Asymptotic normality of pattern counts in conjugacy classes
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Definitions
- Permutations
- Conjugacy invariant permutations
- Patterns

Results
- Uniform case: (Hofer)
- Partial results: (Féray), (Hamaker and Rhoades) and (Kammoun)
- General case: (Dubach) and (Féray and Kammoun)

Proofs
- Comparison techniques
- Weighted dependency graphs

Universality (Aléa days)
- I.I.D.
- Random matrices
- Longest increasing (decreasing) subsequence
- Conjugacy invariant permutations
Word: 2 10 1 6 9 8 7 4 5 3

Cycles: (1, 2, 10, 3)(4, 6, 8)(5, 9)(7)

Matrix:
\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Question: we fix the value of a function, we study another.
Example in LIPN: Bassino et al.

- Condition: Separable i.e. 0 occurrence of the patterns $2413$ and $3142$
- Function to study: Longest increasing subsequence / proportion of other patterns.
Cycle Structure and Spectrum

- # total number of cycles
- #\_i number of cycles of length i

If 0 ≤ p < q and GCD(p, q) = 1, then

\[ e^{\frac{p}{q} 2\pi i} \text{ is } \sum_{r \geq 1} \#_{rq}(\sigma) \]

In particular:

\[ \#(\sigma) = \text{Multiplicity of eigenvalue 1} \]

\[ \text{Tr}(\sigma^k) = \sum_{i\mid k} i\#_i(\sigma) \quad \text{and} \quad k\#_k(\sigma) = \sum_{i\mid k} \text{Tr}(\sigma^i)\mu(i) \]

Where \( \mu(i) \) is the Möbius function defined as:

\[ \mu(i) = \begin{cases} 0 & \text{if } i \text{ is divisible by the square of a prime number,} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct prime numbers.} \end{cases} \]
Conjugacy Classes

The conjugacy class of $\sigma$ is $\{\pi \sigma \pi^{-1}, \pi \in \mathfrak{S}_n\}$.

**Theorem**

Let $\sigma, \rho$ be two permutations.
There is equivalence between:

- $\sigma$ and $\rho$ are in the same conjugacy class
- $\sigma$ and $\rho$ have the same cycle structure, i.e., $\forall i \geq 1, \#_i(\sigma) = \#_i(\rho)$.
- $\sigma$ and $\rho$ have the same spectrum (considering multiplicities)
- $\forall i \geq 1, \text{Tr}(\sigma^i) = \text{Tr}(\rho^i)$. 
Conjugacy invariant

- Definition: $\sigma_n$ is conjugacy invariant if for all $\rho$,
  \[ \rho \sigma_n \rho^{-1} \overset{d}{=} \sigma_n. \]

- $\sigma_n$ is conjugacy invariant if and only if $\mathbb{P}(\sigma_n = \sigma)$ is a function of the cycle structure of $\sigma$. 

Example 1: Ewens
$\mathbb{P}(\sigma_n = \sigma) = \theta^\# \sigma C_n, \theta$.

Example 2: Uniform permutation within a conjugacy class.

Example 3: Uniform Involutions / Derangements.
Morally: Conditioned on the cycle structure, the permutation is chosen uniformly.
Conjugacy invariant

• Definition: $\sigma_n$ is conjugacy invariant if for all $\rho$,

$$\rho \sigma_n \rho^{-1} \overset{d}{=} \sigma_n.$$ 

• $\sigma_n$ is conjugacy invariant if and only if $\mathbb{P}(\sigma_n = \sigma)$ is a function of the cycle structure of $\sigma$.

• Example 1: Ewens

$$\mathbb{P}(\sigma_n = \sigma) = \frac{\theta^{\#\sigma}}{C_{n,\theta}}.$$ 

• Example 2: Uniform permutation within a conjugacy class.

• Example 3: Uniform Involutions / Derangements.

