On the solutions of Knizhnik-Zamolodchikov differential equations by noncommutative Picard-Vessiot theory

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1. **Abstract**: In this work, basing on the algebraic combinatorics on noncommutative formal series with holomorphic coefficients and, on the other hand, a Picard-Vessiot theory of noncommutative differential equations, we give a recursive construction of solutions of Knizhnik-Zamolodchikov equations satisfying asymptotic conditions.
Knizhnik-Zamolodchikov differential equations

Let \((\mathcal{H}(\mathcal{V}), 1_{\mathcal{H}(\mathcal{V})})\) be the ring of holomorphic functions over the manifold \(\mathcal{V} = \mathbb{C}_n^*\), the universal covering of the configuration space of \(n\) points, i.e.
\[
\mathbb{C}_n^* := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}.
\]
Let \(\mathcal{H}(\mathcal{V})\langle\langle T_n \rangle\rangle\) be the ring of noncommutative series over the alphabet \(T_n := \{t_{i,j} \}_{1 \leq i < j \leq n}\) and with coefficients in \(\mathcal{H}(\mathcal{V})\).
The following noncommutative differential equation is so called \(KZ_n\)
\[
dF(z) = \Omega_n(z)F(z), \quad \text{where} \quad \Omega_n(z) := \sum_{1 \leq i < j \leq n} \frac{t_{i,j}}{2i\pi} d\log(z_i - z_j)
\]
for which solutions can be computed by convergent iterations, for the discrete topology\(^2\) of pointwise convergence over \(\mathcal{H}(\mathcal{V})\langle\langle T_n \rangle\rangle\), for instance
\[
F_0(z) = 1_{\mathcal{H}(\mathcal{V})} \quad \text{and} \quad F_l(z) = \int_{z_0}^z \Omega_n(s)F_{l-1}(s).
\]

Remark (dévissage)
\[
\Omega_n(z) = \sum_{1 \leq i < j \leq n-1} \frac{t_{i,j}}{2i\pi} \frac{d(z_j - z_i)}{z_j - z_i} + \sum_{j=1}^{n-2} \frac{t_{i,j}}{2i\pi} \frac{d(z_n - z_j)}{z_n - z_j} + \frac{t_{n-1,n}}{2i\pi} \frac{d(z_n - z_{n-1})}{z_n - z_{n-1}}.
\]

for \(z_n \to z_{n-1}\), c.f. hyperlogarithms

2. \(\forall S, T \in \mathcal{H}(\mathcal{V})\langle\langle T_n \rangle\rangle, d(S, T) = 2^{\varpi(S-T)}\), where \(\varpi\) denotes the valuation, i.e.
   If \(S \neq 0\) then \(\varpi(S) = \inf\{|w|, w \in \text{supp}(S)\}\) else \(+\infty\).
Quadratic relations among $\{t_{i,j}\}_{1 \leq i < j \leq n}$

According to Drinfel’d, $KZ_n$ is completely integrable if $\Omega_n(z)$ is flat, i.e.

$$d\Omega_n(z) - \Omega_n(z) \wedge \Omega_n(z) = 0.$$ 

It turns out that this condition induces the following quadratic relations in $\{t_{i,j}\}_{1 \leq i < j \leq n}$:

$$\mathcal{R}_n = \begin{cases} 
[t_{i,k} + t_{j,k}, t_{i,j}] = 0 & \text{for distinct } i, j, k \text{ and } 1 \leq i < j < k \leq n, \\
[t_{i,j} + t_{i,k}, t_{j,k}] = 0 & \text{for distinct } i, j, k \text{ and } 1 \leq i < j < k \leq n, \\
[t_{i,j}, t_{k,l}] = 0 & \text{for distinct } i, j, k, l \text{ and } \{1 \leq i < j \leq n, 1 \leq k < l \leq n\},
\end{cases}$$

generating the Lie ideal $\mathcal{I}_{\mathcal{R}_n}$.

Solutions of $KZ_n$ belong now to $\mathcal{H}(\mathcal{V})\langle\langle T_n \rangle\rangle/\mathcal{I}_{\mathcal{R}_n}$. 
Examples of $KZ_n$

Example ($KZ_2$ : trivial case)
One has $\mathcal{T}_2 = \{t_{1,2}\}$ and $dF(z) = \Omega_2(z)F(z)$, where
\[ \Omega_2(z) = \left(t_{1,2}/2i\pi\right)d\log(z_1 - z_2), \]
is $F(z_1, z_2) = e^{t_{1,2}/2i\pi}\log(z_1 - z_2) = (z_1 - z_2)^{t_{1,2}/2i\pi} \in \mathcal{H}(\mathbb{C}_*^\infty)\langle\langle\mathcal{T}_2\rangle\rangle$.

Example ($KZ_3$ : simplest non-trivial case)
One has $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and $dF(z) = \Omega_3(z)F(z)$, where
\[ \Omega_3(z) = \frac{1}{2i\pi} \left( t_{1,2} \frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3} \frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3} \frac{d(z_2 - z_3)}{z_2 - z_3} \right). \]
Drinfel’d proposed a following solution on $]0, 1[$
\[ F(z) = (z_1 - z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2i\pi} G\left(\frac{z_3 - z_2}{z_1 - z_2}\right), \]
where $G$ satisfies the following noncommutative differential equation
\[ (DE1) \quad dG(s) = \left(A \frac{ds}{s} - B \frac{ds}{1-s}\right) G(s), \quad \begin{cases} A := t_{1,2}/2i\pi, \\ B := t_{2,3}/2i\pi. \end{cases} \]
He stated that there is a unique solution $G_0$ (resp. $G_1$) satisfying
$G_0(s) \sim_0 e^{A\log(s)} = s^A$ (resp. $G_1(s) \sim_1 e^{-B\log(1-s)} = (1-s)^{-B}$),
and a unique series $\Phi_{KZ}$, so-called Drinfel’d series $^3$, s.t. $G_0 = G_1 \Phi_{KZ}$.

\[ \text{3. Cartier, Gonzalez-Lorca, Racinet defined associators as group like series satisfying the relations duality, pentagonal and hexagonal : } \Phi_{KZ} \text{ is an associator.} \]
**log Φ_{KZ} determined by Drinfel’d**

