Topology of the arc complex

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**Marked surfaces**

**Setting:** Let $S$ be a finite-type, possibly non-orientable surface with finitely many marked points such that

- if $\partial S \neq \emptyset$, then there is at least one marked point on every boundary component;
- interior points can be marked.
Examples of marked surfaces

- Convex polygon, $\mathcal{P}_n$
- Once-punctured polygon, $\mathcal{P}_n^x$
- Orientable crown, $\mathcal{P}_n^\circ$
- Three-holed sphere
- One-holed torus
An arc on $S$ is defined as $\alpha : [0, 1] \hookrightarrow S$ such that $\alpha([0, 1]) \cap S = \{\alpha(0), \alpha(1)\} \subset \mathcal{P}$.
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- We consider isotopy classes of non-trivial arcs;
- 2 classes are *disjoint* if they have two disjoint representatives.
Arc complex

\((S, P)\): a marked surface.

\(\mathcal{A}(S)\): a flag, pure simplicial complex constructed in the following way:

- 0-simplices \(\leftrightarrow\) isotopy classes of embedded arcs,
- For \(k \geq 1\), \(k\)-simplices \(\leftrightarrow\) \((k + 1)\) pairwise disjoint and distinct classes.
Example: a convex polygon $\mathcal{P}_n$, for $n \geq 4$

(a) The arc complex of a hexagon

(b) Two-dimensional associahedron
The arc complex

Important properties:

- The arcs of top-dimensional simplices divide the surface into triangles and at most one once-punctured disk.
- The arc complex is connected.
- For a "generic" surface, the arc complex is locally non-compact with infinite diameter.
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- The arc complex is connected.
- For a "generic" surface, the arc complex is locally non-compact with infinite diameter.
"Generic" example: One-holed torus
Example: orientable crown $P_n^\circ$, for $n \geq 1$

$$A_c(P_1^\circ) = A(P_1^\circ)$$

$$A(P_2^\circ)$$

$$A(P_3^\circ)$$
Example: non-orientable crown $\mathcal{M}_n$, for $n \geq 1$

$\mathcal{A}(\mathcal{M}_1) = \mathcal{A}_C(\mathcal{M}_1)$

$\mathcal{A}(\mathcal{M}_2)$

$\mathcal{A}(\mathcal{M}_3)$
Example: Once-punctured polygon $\mathcal{P}^x_n$, for $n \geq 2$

\[ \mathcal{A}(\mathcal{P}^x_n) \simeq \partial \mathcal{A}(\mathcal{P}^o_n) \simeq \partial \mathcal{A}(\mathcal{M}_n) \]
Crowned hyperbolic surfaces

Topology of the arc complex
**Classical result**: For $n \geq 4$, the arc complex $\mathcal{A}(\mathcal{P}_n)$ of a polygon is a PL-sphere of dimension $n - 4$.

**Theorem (Penner)**

- The arc complex $\mathcal{A}(\Pi_{n})$ of an ideal polygon $\Pi_{n}$ ($n \geq 4$) is a PL-sphere of dimension $n - 4$.
- The arc complex $\mathcal{A}(\Pi_{n}^{\times})$ of an once-punctured ideal polygon $\Pi_{n}^{\times}$ ($n \geq 2$) is a PL-sphere of dimension $n - 2$.

Penner gave a list of surfaces for which the *quotient* arc complex is a sphere.
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Bowditch-Epstein: for $S$ orientable, cellular decomposition of the Teichmüller space using the arc complex and cut locus. (hyperbolic geometry)
Topology of the arc complex: generic case

- **Hatcher**: for $S$ orientable, $\mathcal{A}(S)$ is contractible. (Hatcher flow, combinatorics)
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- **Bowditch-Epstein**: for $S$ orientable, cellular decomposition of the Teichmüller space using the arc complex and cut locus. (hyperbolic geometry)
- **Fomin-Schapiro-Thurston**: for $S$ orientable, the arc complex is a subset of the associated cluster complex. (combinatorics, hyperbolic geometry)
Introduced by Penner to study Teichmüller theory of surfaces decorated with horoballs using combinatorial methods.

- "lambda" lengths of h.c parametrise $\mathcal{D}(S)$
- the a.c gives a cellular decomposition of $\mathcal{D}(S)$
- lambda lengths behave like cluster variables
One particular application

Let $S_{0,3}$ be the three-holed sphere.
One particular application

\[ x = c_1 \alpha + c_2 \beta + c_3 \gamma \]

\( m \in D(S) \quad \rightarrow \quad m' \in D(S) \)

add strips

width of strip [c_3]

lengthens every curve uniformly

admissible deformation
One particular application

**Theorem (Danciger-Guéritaud-Kassel)**

Let $S$ be a compact hyperbolic surface with totally geodesic boundary. Let $m = ([\rho]) \in \mathcal{D}(S)$ be a metric. Fix a choice of strip template $\{(\alpha_g, p_\alpha, w_\alpha)\}_{\alpha \in \mathcal{K}}$ with respect to $m$. Then the restriction of the projectivised infinitesimal strip map $\mathbb{P}f : \mathcal{PA}(S) \to \mathbb{P}^+(T_m \mathcal{D}(S))$ is a homeomorphism on its image $\mathbb{P}^+(\Lambda(m))$.

Here the admissible cone $\Lambda(m)$ consists of all infinitesimal deformations that uniformly lengthen every non-trivial closed geodesic.
Why are admissible deformations important?
Margulis spacetimes

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- **Auslander** (Conjecture): $\Gamma \subset \text{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, discrete s.t. $\mathbb{R}^n / \Gamma$ is a compact manifold $\Rightarrow \Gamma$ is virtually solvable.
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- **D–G–K:** The arc complex parametrises Margulis spacetimes.
Let $S$ be a decorated hyperbolic surface.