Morally: Conditioned on the cycle structure, the permutation is chosen uniformly.
We denote by $D(\sigma) = \{i : \sigma(i+1) < \sigma(i)\}$.
We assume that $(\sigma_n)_{n \geq 1}$ is a sequence of random permutations such that for all $n$, $\sigma_n$ is conjugacy invariant of size $n$.
Furthermore, we suppose that $\frac{\#_1^\sigma n}{n} \to \alpha$.

**Theorem (Kim and Lee 2020)**

$$\frac{\text{card}(D(\sigma_n)) - \frac{(1-\alpha^2)n}{2}}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \frac{1-4\alpha^3+3\alpha^4}{12}).$$

Goal: prove similar results for other functions.
Classical Pattern

Let $\pi$ be a permutation of size $k$. An occurrence of the (classical) pattern $\pi$ in a permutation $\sigma$ is a vector $(i_1, \ldots, i_k)$ with $i_1 < \cdots < i_k$ such that $\sigma(i_1) \cdots \sigma(i_k)$ has the same relative order as the elements of $\pi$.

Examples:

- For the permutation $\sigma = 2173456$, the vector $(i_1, i_2, i_3) = (2, 3, 7)$ is an occurrence of the pattern $\pi = 132$ (176 has the same relative order as $\pi = 132$).
- An occurrence of 21 is an inversion.
- An occurrence of $123 \cdots k$ is an increasing subsequence of length $k$. 
Vincular Pattern

Definition

Let $\pi$ be a permutation of size $k$ and $A$ be a subset of $[k-1]$. An occurrence of the vincular pattern $(\pi, A)$ in a permutation $\sigma$ is a vector $(i_1, \cdots, i_k)$ with $i_1 < \cdots < i_k$ satisfying:

- $(i_1, \cdots, i_k)$ is an occurrence of the classical pattern $\pi$ in $\sigma$.
- For every $s$ in $A$, $i_{s+1} = i_s + 1$.

Examples:

- $(\pi, \emptyset)$: is the classical pattern $\pi$
- An occurrence of $(21, \{1\})$: is a descent
- For the permutation $\sigma = 2173456$, the vector $(i_1, i_2, i_3) = (2, 3, 7)$
  - is an occurrence of the pattern $(\pi = 132, A = \{1\})$
  - not an occurrence of $(\pi = 132, A = \{1, 2\})$

Notation: $\mathcal{N}^{\pi A}(\sigma)$: pattern counts (number of occurrences of the patterns).
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Proofs

  Comparison techniques
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Universality (Aléa days)

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  Conjugacy invariant permutations
Fix $\Pi = (\pi, A)$, and let $k$ be the size of $\pi$.

**Theorem (Hofer (2018))**

We assume that $\sigma_n$ uniform of size $n$

$$\frac{\mathcal{N}_\Pi(\sigma_n) - \mathbb{E}(\mathcal{N}_\Pi(\sigma_n))}{n^{k-\frac{1}{2}-\text{card}(A)}} \xrightarrow{d} \mathcal{N}(0, \sigma^2_\Pi).$$

With

- $\sigma^2_\Pi > 0$.

Generalises:

- Consecutive: Goldstein (2005)
- Monotone: Bonà (2010)
- Classical: Janson et al. (2015)
- Without positivity: Féray (2013)
Recall: Ewens distribution.

\[ \mathbb{P}(\sigma_n = \sigma) = \frac{\theta^{\#\sigma}}{C_{n,\theta}}. \]

Fix \( \Pi = (\pi, A) \), and \( \theta \geq 0 \). Let \( k \) be the size of \( \pi \).

**Theorem (Féray (2013))**

We assume that \( \sigma_n \) follows the Ewens distribution with parameter \( \theta \). Then,

\[ \frac{\mathcal{H}_\Pi(\sigma_n) - \mathbb{E}(\mathcal{H}_\Pi(\sigma_n))}{n^{k - \frac{1}{2} - \text{card}(A)}} \xrightarrow[n \to \infty]{} \mathcal{N}(0, \sigma^2_{\Pi}). \]
Few cycles

Let $\sigma_n$ is conjugacy invariant of size $n$

**Theorem (Kammoun 2020)**

We assume that $\frac{\#(\sigma_n)}{\sqrt{n}} \frac{d}{n \to \infty} 0$.