1. Assuming that \([A, B] = 0\), he proposed an approximation solution for (DE1) over \([0, 1]\), \(z^A(1 - z)^B\) (a group like series) satisfying standard asymptotic conditions. Hence, the logarithm of such approximation solution of \(KZ_3\) belongs to

\[
\mathcal{L}ie_{\mathcal{H}(\mathbb{C}^3_*)} \langle \langle t_{1,2}, t_{1,3}, t_{2,3} \rangle \rangle / [\mathcal{L}ie_{\mathcal{H}(\mathbb{C}^3_*)} \langle \langle t_{1,2}, t_{1,3} \rangle \rangle, \mathcal{L}ie_{\mathcal{H}(\mathbb{C}^3_*)} \langle \langle t_{1,2}, t_{2,3} \rangle \rangle],
\]

2. He also proposed, over \([0, 1]\),

\[
G_0(z) = z^A(1 - z)^B V_0(z) \quad \text{and} \quad G_1(z) = z^A(1 - z)^B V_1(z).
\]

\(V_0\) and \(V_1\) have continuous extensions to \([0, 1]\) and are group like solutions of the following noncommutative differential equation

\[
(\text{DE2}) \quad dS(z) = Q(z)S(z), \quad Q(z) := e^{\text{ad}_B - \log(1 - z)B} e^{\text{ad}_A - \log(z)A} \frac{B}{z - 1} \in \mathfrak{p},
\]

with the initial conditions \(V_0(0) = 1\), \(V_1(1) = 1\) and \(\mathfrak{p}\) is the topological free Lie algebra generated by \(\{\text{ad}^k_A \text{ad}^l_B [A, B]\}_{k, l \geq 0}\).

3. Since \(G_9 = G_1 \Phi_{KZ}\) then the group like series \(\Phi_{KZ}\) equals to \(V(0)V(1)^{-1}\), where \(V\) is a solution of (DE2) and then the coefficients \(\{c_{k,l}\}_{k, l \geq 0}\) of \(\log \Phi_{KZ}\) are obtained, in \(\mathfrak{p}/[\mathfrak{p}, \mathfrak{p}]\), by

\[
\log \Phi_{KZ} = \sum_{k, l \geq 0} c_{k,l} B^{k+1} A^{l+1} = \int_0^1 Q(z)dz \mod [\mathfrak{p}, \mathfrak{p}].
\]
Polylogarithms

Denoting \((X^*, 1_{X^*})\) the monoid generated by \(X = \{x_0, x_1\}\), recall that

\[ L(s) := \sum_{w \in X^*} \text{Li}_w(s)w \in \mathcal{H}(\tilde{B})\langle X \rangle, \quad \text{where} \quad B := \mathbb{C} \setminus \{0, 1\} \]

where \(\text{Li}_\bullet\) is the character of \((\mathcal{H}(\tilde{B})\langle X \rangle, \sqcup, 1_{X^*})\) defined by

\[ \text{Li}_{1_{X^*}} = 1_{\mathcal{H}(\tilde{B})}, \quad \text{Li}_{x_0}(s) = \log(s), \quad \text{Li}_{x_1}(s) = \log(1 - s) \]

and, for any \(x_iw \in \mathcal{L}ynX \setminus X\),

\[ \text{Li}_{x_iw}(s) = \int^s_0 \omega_i(\sigma)\text{Li}_w(\sigma), \quad \text{where} \quad \left\{ \begin{array}{l} \omega_0(s) = ds/s, \\ \omega_1(s) = ds/(1 - s). \end{array} \right. \]

\(\{\text{Li}_l\}_{l \in \mathcal{L}ynX}\) (resp. \(\{\text{Li}_w\}_{w \in X^*}\)) are \(\mathbb{C}\)-algebraically (resp. linearly) free.

By the Friedrichs criteron, \(L\) is group like. Thus\(^4\),

\[ L(s) = \prod_{l \in \mathcal{L}ynX} e^{\text{Li}_l(s)P_l} \quad \text{and then} \quad \left\{ \begin{array}{l} \lim_{z \to 0} L(s) e^{-x_0 \log z} = 1, \\ \lim_{z \to 1} e^{x_1 \log(1-z)} L(s) = \Phi_{KZ}, \end{array} \right. \]

and \(\Phi_{KZ}\) admits \(\{\text{Li}_l(1)\}_{l \in \mathcal{L}ynX \setminus X}\) as convergent locale coordinates

\[ \Phi_{KZ} := \prod_{l \in \mathcal{L}ynX \setminus X} e^{\text{Li}_l(1)P_l} \in \mathbb{R}\langle X \rangle, \quad \text{for} \quad \left\{ \begin{array}{l} x_0 = \frac{t_{1,2}}{2i\pi}, \\ x_1 = -\frac{t_{2,3}}{2i\pi}. \end{array} \right. \]

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\(^4\) \(\{P_l\}_{l \in \mathcal{L}ynT_n}\) is the basis of \(\mathcal{L}ie_{\mathcal{H}(\tilde{B})}\langle X \rangle\) over which are constructed the PBW basis \(\{P_w\}_{w \in T^*_n} \) of \(\mathcal{U}(\mathcal{L}ie_{\mathcal{H}(\tilde{B})}\langle X \rangle)\) and its dual, \(\{S_w\}_{w \in X^*}\), containing the pure transcendence basis \(\{S_l\}_{l \in \mathcal{L}ynX}\).
BACKGROUND ON PV THEORY OF NONCOMMUTATIVE DIFFERENTIAL EQUATIONS
Differential ring of holomorphic functions

- \( \mathcal{V} \): simply connected manifold of \( \mathbb{C}^n \) (\( n > 0 \)).

- \( \mathcal{A} = (\mathcal{H}(\mathcal{V}), \partial_1, \ldots, \partial_n) \): the differential ring of holomorphic functions on \( \mathcal{V} \) and equipped \( 1_{\mathcal{H}(\mathcal{V})} \) as the neutral element.
  
  For any \( f \in \mathcal{H}(\mathcal{V}) \), one has \( df = (\partial_1 f)dz_1 + \ldots + (\partial_n f)dz_n \).

- Let \( \mathcal{C} \) be a sub differential ring of \( \mathcal{A} \) (i.e. \( \partial_i \mathcal{C} \subset \mathcal{C} \), for \( 1 \leq i \leq n \)) and let \( \varsigma \leadsto z \) denotes a path (with fixed endpoints, \((\varsigma, z)\)) over \( \mathcal{V} \), i.e. the parametrized curve \( \gamma : [0, 1] \rightarrow \mathcal{V} \) such that \( \gamma(0) = \varsigma = (\varsigma_1, \ldots, \varsigma_n) \) and \( \gamma(1) = z = (z_1, \ldots, z_n) \).

