Asymptotic expansion for random tensor models

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Plan

1. Introduction
   - Combinatorial Physics
   - Quantum Field Theory (QFT) and Combinatorial QFT
   - Random matrices

2. The multi-orientable (MO) tensor model
   - Ribbon jackets and the large $N$ limit
   - The dominant order
   - The double scaling limit

3. The quartic $O(N)^3$—invariant model

4. The prismatic tensor model

5. Conclusions
Combinatorial Physics

problems in **Theoretical Physics** successfully tackled using **Combinatorial** methods

problems in **Combinatorics** successfully tackled using **Theoretical Physics** methods

(most of) *this talk*: example of the first case

*combinatorial techniques*:
- analysis of the general term in an asymptotic expansion
- analytic analysis of the singularities of the relevant generating series

*physical problem*: implementation of the celebrated double scaling mechanism for various random tensor models
Quantum Field Theory (QFT) - quantum description of particles and their interactions

description compatible with Einstein’s theory of special relativity

QFT formalism applies to:

- Standard Model of elementary particle physics
- statistical physics (statistical QFT)
- condensed matter physics
- etc.

great experimental success!
(real or complex) fields $\Phi : \mathbb{R}^4 \to \mathbb{R}$ or $\mathbb{C}$ (4–dimensional QFT)

**action of a QFT model** $(S(\phi))$

quadratic part (propagation) + non-quadratic part (cubic, quartic, etc.)

**partition function**: $Z = \int \mathcal{D}\Phi e^{-S(\Phi)}$

perturbative expansion (Taylor expansion) of the partition function $Z$ in the coupling constant $\lambda$

**Feynman graphs** associated to the terms of the expansion

**example of a Feynman graph of the $\Phi^4$ model**:

Feynman graphs $\rightarrow$ Feynman amplitudes
Combinatorial QFT

the scalar field $\phi$ is not a function of space-time (there is no space-time)!
real field $\phi \in \mathbb{R}$ (or complex field $\phi \in \mathbb{C}$)

*partition function:*

$$Z = \int_{\mathbb{R}} d\phi \ e^{-\frac{1}{2} \phi^2 + \frac{\lambda}{4!} \phi^4}.$$

*perturbation theory* - formal series in $\lambda$

$\rightarrow$ (abstract) Feynman graphs and Feynman amplitudes
One (still) needs to evaluate integrals of type

$$
\frac{\lambda^n}{n} \int d\phi \ e^{-\phi^2/2} \left( \frac{\phi^4}{4!} \right)^n.
$$

one can (still) use standard QFT techniques:

$$
\int d\phi \ e^{-\phi^2/2} \phi^{2k} = \frac{\partial^{2k}}{\partial J^{2k}} \int d\phi \ e^{-\phi^2/2+J\phi} |_{J=0} = \frac{\partial^{2k}}{\partial J^{2k}} e^{J^2/2} |_{J=0}.
$$

$J$ - the source

0–dimensional QFT - interesting "laboratories" for testing theoretical physics tools


From scalars to matrices

Definition
A random matrix is a matrix of given type and size whose entries consist of random numbers from some specified distribution.

Random matrices & combinatorics:

counting maps theorems (via matrix integral techniques)

\[ \int f(\text{matrix of dim } N) = \sum_{g} N^{2-2g} A_g \]

\(A_g\) - some weighted sum encoding maps of genus \(g\)
(this depends on the choice of \(f\) - the physical model)


Random matrices in mathematics & physics

- **mathematics**
  - non-commutative probabilities
  - the Kontsevich matrix model - the Witten conjecture: rigorous approach to the moduli space of punctured Riemann surfaces
    - E. Witten, *Nucl. Phys. B* (1990),
  - *etc.*

- **physics**: nuclear physics (spectra of heavy atoms), particle physics (quantum chromodynamics), 2-dimensional quantum gravity, string theory *etc.*

  - Wishart, *Biometrika* (1928)
  - M. L. Mehta, *Random Matrices*, Elsevier ('04)
Other applications of random matrices