Then, $\frac{\mathcal{N}(\sigma_n) - \mathbb{E}(\mathcal{N}(\sigma_n))}{n^{k-\frac{1}{2}} \text{card}(A)} \frac{d}{n \to \infty} \mathcal{N}(0, \sigma^2_{\Pi})$.

**Theorem (Hamaker and Rhoades (2022))**

We assume that: for all $i \#_i(\sigma_n) \frac{d}{n \to \infty} 0$.

Then, $\frac{\mathcal{N}(\sigma_n) - \mathbb{E}(\mathcal{N}(\sigma_n))}{n^{k-\frac{1}{2}} \text{card}(A)} \frac{d}{n \to \infty} \mathcal{N}(0, \sigma^2_{\Pi})$.

If we combine both techniques.

**Theorem (Not written anywhere)**

We assume that: for all $i \#_i(\sigma_n) \frac{d}{n \to \infty} 0$.

Then, $\frac{\mathcal{N}(\sigma_n) - \mathbb{E}(\mathcal{N}(\sigma_n))}{n^{k-\frac{1}{2}} \text{card}(A)} \frac{d}{n \to \infty} \mathcal{N}(0, \sigma^2_{\Pi})$.
Our result

Fix $\Pi = (\pi, A)$,

Theorem (Féray and Kammoun (2023))

We assume that $\sigma_n$ is conjugacy invariant of size $n$ and that $\frac{#1(\sigma_n)}{n} \xrightarrow{d} \alpha$, $\frac{#2(\sigma_n)}{n} \xrightarrow{d} \beta$. Then

$$\frac{\eta^\Pi(\sigma_n) - \mathbb{E}(\eta^\Pi(\sigma_n))}{n^{k-\frac{1}{2}-\text{card}(A)}} \xrightarrow{d} \mathcal{N}(0, \sigma^2_{\Pi,\alpha\beta}).$$

Moreover, if $A = \emptyset$, then $\sigma^2_{\Pi,\alpha,\beta} = 0$ if and only if $(\alpha, \beta) = (1, 0)$.

Remarks:

- Hofer (2018) implies that $\sigma^2_{\Pi,0,0} > 0$ for any $\Pi$.
- It is easy to see that $\sigma^2_{\Pi,1,0} = 0$ for any $\Pi$. (Identity)
- $\sigma^2_{\Pi,\alpha\beta}$ is a polynomial in $(\alpha & \beta)$. (Hamaker and Rhoades (2022))
- Dubach (2024) proved the same result for classical patterns ($A = \emptyset$) + speed of convergence.

Conjecture: for any $\Pi$, $\sigma^2_{\Pi,\alpha,\beta} = 0$ if and only if $(\alpha, \beta) = (1, 0)$.

Questions: for which patterns, $\sigma^2_{\Pi,\alpha,\beta}$ does not depend on $\beta$? (consecutive)?
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Comparison techniques

• Initially for the longest increasing subsequence / RSK (Kammoun 2018).

• Works for other combinatorial structures (coloured permutations, k-arrangements, etc.)

We give the proof of

**Theorem (Kammoun 2020)**

We assume that  \[ \frac{\#(\sigma_n)}{\sqrt{n}} \xrightarrow{d} 0. \]

Then,  \[ \frac{\Pi(\sigma_n) - \mathbb{E}(\Pi(\sigma_n))}{n^{k-\frac{1}{2} - \text{card}(A)}} \xrightarrow{d} \mathcal{N}(0, \sigma^2_{\Pi}). \]
Simple random walk a directed version of the Cayley graph of $\mathfrak{S}_n$.