- For any integers \( i, j \) such that \( 1 \leq i < j \leq n \), let \( \omega_{i,j} \) denote the 1-differential forms\(^5\), in \( \Omega^1(\mathcal{V}) \), \( \omega_{i,j} = d\xi_{i,j} \), with \( \xi_{i,j} \in \mathcal{C} \).

**Example** \( (\xi_{i,j}(z) = \log(z_i - z_j), 1 \leq i < j \leq n) \)

Let \( \mathcal{C}_0 := \mathbb{C}[(\partial_1 \xi_{i,j})^{\pm1}, \ldots, (\partial_n \xi_{i,j})^{\pm1}]_{1 \leq i < j \leq n} \).

Then \( \mathcal{C}_0 \) is a sub differential ring of \( \mathcal{A} \).

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\( ^5 \) Over \( \mathcal{V} \), the holomorphic function \( \xi_{i,j} \) is called a primitive for \( \omega_{i,j} \) which is said to be a exact form and then is a closed form (i.e. \( d\omega_{i,j} = 0 \)).
Notations

► \((\mathcal{T}_n^*, 1_{\mathcal{T}_n^*})\) is the free monoid generated by \(\mathcal{T}_n\).

► \(A\langle\langle \mathcal{T}_n \rangle\rangle\) (resp. \(A\langle \mathcal{T}_n \rangle\)) is the set of series (resp. polynomials) over \(\mathcal{T}_n\) with coefficients in \(A\). \(\text{Lyn} \mathcal{T}_n\) (resp. \(\text{Lyn} \mathcal{T}\)) is the set of Lyndon words over \(\mathcal{T}_n\) (resp. \(\mathcal{T}\)).

► \(T_k := \{t_{j,k}\}_{1 \leq j \leq k-1}\), \(\mathcal{T} := \{T_2, \ldots, T_n\}\) s.t. \(T_k = T_k \sqcup T_{k-1}\), \(k \leq n\). \(|\mathcal{T}_n| = n(n-1)/2\) and \(|\mathcal{T}_n| = n - 1\). If \(n \geq 4\) then \(|\mathcal{T}_{n-1}| \geq |\mathcal{T}_n|\).

Example

► \(\mathcal{T}_5 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{1,5}, t_{2,3}, t_{2,4}, t_{2,5}, t_{3,4}, t_{3,5}, t_{4,4}\}\), one has \(T_5 = \{t_{1,5}, t_{2,5}, t_{3,5}, t_{4,5}\}\) and \(\mathcal{T}_4\).

► \(\mathcal{T}_4 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,4}\}\), one has \(T_4 = \{t_{1,4}, t_{2,4}, t_{3,4}\}\) and \(\mathcal{T}_3\).

► \(\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}\), one has \(T_3 = \{t_{1,3}, t_{2,3}\}\) and \(\mathcal{T}_2 = \{t_{1,2}\}\).

► In \((A\langle\langle \mathcal{T}_n \rangle\rangle, \partial_1, \ldots, \partial_n)\), for any \(S \in A\langle\langle \mathcal{T}_n \rangle\rangle\), one defines

\[
\partial_i S = \sum_{w \in \mathcal{T}_n^*} (\partial_i \langle S \mid w \rangle) w \quad \text{and} \quad dS = \sum_{i=1}^{n} (\partial_i S) dz_i.
\]

\(\text{Const}(A) = \mathbb{C}.1_{\mathcal{H}(\Omega)}\) and \(\text{Const}(A\langle\langle \mathcal{T}_n \rangle\rangle) = \mathbb{C}\langle\langle \mathcal{T}_n \rangle\rangle\).
Lazard elimination: $\mathcal{L}ie_A\langle T_n \rangle = \mathcal{I}_n \oplus \mathcal{L}ie_A\langle T_n \rangle$

Let $\rho$ the right normed bracketing which is the unique linear endomorphism of $A\langle\langle T_n \rangle\rangle$ defined, by $\rho(1_{T_n^\ast}) = 0$ and, for $w = t_1 \ldots t_k \in T_n^\ast$, by

$$\rho(w) = [t_1, [\ldots, [t_{k-1}, t_k]] \ldots] = \text{ad}_{t_1} \ldots \text{ad}_{t_{k-1}} t_k.$$ 

$\mathcal{I}_n$ : Lie subalg. generated by $\{\text{ad}_{T_n}^k t_{i,j}\}_{t_i,j \in T_{n-1}} = \{(-1)^{|w|} \rho(vt)/|v|!\}_{v \in T_{n-1}^\ast}$.

By PBW, $U(\mathcal{I}_n)$ is freely generated by

$$\{\text{ad}_{T_n}^{k_1} t_1 \ldots \text{ad}_{T_n}^{k_p} t_p\}_{t_1, \ldots, t_p \in T_{n-1}}^{k_1, \ldots, k_p \geq 0, p \geq 0}$$

which are associated to the following family of polynomials of $U(\mathcal{I}_n)^\vee$

$$\{t_1(\bar{T}_n^{k_1} \bowtie (\ldots \bowtie (t_p \bar{T}_n^{k_p}) \ldots))\}_{t_1, \ldots, t_p \in T_{n-1}}^{k_1, \ldots, k_p \geq 0, p \geq 0},$$

$$\{t_1(\bar{v}_1 \bowtie (\ldots \bowtie (t_p \bar{v}_p) \ldots))\}_{v_1 \in T_n^{k_1}, \ldots, v_p \in T_n^{k_p}, t_1, \ldots, t_k \in T_{n-1}}^{k_1, \ldots, k_p \geq 0, p \geq 0},$$

$$\{(t_1 \bar{v}_1) \circ \ldots \circ (t_p \bar{v}_p)\}_{v_1 \in T_n^{k_1}, \ldots, v_p \in T_n^{k_p}, t_1, \ldots, t_k \in T_{n-1}}^{k_1, \ldots, k_p \geq 0, p \geq 0},$$

$$\{(t_1 \bar{T}_n^{k_1}) \circ \ldots \circ (t_p \bar{T}_n^{k_p})\}_{t_1, \ldots, t_p \in T_{n-1}}^{k_1, \ldots, k_p \geq 0, p \geq 0},$$

where $\bar{T}_n^k = \{\bar{v} \in T_n^k, |v| = k\}$ and the composite operator $\circ$ is defined, for any $H$ and $R \in A\langle\langle T_n \rangle\rangle$ and $t \in T_{n-1}$, by

If $R \neq 1_{T_n^\ast}$ then $(tH) \circ R = t(H \bowtie R)$ else $(tH) \circ R = tH$.