- spacing between perched birds (parked cars)


More on matrix integral techniques


B. Eynard, "Counting Surfaces" (Springer) etc.

\( M - N \times N \) Hermitian matrix

*The partition function:*

\[
Z := \int dMe^{-\frac{1}{2}\text{Tr} M^2 + \frac{\lambda}{\sqrt{N}} \text{Tr} M^3}.
\]

\( dM := \prod_i dM_{ii} \prod_{i<j} d\text{Re}M_{ij} \text{Im}M_{ij} \) (the measure)

QFT perturbative expansion in \( \lambda \) - Feynman ribbon graphs (dual to 2-dimensional triangulations)

The partition function \( Z \) generates random triangulations - a generating function
Duality ribbon graphs $\langle \rightarrow \rangle 2D$ random triangulations

the triangulation building block: the triangle (the $2D$ simplex)
dual of a triangle - a ribbon vertex of valence 3
Feynman graphs of matrix models are ribbon graphs or \((2D)\) maps
the matrix amplitude can be combinatorially computed - in terms
of number of vertices \((V)\), edges and faces \((F)\) of the graph

\[
\mathcal{A} = \lambda^V N^{-\frac{1}{2}} V + F = \lambda^V N^{2-2g}
\]

(since \(E = \frac{3}{2} V\))
The partition function supports a \(1/N\) expansion:

\[
Z = N^2 Z_0(\lambda) + Z_1(\lambda) + \ldots = \sum_{g=0}^{\infty} N^{2-2g} Z_g(\lambda)
\]

\(Z_g\) gives the contribution from surfaces of genus \(g\)

In the \(N \to \infty\) limit, only planar surfaces survive
- dominant graphs - (\textit{triangulations of the sphere }\(S^2\))

E. Brézin \textit{et al.}, \textit{Commun. Math. Phys.} ('78),

The double scaling limit for matrix models

The successive coefficient functions $Z_g(\lambda)$ as well diverge at the same critical value of the coupling $\lambda = \lambda_c$
the leading singular piece of $Z_g$:

$$Z_g(\lambda) \propto f_g(\lambda_c - \lambda)^{(2-\gamma_{\text{str}})\chi/2} \quad \text{with} \quad \gamma_{\text{str}} = -\frac{1}{2} \quad \text{(pure gravity)}$$

contributions from higher genera ($\chi < 0$) are enhanced as $\lambda \to \lambda_c$
$$\kappa^{-1} := N(\lambda - \lambda_c)^{(2-\gamma_{\text{str}})/2}$$
the partition function expansion:

$$Z = \sum_g \kappa^{2g-2} f_g$$

double scaling limit: $N \to \infty$, $\lambda \to \lambda_c$ while holding fixed $\kappa$
coherent contribution from all genus surfaces

Question:

How much of these celebrated 2D results generalize to 3D?
Tensor models were introduced already in the 90’s - replicate in dimensions higher than 2 the success of random matrix models:
J. Ambjorn et. al., *Mod. Phys. Lett.* (’91),

natural generalization of matrix models

matrix $\rightarrow$ rank three tensor
From a tetrahedron to a 4–valent tensor vertex

\[
\begin{align*}
\tilde{\phi}_{iln} & \quad n & & m & \tilde{\phi}_{mlk} \\
\phi_{mjn} & & & l & \phi_{ijk} \\
i & j & & k & \\
\end{align*}
\]
tensor graphs - 3D maps

the triangulation building block: the tetrahedron (the 3D simplex)
dual of a tetrahedron - a tensor vertex of valence 4
4-dimensional models