- If we start from any conjugacy invariant measure, the stationary measure is Ewens with parameter 0.
- In each step, $\mathcal{N}^\Pi$ varies at most by $\frac{2}{k!} n^{k - \text{card}(A) - 1}$.

$$\left| \mathcal{N}^\Pi(\sigma_n) - \mathcal{N}^\Pi(\sigma_n^{\text{unif}}) \right| \leq \left| \mathcal{N}^\Pi(\sigma_n) - \mathcal{N}^\Pi(\sigma_{0,n}^{Ew}) \right| + \left| \mathcal{N}^\Pi(\sigma_{0,n}^{Ew}) - \mathcal{N}^\Pi(\sigma_n^{\text{unif}}) \right|$$

$$\leq \frac{2}{k!} n^{k - \text{card}(A) - 1} \left( \# \sigma_n + \# \sigma_n^{\text{unif}} \right) \approx \log(n)$$

We want that $\left| \mathcal{N}^\Pi(\sigma_n) - \mathcal{N}^\Pi(\sigma_n^{\text{unif}}) \right| = o(n^{k - \text{card}(A) - \frac{1}{2}})$.

It is sufficient that $\# \sigma_n = o(\sqrt{n})$. 
Weighted dependency graphs

Initially developed by Féray (2018).
Works for other combinatorial structures.
We give a proof of

**Theorem (Féray and Kammoun (2023))**

*We assume that* $\sigma_n$ *is conjugacy invariant of size* $n$ *and that* $\frac{\#_1(\sigma_n)}{n} \xrightarrow{d} \alpha$, $\frac{\#_2(\sigma_n)}{n} \xrightarrow{d} \beta$. *Then*

$$\frac{\mathcal{H}^\Pi(\sigma_n) - \mathbb{E}(\mathcal{H}^\Pi(\sigma_n))}{n^{k-\frac{1}{2} - \text{card}(A)}} \xrightarrow{d} \mathcal{N}(0, \sigma_{\Pi,\alpha,\beta}^2).$$
Cumulants

**Definition**

\[ \kappa_r(X_1, \ldots, X_r) = [t_1 t_2 \cdots t_r] \log(E(e^{\sum_{j=1}^n t_j X_j})) \]

For simplicity, we write \( \kappa_r(X) := \kappa_r(X, \cdots, X) \).

- \( X \sim \mathcal{N}(m, \sigma^2) \) if and only if for all \( r \geq 3 \), \( \kappa_r(X) = 0 \)
- If \( X_1 \) and \( X_2 \) are independent, then \( \kappa_r(X_1 + X_2) = \kappa_r(X_1) + \kappa_r(X_2) \)
- \( \kappa_r(X + C) = \kappa_r(X) \) if \( r \geq 2 \)
- \( \kappa_r(\alpha X) = \alpha^r \kappa_r(X) \)
- If \( \{X_1, \ldots X_i\} \) and \( \{Y_{i+1}, \ldots Y_r\} \) are independent (and non-empty), then \( \kappa_r(X_1, \ldots, X_i, Y_{i+1}, \ldots, Y_r) = 0 \)

**Proof of the CLT** For \( r \geq 3 \)

\[
K_r \left( \frac{\sum_{i=1}^n X_i - n \mathbb{E}(X_1)}{\sqrt{n}} \right) = K_r \left( \frac{\sum_{i=1}^n X_i}{\sqrt{n}} \right) = \frac{1}{n^{r/2}} \sum_{i=1}^n \kappa_r(X_i) = \frac{n}{n^{r/2}} \kappa_r(X_1) = o(1)
\]
Weak dependency

• If \{X_1, \ldots, X_r\} are "weakly dependent", then \(\kappa_r(X_1, \ldots, X_r) \approx 0\).

• Dependency graphs: a graph with weights on the edges. Vertices are indexed by random variables, and weights measure the "dependency".

• If the weights are sufficiently "small", we have a CLT for the sum of the variables.
Uniform Permutation

• Example: $\sigma_n$ is uniform and $A_{i,j} = 1[\sigma_n(i) = j]$.

• If $i \neq j$ and $k \neq m$, then

$$\mathbb{E}(A_{i,k}A_{j,m}) = \frac{1}{n(n-1)} \approx \frac{1}{n^2} = \mathbb{E}(A_{i,k})\mathbb{E}(A_{j,m}).$$

• if $k \neq m$, then $\mathbb{E}(A_{i,k}A_{i,m}) = 0$ and $\mathbb{E}(A_{i,k})\mathbb{E}(A_{j,m}) = \frac{1}{n^2}$.