6. $\bar{v}$ is the polynomial $t_1 \bowtie \ldots \bowtie t_k$ associated to $v = t_1 \ldots t_k.$
Lexicographic ordering

\( \text{Lie}_A\langle T_n \rangle \) is the set of Lie polynomials over \( T_n \) with coefficients in \( A \) and is equipped with the basis \( \{ P_l \}_{l \in \text{Lyn}T_n} \) over which are constructed the PBW basis \( \{ P_w \}_{w \in T_n^*} \) of \( \mathcal{U}(\text{Lie}_A\langle T_n \rangle) \) and its dual, \( \{ S_w \}_{w \in T_n^*} \), containing the pure transcendence basis \( \{ S_l \}_{l \in \text{Lyn}T_n} \) of \( \mathcal{U}(\text{Lie}_A\langle T_n \rangle), \sqcup, 1_{T_n^*} \).

Example (in \( KZ_3 \), \( T_3 = \{ t_{1,2}, t_{1,3}, t_{2,3} \} \) and \( t_{1,2} < t_{1,3} < t_{2,3} \))

\[ \forall k \geq 0, i = 1 \text{ or } 2, \quad t_{1,2}^k t_{i,3} \in \text{Lyn}T_3, \quad P_{t_{1,2}^k t_{i,3}} = \text{ad}_{t_{1,2}}^k t_{i,3}, \quad S_{t_{1,2}^k t_{i,3}} = t_{1,2}^k t_{i,3}. \]

In the sequel, let \( \text{Lyn}T_n \) (resp. \( T_k \)) be the set of Lyndon words over \( T_n \) (resp. \( T_k \)) equipped the following total order over \( T_k \) (\( n \geq k \geq 2 \)) :

\[ t_{1,k} \succ \ldots \succ t_{k-1,k}, \quad T_2 \succ \ldots \succ T_n, \quad \text{Lyn}T_2 \succ \ldots \succ \text{Lyn}T_n. \]

By the standard factorization \(^8\) of Lyndon words, one has

\[ \text{Lyn}T_{n-1} \succ \text{Lyn}T_n \cdot \text{Lyn}T_{n-1} \succ \text{Lyn}T_n, \]

More generally, for any \((t_1, t_2) \in T_{k_1} \times T_{k_2}, 2 \leq k_1 < k_2 \leq n\), one also has

\[ t_2 t_1 \in \text{Lyn}T_{k_2} \subset \text{Lyn}T_n \quad \text{and} \quad t_2 \prec t_2 t_1 \prec t_1. \]

\(^7\) in which one defines \( \Delta_{\sqcup} x = x \otimes 1_{T_n^*} + 1_{T_n^*} \otimes x \), or equivalently,

\[ u \sqcup 1_{T_n^*} = 1_{T_n^*} \sqcup u = u \quad \text{and} \quad xu \sqcup yv = x(u \sqcup yv) + y(xu \sqcup v). \]

\(^8\) \( i.e. st(l) = (l_1, l_2), \) where \( l_2 \) is the longest nontrivial proper right factor of a Lyndon word \( l \), or equivalently, its smallest such for the lexicographic ordering.
1. If $l \in \mathcal{L}yn T_{k-1}$ and $t \in T_k$, $2 \leq k \leq n$ then $tl \in \mathcal{L}yn T_n$ and $t < tl < l$.

2. If $l_1 \in \mathcal{L}yn T_{k_1}$ and $l_2 \in \mathcal{L}yn T_{k_2}$ (for $2 \leq k_1 < k_2 \leq n$) then $l_2 l_1 \in \mathcal{L}yn T_{k_2} \subset \mathcal{L}yn T_n$ and $l_2 < l_2 l_1 < l_1$.

3. If $l_1 \in \mathcal{L}yn T_k$ and $l_2 \in \mathcal{L}yn T_{k-1}$ (for $2 \leq k_1 < k_2 \leq n$) then $l_1 l_2 \in \mathcal{L}yn T_n$ and $l_1 < l_1 l_2 < l_2$.

In $\mathcal{A}(T_n) \hat{\otimes} \mathcal{A}(T_n)$, let $\nabla S = S - 1_{T_n^*} \otimes 1_{T_n^*}$. The diagonal series is defined by

$$D_{T_n} := M^*,$$

with

$$M := \sum_{t \in T_n} t \otimes t,$$

and is the unique solution of $\nabla S = M S$ and $\nabla S = S M$. Then

$$D_{T_n} = D_{T_n-1} \left( \prod_{l_1 = l, l_2 \in \mathcal{L}yn T_{n-1}, l_1 \in \mathcal{L}yn T_n} e^{S_l \otimes P_l} \right) D_{T_n}, \quad \text{for} \quad n > 2.$$
More about notations

Let us back to the relations

\[ \mathcal{R}_n = \begin{cases} 
[t_{i,k} + t_{j,k}, t_{i,j}] = 0 & \text{for distinct } i, j, k \quad \text{and } 1 \leq i < j < k \leq n, \\
[t_{i,j} + t_{i,k}, t_{j,k}] = 0 & \text{for distinct } i, j, k \quad \text{and } 1 \leq i < j < k \leq n, \\
[t_{i,j}, t_{k,l}] = 0 & \text{for distinct } i, j, k, l \quad \text{and } \{ 1 \leq i < j \leq n, 1 \leq k < l \leq n, \}.
\]

generating the Lie ideal \( \mathcal{J}_{\mathcal{R}_n} \).

- The monoid (resp. the set of Lyndon words) generated by \( T_n \) satisfying the relations \( \mathcal{R}_n \) is denoted by \( \langle T_n^*; \mathcal{J}_{\mathcal{R}_n} \rangle \) (resp. \( \langle \text{Lyn}T_n; \mathcal{J}_{\mathcal{R}_n} \rangle \)).

- The set of noncommutative polynomials (resp. series) with coefficients in \( A \), over \( T_n \), satisfying \( \mathcal{R}_n \), is denoted by \( A\langle T_n \rangle / \mathcal{J}_{\mathcal{R}_n} \) (resp. \( A\langle \langle T_n \rangle \rangle / \mathcal{J}_{\mathcal{R}_n} \)).