**Aim:** To parametrise *decorated* Margulis spacetimes using the arc complex of decorated hyperbolic surfaces.

**Theorem (P.)**

Let $S$ be a finite-type decorated surface with a metric $m \in \mathcal{D} \left( \prod_n \right)$. Then the projectivised strip map $\mathbb{P} f : \mathcal{PA}(S) \to \mathbb{P}^+ (T_m \mathcal{D}(S))$ is a homeomorphism on its image $\mathbb{P}^+ (\Lambda(m))$.

Here $\Lambda(m)$ is the set of deformations uniformly lengthening all horoball connections.
Decorated surfaces to bicolleourings

Non-trivial bicollouring of marked points with blue and red: at least one R-R diagonal.

The subcomplex $\mathcal{Y}$ generated by $G - G$, $R - G$ diagonals is isomorphic to the arc complex of the decorated surface.
Examples

Rejected R-R diagonals

The subcomplex $\mathcal{Y}(P_6)$
Examples

Rejected R-R diagonals

The subcomplex $\mathcal{Y}(\mathcal{P}_4^\times)$
Contributions

Theorem (P.)

Let $\mathcal{P}_n$ (resp. $\mathcal{P}_n^\times$) be a polygon with a non-trivial bicolouring. Then the subcomplex $\mathcal{Y}(\mathcal{P}_n)$ (resp. $\mathcal{Y}(\mathcal{P}_n^\times)$) is a shellable closed ball of dimension $2n - 4$ (resp. $2n - 2$).

Theorem (P.)

Let $S = \mathcal{P}_n^\circ$, $\mathcal{M}_n$, where $n \geq 1$ with any bicoloring. Then, the subcomplex $\mathcal{Y}(S)$ is a collapsible combinatorial ball of dimension $n - 1$.

In fact, we show something stronger...
Let $X$ be a pure simplicial complex of dimension $n$.

**Definition**

A shelling order is an ordering of the maximal simplices $\{C_1, C_2 \ldots\}$ of $X$ such that $C_k \cap (\bigcup_{i=1}^{k-1} C_i)$ is a pure simplicial complex of dimension $n - 1$.

A complex is called *shellable* if there exists a shelling order.

**Ex:**

![Example](image)

$\dim C_1 \cap C_2 = 1$

**Non-ex:**

![Non-example](image)

$\dim C_1 \cap C_2 = 0$
Shellability: Example

Danaraj-Klee: Any shellable pseudomanifold with boundary is PL-homeomorphic to a closed ball.
Shellability of the arc complex

**Theorem (P.)**

Let $\mathcal{P}_n$ (resp. $\mathcal{P}_n^\times$) be a polygon with a non-trivial bicolouring. Then the subcomplex $\mathcal{Y}(\mathcal{P}_n)$ (resp. $\mathcal{Y}(\mathcal{P}_n^\times)$) is a shellable closed ball of dimension $2n - 4$ (resp. $2n - 2$).

**Corollary**

- For $n \geq 3$, the arc complex of a decorated polygon is a closed ball of dimension $2n - 4$.
- For $n \geq 1$, the arc complex of a decorated once-punctured polygon is a closed ball of dimension $2n - 2$. 
Collapsibility

Let $X$ be a finite simplicial complex.

**Definition**

Let $\sigma, \tau$ be two simplices of $X$ such that

- $\sigma \subset \tau,$
- $\tau$ is the unique maximal simplex containing $\sigma.$

Then $X$ is said to be *collapsing onto* $X \setminus \{\sigma, \tau\}.$ A complex $X$ is said to be collapsible if there is a finite sequence of collapses ending at a 0-simplex.
Strong collapsibility

Definition

Let $X$ be a finite simplicial complex. A 0-simplex $v \in X$ is *vertex-dominated* by another 0-simplex $v'$ if $\text{Link}(X, v) = v' \otimes L$. In this case, $X$ is said to *strongly collapse* onto $X \setminus v$.

A finite complex is *strongly collapsible* if there is a finite sequence of strong collapses terminating at a 0-simplex.
Strong collapsibility and the arcs

An arc $v$ is vertex-dominated by an arc $v'$ if any triangulation containing the arc $v$ also contains $v'$.
Strong collapsibility: Illustration

A coincidence in dimension two...

\[ \mathcal{A}(M_3) \quad \Rightarrow \quad \mathcal{A}(P_3^\circ) \quad \Rightarrow \quad \mathcal{A}(S_{0,3}) \]

--- vertex domination
Collapsibility of the arc complexes

Theorem (P.)

For $n \geq 1$,

- $A(P_n^\circ)$ is strongly collapsible.
- $A(M_n)$ collapses onto $A_C(M_n)$.
- $A_C(M_n)$ is strongly collapsible.
- $A(M_n)$ is collapsible but not strongly collapsible.

The statements remain true even if we put a bicolouring on the marked points.
Walls of the admissible cone

\[
\begin{align*}
\{d_{\gamma}(m) = 0\} & \quad \{d_{\gamma}(m) = 0\} \\
\{f_{c1}(m)\} & \quad \{f_{c1}(m)\} \\
\{f_{b1}(m)\} & \quad \{f_{b1}(m)\} \\
\{f_{d1}(m)\} & \quad \{f_{d1}(m)\}
\end{align*}
\]
What next?

- Is $\mathcal{Y}(\mathcal{P}_n)$ or $\mathcal{Y}(\prod_n^x)$ collapsible for any bicolouring?
- Collapsibility of infinite arc complexes: arborescence (Adiprasito–Funar).
- How to interpret collapsibility in terms of hyperbolic geometry?