For any $U = (i_\ell, j_\ell)_{1 \leq \ell \leq r}$, let $G(U)$, be the complete graph with vertices $U$ and the weight of $((i,j), (k,l))$ is

$$\begin{cases} 
1 & \text{if } i = k \text{ or } j = l \\
\frac{1}{n} & \text{otherwise.}
\end{cases}$$

For example, if $U = ((1,4), (1,2), (4,3), (1,2))$, $G(U)$
**Uniform Permutation**

**Theorem (Féray 2018)**

For all $r \geq 1$, there exists $C_r$ such that: For all integers $n$, for all $U = (i_\ell, j_\ell)_{1 \leq \ell \leq r}$

$$\kappa_r(A_{i_1,j_1}, \ldots, A_{i_r,j_r}) \leq C_r M(U) n^{-\text{card}(U)}$$

where

- $M(U)$ is the maximum weight of a spanning tree of $G(U)$.
- $\text{card}(U)$ is the number of distinct elements in $U$.

For example, if $U = ((1,4), (1,2), (4,3), (1,2))$, $G(U) =

![Graph](image)

For all $n$, $\kappa_r(A_{1,4}, A_{1,2}, A_{4,3}, A_{1,2}) \leq C_4 \frac{1}{n} n^{-3} = C_4 n^{-4}$
New graphs

- $G^1(U)$, the complete graph with vertices $U$ and the weight of $((i, j), (k, l))$ is 1 if $i = k$ or $j = l$ or $i = j$ or $k = l$, and $\frac{1}{n}$ otherwise.

For example, if $U = ((1, 4), (1, 2), (4, 3), (1, 2))$, $G^1(U) =$

- $G^2(U) := ([n], E = U)$

For example, if $U = ((1, 4), (1, 2), (4, 3), (1, 2))$, $G^2(U) =$
Uniform Permutation within a Conjugacy Class

\( \sigma^\lambda_n \) is uniform within the conjugacy class \( \lambda \) and \( A_{i,j} = 1[\sigma^\lambda_n(i) = j] \).

**Theorem (Féray and Kammoun 2023)**

For all \( r \geq 1 \), there exists \( C_r \) such that: For all integers \( n \), for all \( U = (i_\ell,j_\ell)_{1 \leq \ell \leq r} \)

\[
\kappa_r(A_{i_1,j_1}, \ldots, A_{i_r,j_r}) \leq C_r M(U) n^{CC(U) - \text{card}(U)}
\]

where

- \( M(U) \) is the maximum weight of a spanning tree of \( G^1(U) \), the complete graph with vertices \( U \) and the weight of \( ((i,j),(k,l)) \) is 1 if \( i = k \) or \( j = l \) or \( i = j \) or \( k = l \), and \( \frac{1}{n} \) otherwise.
- \( \text{card}(U) \) is the number of distinct elements in \( U \).
- \( CC(U) \) the number of nontrivial connected components in the graph \( G^2(U) = ([n], E = U) \)
Application: Patterns

If we denote by $X^{(\pi,A)}$ the number of occurrences of the pattern $(\pi,A)$, we have

$$X^{(\pi,A)}(\sigma_n) = \sum_{i_1<\ldots<i_k} \sum_{j_1,\ldots,j_k \text{ for } s \in A} A_{i_1,j_1} \cdots A_{i_k,j_k}.$$

To conclude: The magic of weighted dependency graphs: We can "easily" move from controlling mixed cumulants of $\{A_{i,j} : (i,j) \in [n]^2\}$ to controlling mixed cumulants of $\{A_{i_1,i_2} \cdots A_{i_r,j_r} : (i_1,j_1,\ldots,i_r,j_r) \in [n]^{2r}\}$. We obtain

$$\kappa_r(X^{(\pi,A)}(\sigma_n)) \leq C_{k,r} n^{r(k-\text{card}(A)-1)+1},$$

and thus

$$\kappa_r \left( \frac{X^{(\pi,A)}(\sigma_n) - \mathbb{E}(X^{(\pi,A)}(\sigma_n))}{n^{k-\text{card}(A)-\frac{1}{2}}} \right) \leq C_{k,r} n^{1-\frac{r}{2}}.$$
Motivation: universality

Central Limit Theorem
Let \( X_1, X_2, \ldots, X_n \) be i.i.d with \( \text{Var}(X_i) = \sigma^2 < +\infty \). Then,

\[
\sqrt{n}\left(\frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}(X_1)\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2)
\]

The limit is universal (does not depend on the distribution of \( X_i \)).