- The set of Lie polynomials (resp. Lie series) with coefficients in \( A \), over \( T_n \), satisfying \( \mathcal{R}_n \), is denoted by \( \text{Lie}_A \langle T_n \rangle / \mathcal{J}_{\mathcal{R}_n} \) (resp. \( \text{Lie}_A \langle \langle T_n \rangle \rangle / \mathcal{J}_{\mathcal{R}_n} \)).

- \( H_\boxplus (T_n) / \mathcal{J}_{\mathcal{R}_n} \) denotes \( (A\langle T_n \rangle / \mathcal{J}_{\mathcal{R}_n}, \text{conc}, \Delta_\boxplus, 1_{T^*_n}) \).
Iterated integrals and Chen series

The iterated integral associated, of the 1-differential forms \( \{ \omega_{i,j} \}_{1 \leq i < j \leq n} \) and along the path \( \varsigma \leadsto z \), is given by \( \alpha^\varsigma(1_{T^*_n}) = 1_{\mathcal{H}(V)} \) and, for any \( w = t_{i_1,j_1} t_{i_2,j_2} \cdots t_{i_k,j_k} \in T^*_n \),

\[
\alpha^\varsigma(w) := \int^z_\varsigma \omega_{i_1,j_1}(s_1) \int^s_\varsigma \omega_{i_2,j_2}(s_2) \cdots \int^{s_k-1}_\varsigma \omega_{i_k,j_k}(s_k) \in \mathcal{H}(V),
\]

where \( (\varsigma, s_1 \ldots, s_{k-1}, z) \) is a subdivision of \( \varsigma \leadsto z \).

The Chen series, of the differential forms \( \{ \omega_{i,j} \}_{1 \leq i < j \leq n} \) and along a path \( \varsigma \leadsto z \), is the following noncommutative generating series

\[
C_{\varsigma \leadsto z} := \sum_{w \in T^*_n} \alpha^\varsigma(w) w \in \mathcal{H}(V) \langle \langle T^*_n \rangle \rangle.
\]

Proposition

1. \( \forall u, v \text{ in } T^*_n, \alpha^\varsigma(u \uplus v) = \alpha^\varsigma(u) \alpha^\varsigma(v) \) (Chen’s lemma).

2. \( \forall t \in T_n, k \geq 0, \alpha^\varsigma(t^k) = (\alpha^\varsigma(t))^k / k! \) and then \( \alpha^\varsigma(t^*) = e^{\alpha^\varsigma(t)} \).

3. For any compact \( K \subset V \), there is \( c > 0 \) and a morphism of monoids \( \mu : T^*_n \rightarrow \mathbb{R}_{\geq 0} \) s.t. \( \|C_{\varsigma \leadsto z} w\|_K \leq c \mu(w) |w|^{-1} \), for \( w \in T^*_n \), and then \( C_{\varsigma \leadsto z} \) is said to be exponentially bounded from above.
Basic triangular theorem over a differential ring

Let $\mathcal{C}$ be a sub differential ring of $\mathcal{A}$. For any $S \in \mathcal{C}\ll\mathcal{T}_n\gg$, let $\mathcal{F}(S) := \text{span}_{\mathcal{C}}\{\langle S|w\rangle\}_{w \in \mathcal{T}_n^*}$

**Lemma**

*The following assertions are equivalent* \(^9\)

1. The following map is injective

   $$(\mathcal{C}\ll\mathcal{T}_n\gg, \oplus, 1_{\mathcal{T}_n^*}) \longrightarrow (\mathcal{H}(\mathcal{V}), *, 1_{\mathcal{H}(\mathcal{V})}), \quad w \longmapsto \alpha^*_\varsigma(w).$$

2. $\{\alpha^*_\varsigma(w)\}_{w \in \mathcal{T}_n^*}$ is linearly free over $\mathcal{C}$.

3. $\{\alpha^*_\varsigma(l)\}_{l \in \mathcal{L}_{\text{yn}}\mathcal{T}_n}$ is algebraically free over $\mathcal{C}$.

4. $\{\alpha^*_\varsigma(t)\}_{t \in \mathcal{T}_n}$ is algebraically free over $\mathcal{C}$.

5. $\{\alpha^*_\varsigma(t)\}_{t \in \mathcal{T}_n \cup \{1_{\mathcal{T}_n^*}\}}$ is linearly free over $\mathcal{C}$.

6. For any $\mathcal{C} \in \mathcal{L}_{\text{ie}}\ll\mathcal{T}_n\gg$, there is an automorphism $\psi$ of $\mathcal{F}(C_{\varsigma \rightsquigarrow z})$ such that $\psi(C_{\varsigma \rightsquigarrow z}) = C_{\varsigma \rightsquigarrow z}e^\mathcal{C}$.

---

\(^9\) This is the abstract form, over ring, of (Deneufchâtel, Duchamp, HNM & Solomon, 2011).
Noncommutative differential equations

\[(NCDE) \quad dS = M_n S, \quad \text{where} \quad M_n = \sum_{1 \leq i < j \leq n} \omega_{i,j} t_{i,j}.\]

Proposition

1. \(C_\zeta \mapsto z\), satisfying \((NCDE)\), is group-like and \(\log C_\zeta \mapsto z\) is primitive:
   \[
   C_\zeta \mapsto z = \prod_{l \in \mathcal{L} \in T_n} e^{\alpha_{\zeta}^z(S_l) P_l} \quad \text{and} \quad \log C_\zeta \mapsto z = \sum_{w \in T_n^*} \alpha_{\zeta}^z(w) \pi_1(w),
   \]
   where \(\pi_1(w) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in T_n T_n^*} \langle w | u_1 \sqcup \ldots \sqcup u_k \rangle u_1 \ldots u_k.\)

2. Let \(C \in \mathbb{C} \langle \langle T_n \rangle \rangle, \langle C | 1_{T_n^*} \rangle = 1\). Then \(C_\zeta \mapsto z C\) satisfies \((NCDE)\).
   Moreover, \(C_\zeta \mapsto z C\) is group-like if and only if \(C\) is group-like.

From this, it follows that the differential Galois group of \((NCDE)\) + group-like solutions is the group \(\{ e^C \} \subseteq \text{Lie}_{\mathbb{C}}_{1_{\mathfrak{H}(\mathcal{V})}} \langle \langle \mathcal{H} \rangle \rangle \). Which leads to

the definition of the PV extension related to \((NCDE)\) as \(\hat{C}_0 \mathcal{X} \{ C_{\zeta \mapsto z} \}\).

