}} \Psi^{m-1,\mathcal{S}} \sum_{\mathcal{T}\in\mathbb{T}^{[m]}:\mathcal{S}\nearrow\mathcal{T}} \pi(\mathcal{S},\mathcal{T})\alpha_{\mathcal{T}} = 0$$

$\pi(\mathfrak{s},\mathfrak{t})$ (resp. $\pi(\mathcal{S},\mathcal{T})$) : probability to get from $\mathfrak{s}$ to $\mathfrak{t}$ (resp. $\mathcal{S}$ to $\mathcal{T}$).

Because $(\Psi^{m-1,\mathcal{S}})_{\mathcal{S}\in\mathbb{T}^{[m-1]}}$ is linearly independent :

$$\iff \forall\mathcal{S}\in\mathbb{T}^{[m-1]},\ \sum_{\mathcal{T}\in\mathbb{T}^{[m]}:\mathcal{S}\nearrow\mathcal{T}} \pi(\mathcal{S},\mathcal{T})\alpha_{\mathcal{T}} = 0$$

Last slide.

Thank you for your attention.