Symmetry/independence + control = universality
Fisher-Tippett-Gnedenko Theorem

Let \( X_1, X_2, \ldots, X_n \) be i.i.d and \( M_n = \max(X_1, X_2, \ldots, X_n) \).
Suppose there exist constants \( a_n > 0 \) and \( b_n \) such that, for every real \( x \),

\[ \mathbb{P} \left( \frac{M_n - b_n}{a_n} \leq x \right) \to G(x) \]

where \( G(x) \) is a non-degenerate cumulative distribution function. Then, \( G \) is
the cumulative distribution function of a Gumbel, Fréchet, or Weibull
variable.
The limit fluctuations depend on the tail of the distribution of \( X_1 \).

**Symmetry/Independence + Control = Universality**
Let’s define the symmetric matrix $M$ as

$$M = \frac{1}{\sqrt{n}} \begin{bmatrix}
a_{1,1} & a_{1,2} & \ldots & \ldots & a_{1,n} \\
a_{1,2} & a_{2,2} & \ldots & \ldots & a_{2,n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
a_{1,n} & a_{2,n} & \ldots & \ldots & a_{n,n}
\end{bmatrix}$$

The entries $\{a_{i,j}\}_{1 \leq i \leq j \leq n}$ are i.i.d. such that $\mathbb{E}(a_{1,1}) = 0$ and $\mathbb{E}(a_{1,1}^2) = 1$.

Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $M$. 
Histogram of Eigenvalues

Gaussian entries

Entries 1 or −1
"The histogram of eigenvalues is not far from a semi-circle"

**Theorem**

The empirical spectral measure of the eigenvalues of $M$

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i},$$

converges weakly to the semi-circular law of Wigner as $n$ tends to infinity.
But also*,

- The largest eigenvalue converges to 2
- The fluctuations of the largest eigenvalue are of Tracy-Widom type
- Large deviations of the largest eigenvalues are universal
- The joint limit fluctuations of the first $k$ eigenvalues are universal
- The local limit laws are universal
- The fluctuations of the number of points in $[a,b]$ are universal

*Some conditions apply on the moments / the tail of the distribution

And for random permutations?
Longest Decreasing Subsequence

- \((\sigma(i_1), \ldots, \sigma(i_k))\) is a decreasing subsequence of \(\sigma\) if \(i_1 < i_2 < \cdots < i_k\) and \(\sigma(i_1) > \cdots > \sigma(i_k)\).
Longest Decreasing Subsequence

- $(\sigma(i_1), \ldots, \sigma(i_k))$ is a decreasing subsequence of $\sigma$ if $i_1 < i_2 < \cdots < i_k$ and $\sigma(i_1) > \cdots > \sigma(i_k)$.
- LDS($\sigma$): The length of the longest decreasing subsequence of $\sigma$. 

Example: $\sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 6 \ 1 \ 8 \ 7 \ 5 \ 2 \ 4 \ 3)$

LDS(\sigma) = 5
Longest Decreasing Subsequence

- $(\sigma(i_1), \ldots, \sigma(i_k))$ is a decreasing subsequence of $\sigma$ if $i_1 < i_2 < \cdots < i_k$ and $\sigma(i_1) > \cdots > \sigma(i_k)$.
- $\text{LDS}(\sigma)$: The length of the longest decreasing subsequence of $\sigma$.
- Example:

\[
\sigma = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
6 & 1 & 8 & 7 & 5 & 2 & 4 & 3
\end{pmatrix}
\]