10. \(M_n \in \Omega^1(\mathcal{V}) \langle T_n \rangle \) and \(\Delta_{\sqcup \ldots \sqcup} M_n = 1_{T_n^*} \otimes M_n + M_n \otimes 1_{T_n^*}.\)
11. In fact, the Hausdorff group (group of characters) of \((\mathcal{A} \langle T_n \rangle, \sqcup, 1_{T_n^*})\).
ALGORITHMIC AND COMPUTATIONAL ASPECTS OF SOLUTIONS OF $KZ_n$ BY DEVISSAGE
Solutions of (\textbf{NCDE}) by \( \{ V_m(\varsigma, z) \}_{m \geq 0} \) (1/2)

\[
V_m(\varsigma, z) = V_0(\varsigma, z) \sum_{t_i, j \in T_{n-1}} \int_{\varsigma}^{z} e^{\sum_{t \in T_n} \text{ad}_{-\alpha_{\varsigma}^t} \omega_{i, j}(s) t_{i, j} V_{m-1}(\varsigma, s)},
\]

\[
V_0(\varsigma, z) = \prod_{l \in L \cap T_n} e^{\alpha_{\varsigma}^z (S_l) P_l} \mod [\mathcal{L} \mathcal{e}_A \langle T_n \rangle, \mathcal{L} \mathcal{e}_A \langle T_n \rangle]
\]

\[
= e^{\sum_{t \in T_n} \alpha_{\varsigma}^z(t) t}.
\]

1. \((\alpha_{\varsigma}^z \otimes \text{Id}) \mathcal{D}_{T_n}\) satisfies the differential equation \( \text{d} F = N_{n-1} F \), where.

\[
N_{n-1} := \sum_{k=1}^{n-1} \omega_{k, n} t_{k, n} \in \mathcal{L} \mathcal{e}_\Omega^1(\nu) \langle T_n \rangle.
\]

2. \( V_0 \) satisfies the partial differential equation \( \partial_n f = N_{n-1} f \).

3. For any \( m \geq 1 \), on obtains explicitly

\[
V_m(\varsigma, z) = \sum_{w = t_{i_1, j_1} \cdots t_{i_m, j_m} \in T^*_{n-1}} \int_{\varsigma}^{z} \omega_{i_1, j_1}(s_1) \cdots \int_{\varsigma}^{s_{m-1}} \omega_{i_m, j_m}(s_m) \kappa_w(\varsigma, s_1, \cdots, s_m),
\]

where (using the identity \( e^{-a} be^a = e^{\text{ad}_{-a} b} \))

\[
V_0(\varsigma, z)^{-1} \kappa_w(\varsigma, s_1, \cdots, s_m)
\]

\[
= \prod_{p=1}^{m} e^{\text{ad}_{-\sum_{t \in T_n} \alpha_{\varsigma}^t(t)} t_{i_p, j_p}} = \sum_{q_1, \cdots, q_k \geq 0} \prod_{p=1}^{m} \frac{1}{q_p!} \text{ad}_{q_p}^{q_p} - \sum_{t \in T_n} \alpha_{\varsigma}^{s_p}(t) t_{i_p, j_p}.
\]
Solutions of \((NCDE)\) by \(\{V_m(\varsigma, z)\}_{m\geq 0}\) (2/2)

Proposition

1. \((NCDE)\) admits \(V_0(\varsigma, z)G(\varsigma, z)\) as solution, with
\[
G(\varsigma, z) = (\alpha^z_\varsigma \otimes \text{Id}) \sum_{k \geq 0} \sum_{\substack{v_1, j_1, \ldots, v_k, j_k \in T^*_n \\text{ s.t. } \ t_1, j_1, \ldots, t_{k}, j_k \in T_{n-1}}} \frac{(-1)^{|v_1, j_1| \ldots |v_k, j_k|}}{|v_1, j_1| \ldots |v_k, j_k|}
\]
\[
(t_{i_1, j_1} \tilde{v}_{i_1, j_1}) \circ \cdots \circ (t_{i_k, j_k} \tilde{v}_{i_k, j_k}) \otimes \rho(v_{i_1, j_1} t_{i_1, j_1}) \cdots \rho(v_{i_k, j_k} t_{i_k, j_k})
\]

2. There is a diffeomorphism \(g\) of \(V\) s.t. \(G(\varsigma, z)\) is group like series and is the Chen series, along the path \(g(\varsigma \rightsquigarrow z)\) and of the differential forms \(\{\omega_{i,j}\}_{1 \leq i < j \leq n-1}\), and then satisfies
\[
dS = M^*_{n-1}S, \quad \text{where} \quad M^*_{n-1} = \sum_{1 \leq i < j \leq n-1} g^*\omega_{i,j} t_{i,j} \in \text{Lie}_{\Omega^1(V)}\langle T_{n-1}\rangle.
\]

3. If the restricted \(\rightsquigarrow\)-morphism \(\alpha^z_{\varsigma}\), on \(\mathbb{C}\langle T_n\rangle\), is injective then there is a primitive series \(C \in \text{Lie}_{\mathbb{C}}\langle\langle T_{n-1}\rangle\rangle\) such that
\[
G(\varsigma, z) = \left( \sum_{w \in T^*_n} \alpha^z_{\varsigma}(w)w \right) e^C.
\]
Solutions of $KZ_n \ (n \geq 4)$

For any $1 \leq i < j \leq n - 1$, let $(P_{i,j}) : z_i - z_j = 1$.

Theorem $(\omega_{i,j}(z) = d \log(z_i - z_j), t_{i,j} \leftarrow t_{i,j}/2i\pi)$

For $z_n \rightarrow z_{n-1}$, solution of $dF = M_nF$ can be put in the form $f(z)G(z_1, \ldots, z_{n-1})$ such that

1. $f(z) \sim (z_{n-1} - z_n)^{t_{n-1,n}}$ satisfying $\partial_n f = N_{n-1} f$, where
$$N_{n-1}(z) = \sum_{k=1}^{n-1} t_{k,n} \frac{dz_n}{z_n - z_k} = \sum_{k=1}^{n-1} t_{k,n} \frac{ds}{s - s_k}, \quad \text{with} \quad \begin{cases} s = z_n, \\ s_k = z_n - z_k. \end{cases}$$

2. $G(z_1, \ldots, z_{n-1})$ is solution of $dS = M_{n-1}^{t_{n-1,n}}S$, where
$$M_{n-1}^{t_{n-1,n}}(z) \sim \sum_{1 \leq i < j \leq n-1} \varphi^{(\varsigma,z)}_{t_{n-1,n}}(t_{i,j}) d \log(z_i - z_j),$$
$$\varphi^{(\varsigma,z)}_{t_{n-1,n}}(t_{i,j}) = e^{\text{ad}^- \sum_{1 \leq k < n} \log(z_k - z_{n-1})t_{k,n}} t_{i,j} \mod J_{R_n}.$$ 

Moreover, $M_{n-1}^{t_{n-1,n}}$ exactly coincides with $M_{n-1}$ in the intersection of affine planes $\bigcap_{1 \leq i < n-1} (P_{i,n-1})$.

Conversely, if $f$ satisfies $\partial_n f = N_{n-1} f$ and $G(z_1, \ldots, z_{n-1})$ satisfies $dS = M_{n-1}^{t_{n-1,n}}S$ then $f(z)G(z_1, \ldots, z_{n-1})$ satisfies $dF = M_nF$. 
Solutions of $KZ_n$ $(n \geq 4)$ with asymptotic conditions

Let $F_{*} : (\mathbb{C} \langle \mathcal{T}_n \rangle, \sqcup, 1_{\mathcal{T}_n^{*}}) \to (\mathcal{H}(\mathcal{V}), *, 1_{\mathcal{H}(\mathcal{V})})$ be the character defined by $F_{1_{\mathcal{T}_n^{*}}} = 1_{\mathcal{H}(\mathcal{V})}$, $\forall t_{i,j} \in \mathcal{T}_n$, $F_{t_{i,j}}(z) = \log(z_i - z_j)$, $\forall t_{i,j} w \in \text{Lyn} \mathcal{T}_n \setminus \mathcal{T}_n$,

$$F_{t_{i,j} w}(z) = \int_0^z \omega_{i,j}(s) F_w(s), \text{ where } \omega_{i,j}(z) = d \log(z_i - z_j).$$

Corollary ($\omega_{i,j}(z) = d \log(z_i - z_j)$, $t_{i,j} \leftarrow t_{i,j}/2i\pi$)

1. $\{F_t\}_{t \in \mathcal{T}_n \cup \{1_{\mathcal{T}_n^{*}}\}}$ are $C_0$-linearly free.

2. The graph of $F_{*}$, $F$, is unique solution of $dF = M_n F$ and

$$F(z) = \prod_{l \in \text{Lyn} \mathcal{T}_n} e^{F_{s_l}(z) P_l} \sim_{z_i \sim z_{i-1}}^{1 \leq i \leq n} (z_{i-1} - z_i)^{t_{i-1,i}} G_i(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$$

where $G_i(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ satisfies $dS = M_{n-1}^{t_{*}^{n}} S$ and, for $y_1 = z_1, \ldots, y_{i-1} = z_{i-1}, y_i = z_{i+1}, \ldots, y_{n-1} = z_n$, one has

$$M_{n-1}^{t_{*}^{n}}(y) = \sum_{1 \leq i < j \leq n-1} e^{ad - \sum_{1 \leq k \leq n-1} \log(y_k - y_{n-1}) t_{k,n}} t_{i,j} d \log(y_i - y_j) \mod \mathcal{J}_{R_n}$$

and $M_{n-1}^{t_{*}^{n}}$ exactly coincides with $M_{n-1}$ in $\bigcap_{1 \leq k < n-1} (P_{i,n-1})$.

3. In $\text{Lie}_A \langle \mathcal{T}_n \rangle/[\text{Lie}_A \langle \mathcal{T}_n \rangle, \text{Lie}_A \langle \mathcal{T}_n \rangle]$, one has

$$F(z) = e^{\sum_{i=1}^{n-1} \log(z_n - z_i) t_{i,n}} \sum_{k \geq 0, l_1, \ldots, l_k \geq 0} F(t_1 \bar{T}_n^{l_1}) \circ \cdots \circ (t_k \bar{T}_n^{l_k})(z) \prod_{1 \leq j \leq k} \text{ad}_{-T_n}^{l_j} t_j.$$
**KZ$_3$ : Simplest non-trivial case (1/3)**

One has $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and

$$\Omega_3(z) = \frac{1}{2i\pi} \left( t_{1,2} \frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3} \frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3} \frac{d(z_2 - z_3)}{z_2 - z_3} \right).$$

Solution of $dF(z) = \Omega_3(z) F(z)$ can be computed as limit of the sequence $\{F_l\}_{l \geq 0}$, in $\mathcal{H}(\mathbb{C}_*^3) \langle \langle \mathcal{T}_3 \rangle \rangle$, by convergent Picard’s iteration:

$$F_0(z) = 1_{\mathcal{H}(\mathcal{V})} \quad \text{and} \quad F_l(z) = \int_0^z \Omega_3(s) F_{l-1}(s).$$

Let us compute, by another way, a solution of $dF(z) = \Omega_3(z) F(z)$ as the limit of the sequence $\{V_l\}_{l \geq 0}$, in $\mathcal{H}(\mathbb{C}_*^3) \langle \langle \mathcal{T}_3 \rangle \rangle$, iteratively obtained by

$$V_0(z) = e^{(t_{1,2}/2i\pi) \log(z_1 - z_2)},$$

$$V_l(z) = \int_0^z e^{(t_{1,2}/2i\pi)(\log(z_1 - z_2) - \log(s_1 - s_2))} \tilde{\Omega}_2(s) V_{l-1}(s)$$

$$= V_0(z) \int_0^z e^{-(t_{1,2}/2i\pi) \log(s_1 - s_2)} \tilde{\Omega}_2(s) V_{l-1}(s),$$

with $\tilde{\Omega}_2(z) = \frac{1}{2i\pi} \left( t_{1,3} \frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3} \frac{d(z_2 - z_3)}{z_2 - z_3} \right).$
Explicit solution is $F = V_0 G$, where $V_0(z) = (z_1 - z_2)^{t_{1,2}/2i\pi}$ and

$$G(z) = \sum_{t_{1,j_1} \cdots t_{m,j_m} \in \{t_{1,3}, t_{2,3}\} \atop m \geq 0} \int_0^z \omega_{i_1,j_1}(s_1) \varphi^{s_1}(t_{i_1,j_1}) \cdots \int_0^{s_{m-1}} \omega_{i_m,j_m}(s_m) \varphi^{s_m}(t_{i_m,j_m}),$$

where $\omega_{1,3}(z) = d \log(z_1 - z_3)$ and $\omega_{2,3}(z) = d \log(z_2 - z_3)$ and $\varphi$ is the following automorphism of Lie algebra, $\mathcal{L}ie_{\mathcal{H}(\mathbb{C}_n^*)} \langle T_3 \rangle$,

$$\varphi^z = e^{\text{ad}-(t_{1,2}/2i\pi) \log(z_1 - z_2)} = \sum_{k \geq 0} \frac{\log^k(z_1 - z_2)}{(-2i\pi)^k k!} \text{ad}_{t_{1,2}}^k.$$