$LDS(\sigma) = 5$. 
Longest Decreasing Subsequence: Universality

We assume that $\sigma_n$ is conjugation invariant and $\frac{\#_1(\sigma_n)}{n} \to \alpha$

Theorem (Dubach (2024+))

$$\frac{\text{LDS}(\sigma_n)}{\sqrt{n}} \xrightarrow{d} 2 \sqrt{1 - \alpha}$$

Theorem (Kammoun 2018)

If

$$n^{-\frac{1}{6}} \min_{1 \leq i \leq n} \left( \left( \sum_{j=1}^{i} #_j(\sigma_n) \right) + \frac{\sqrt{n}}{i} \sum_{j=i+1}^{n} #_j(\sigma_n) \right) \xrightarrow{p}$$

0, then,

$$\frac{\text{LDS}(\sigma_n) - 2 \sqrt{n}}{\sqrt[6]{n}} \xrightarrow{d} \text{Tracy Widom}$$
**Theorem (Guionnet, Kammoun 2023)**

If $\sigma_n$ is conjugacy invariant and $\#(\sigma_n) = o(\sqrt{n})$. Then, $\frac{\text{LDS}(\sigma_n)}{\sqrt{n}}$ satisfies a LD principle

- with speed $\sqrt{n}$ and rate function $J_{\text{LDS}, \frac{1}{2}}$.
- with speed $n$ and rate function $J_{\text{LDS}, 1}$

With,

$$J_{\text{LDS}, \frac{1}{2}}(x) = \begin{cases} 2x \cosh^{-1} \frac{x}{2} & \text{if } x > 2 \\ +\infty & \text{if } x \leq 2 \end{cases}.$$  

$$J_{\text{LDS}, 1}(x) = \begin{cases} -1 + \frac{x^2}{4} + 2 \ln \left(\frac{x}{2}\right) - \left(2 + \frac{x^2}{2}\right) \ln \left(\frac{2x^2}{4+x^2}\right) & \text{if } 0 < x \leq 2 \\ 0 & \text{if } x > 2 \\ +\infty & \text{if } x \leq 0 \end{cases}.$$
In other words: if $\sigma_n$ is conjugation invariant and $\#(\sigma)$ "is low" then

$$-\log \left( \mathbb{P} \left( \frac{\text{LDS}(\sigma_n)}{\sqrt{n}} \approx x \right) \right) \approx \begin{cases} 
\left( -1 + \frac{x^2}{4} + 2 \ln \left( \frac{x}{2} \right) - \left( 2 + \frac{x^2}{2} \right) \ln \left( \frac{2x^2}{4+x^2} \right) \right) n & \text{if } x \in ]0, 2[ \\
2x \cosh^{-1} \left( \frac{x}{2} \right) \sqrt{n} & \text{if } x > 2 \\
+\infty & \text{if } x \leq 0 \\
0 & \text{if } x = 2
\end{cases}$$

The same phenomenon appears for $\lambda_1$ (Wigner Matrices).
What we know

**Type 1: Local events**
- $\mathbb{P}(S \subset D(\sigma))$
- $\mathbb{P}(\sigma(10) > 10)$

**Type 2: LLN / first order / global convergence**
- $\frac{\gamma_{\Pi}}{n^{k - \text{card}(A)}}$
- LDS $\sqrt{n}$

The limit depends only on $\frac{\#_1}{n}$

**Type 3: fluctuations (Poisson / Normal)**
- $\text{Tr}((\sigma_n \rho_n \pi_n \sigma_n^{-1} \rho_n^{-1} \pi_n)^{2024})$
- $\frac{\gamma_{\Pi} - E(\gamma_{\Pi})}{n^{k - \text{card}(A) - \frac{1}{2}}}$

The limit depends on $\frac{\#_1}{n}$ and $\frac{\#_2}{n^\alpha}$ for some $\alpha$

**Type 4: others**
- LDS $-2 \sqrt{n} \frac{1}{n^{\frac{1}{6}}}$

- Large deviations.

Universality if $\#$ is low.
There is still much work to be done.
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Merci de votre attention