Since $t_{1,2} \prec t_{1,3} \prec t_{2,3}$ and, for $k \geq 0$ and $i = 1$ or $2$, $t_{1,2}^k t_{i,3} \in \text{LynT}_3$ then

$$P_{t_{1,2}^k t_{i,3}} = \text{ad}_{t_{1,2}}^k t_{i,3} \quad \text{and} \quad S_{t_{1,2}^k t_{i,3}} = t_{1,2}^k t_{i,3}$$

and then

$$\varphi^z(t_{i,3}) = \sum_{k \geq 0} \frac{\log^k(z_1 - z_2)}{(-2i\pi)^k k!} P_{t_{1,2}^k t_{i,3}}, \quad \bar{\varphi}^z(t_{i,3}) = \sum_{k \geq 0} \frac{\log^k(z_1 - z_2)}{(-2i\pi)^k k!} S_{t_{1,2}^k t_{i,3}},$$

where $\bar{\varphi}$ (adjoint to $\varphi$) is the following automorphism of $(\mathcal{A}\langle T_3 \rangle, \varpi, 1_{T_3^*})$

$$\bar{\varphi}^z = e^{-(t_{1,2}/2i\pi) \log(z_1 - z_2)} = \sum_{k \geq 0} \frac{\log^k(z_1 - z_2)}{(-2i\pi)^k k!} t_{1,2}^k.$$
Belonging to $\mathcal{H}(\mathbb{C}^*_3 \langle \mathcal{T}_3 \rangle)$, $G$ satisfies $dG(z) = \tilde{\Omega}_2(z)G(z)$, where

$$\tilde{\Omega}_2(z) = \frac{1}{2i\pi} \left( \varphi^z(t_{1,3}) \frac{d(z_1 - z_3)}{z_1 - z_3} + \varphi^z(t_{2,3}) \frac{d(z_2 - z_3)}{z_2 - z_3} \right).$$

In the affine plan $(P_{1,2}) : z_1 - z_2 = 1$, one has

$$\log(z_1 - z_2) = 0 \quad \text{and then} \quad \varphi \equiv \text{Id}.$$

Setting $x_0 = t_{1,3}/2i\pi$, $x_1 = -t_{2,3}/2i\pi$ and $z_1 = 1, z_2 = 0, z_3 = s$, one has

$$\tilde{\Omega}_2(z) = \frac{1}{2i\pi} \left( t_{1,3} \frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3} \frac{d(z_2 - z_3)}{z_2 - z_3} \right) = x_1 \frac{ds}{1 - s} + x_0 \frac{ds}{s}.$$

$KZ_3$ admits then the noncommutative generating series of polylogarithms, $L$, as the actual solution satisfying the Drinfel'd asymptotic conditions.

Via $L$ and the homographic substitution $g : z_3 \mapsto (z_3 - z_2)/(z_1 - z_2)$, mapping $\{z_2, z_1\}$ to $\{0, 1\}$, $L((z_3 - z_2)/(z_1 - z_2))$ is a particular solution of $KZ_3$, in $(P_{1,2})$. So is $L((z_3 - z_2)/(z_1 - z_2))(z_1 - z_2)^{(t_{1,2} + t_{1,3} + t_{2,3})/2i\pi}$.

To end with $KZ_3$, by braid relations, $[t_{1,2} + t_{2,3} + t_{1,3}, t] = 0$, for $t \in \mathcal{T}_3$, meaning that $t$ commutes with $(z_1 - z_2)^{(t_{1,2} + t_{2,3} + t_{1,3})/2i\pi}$ and then $\mathcal{A} \langle \mathcal{T}_3 \rangle$ commutes with $(z_1 - z_2)^{(t_{1,2} + t_{1,3} + t_{2,3})/2i\pi}$.

Thus, $KZ_3$ also admits $(z_1 - z_2)^{(t_{1,2} + t_{1,3} + t_{2,3})/2i\pi} L((z_3 - z_2)/(z_1 - z_2))$ as a particular solution in $(P_{1,2})$. 
Other example of non-trivial case: $KZ_4 (t_{i,j} \leftarrow t_{i,j}/2i\pi)$

For $n = 4$, one has $\mathcal{T}_4 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,4}\}$ and then $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and $\mathcal{T}_4 = \{t_{1,4}, t_{2,4}, t_{3,4}\}$. Then

$$\varphi^{(\varsigma,z)}_{T_4} = e^{\text{ad} - \sum_{t \in \mathcal{T}_4} \alpha(z)_{t} t},$$

and for any $t_{i,j} \in \mathcal{T}_3$,

$$\varphi^{(\varsigma,z)}_{t_{i,j}}(t_{i,j}) = \varphi^{(\varsigma,z)}_{T_4}(t_{i,j}) \mod J_{\mathcal{R}_n}.$$

If $z_4 \to z_3$ then

$$F(z) = V_0(z)G(z_1, z_2, z_3), \quad \text{where} \quad V_0(z) = e^{\sum_{1 \leq i \leq 4} t_{i,4} \log(z_i - z_4)}$$

and $G(z_1, z_2, z_3)$ satisfies $dS = M_{3}^{t_{i,j}}S$ with

$$M^{t_{i,j}}_{3}(z) = \varphi^{(z_0,z)}_{t_{i,j}}(t_{1,2}) d \log(z_1 - z_2)$$
$$+ \varphi^{(z_0,z)}_{t_{i,j}}(t_{1,3}) d \log(z_1 - z_3)$$
$$+ \varphi^{(z_0,z)}_{t_{i,j}}(t_{2,3}) d \log(z_2 - z_3).$$

Considering $(P_{1,4}) : z_1 - z_4 = 1, \quad (P_{2,4}) : z_2 - z_4 = 1, \quad (P_{3,4}) : z_3 - z_4 = 1$, in the intersection $(P_{1,3}) \cap (P_{2,3})$, one has $\log(z_1 - z_3) = \log(z_2 - z_3) = 0$ and $\varphi_{t_{i,j}} \equiv \text{Id}$ and then $M^{t_{i,j}}_{3}$ exactly coincides with $M_3$. 
Bibliography