4D vertex (dual image of a 4–simplex (5–cell)):
QFT-inspired simplification - the colored tensor model

highly non-trivial combinatorics and topology
→ a QFT simplification of these models - colored tensor models


a quadruplet of complex fields \((\phi^0, \phi^1, \phi^2, \phi^3)\);

\[
S[\{\phi^i\}] = S_0[\{\phi^i\}] + S_{int}[\{\phi^i\}]
\]

\[
S_0[\{\phi^i\}] = \frac{1}{2} \sum_{p=0}^{3} \sum_{i,j,k=1}^{N} \phi^p_{ijk} \phi^p_{ijk}
\]

\[
S_{int}[\{\phi^i\}] = \frac{\lambda}{4} \sum_{i,j,k,i',j',k'=1}^{N} \phi^0_{ijk} \phi^1_{i'j'k} \phi^2_{i'jk'} \phi^3_{k'j'i} + \text{c. c.}
\]

the indices \(0, \ldots, 3\) - color indices.

Various results

- double-scale limit mechanism

  1. combinatorial methods - analysis of the general term of the large $N$ asymptotic expansion and analytic analysis of the singularities of the relevant generating series
     

  2. QFT methods
     
     S. Dartois et. al., JHEP (2013), V. Bonzom et. al., JHEP (2014)

- Connes-Kreimer Hopf algebraic reformulation of tensor renormalizability


- loop vertex expansion of the perturbative series


- etc.
Multi-Orientable (MO) models


edge and (valence 4) vertex of the model:
Example of an MO tensor graph:
A jacket of an MO graph is the graph made by excluding one type of strands throughout the graph. The outer jacket $\bar{c}$ is made of all outer strands, or equivalently excludes the inner strands (the green ones); jacket $\bar{a}$ excludes all strands of type $a$ (the red ones) and jacket $\bar{b}$ excludes all strands of type $b$ (the blue ones). Such a splitting is always possible.
Example of jacket subgraphs

A MO graph with its three jackets $\bar{a}$, $\bar{b}$, $\bar{c}$

one can prove that each jacket of an MO tensor graph is a ribbon graph (or 2D map)
Euler characteristic & degree of MO tensor graphs

ribbon graphs - orientable or non-orientable surfaces.

Euler characteristic formula:

$$\chi(\mathcal{J}) = V_{\mathcal{J}} - E_{\mathcal{J}} + F_{\mathcal{J}} = 2 - k_{\mathcal{J}},$$

$k_{\mathcal{J}}$ is the non-orientable genus,
$V_{\mathcal{J}}$ is the number of vertices,
$E_{\mathcal{J}}$ the number of edges and
$F_{\mathcal{J}}$ the number of faces.

If the surface is orientable, $k$ is even and equal to twice the orientable genus $g$

the degree of an MO tensor graph $\mathcal{G}$:

$$\omega(\mathcal{G}) := \sum_{\mathcal{J}} \frac{k_{\mathcal{J}}}{2} = 3 + \frac{3}{2} V_{\mathcal{G}} - F_{\mathcal{G}},$$

the sum over $\mathcal{J}$ running over the three jackets of $\mathcal{G}$. 

Large $N$ expansion of the MO tensor model

generalization of the random matrix asymptotic expansion in $N$

One needs to count the number of faces of the tensor graph

This can be achieved using the graph’s jackets (ribbon subgraphs)

The tensor partition function writes as a formal series in $1/N$:

$$
\sum_{\omega \in \mathbb{N}/2} C[^{\omega}] (\lambda) N^{3-\omega},
$$

$$
C[^{\omega}] (\lambda) = \sum_{G, \omega(G) = \omega} \frac{1}{s(G)} \lambda^{v_G}.
$$

the role of the genus is played by the degree
Theorem

The MO model admits a $1/N$ expansion whose dominant graphs are the “melonic” ones.
More on melonic tensor graphs

- they maximize the number of faces for a given number of vertices.
- they correspond to a particular class of triangulations of the sphere $S^3$. 
Combinatorial analysis of the general term of the expansion

- for the colored tensor model
  *Annales IHP D Comb., Phys. & their Interactions* (2016)

- for the MO tensor model

adaptation of the Gurău-Schaeffer combinatorial approach for the MO case

combinatorial analysis leading to the implementation of the double scaling mechanism
An external strand is called left (L) if it is on the left side of a positive half-edge or on the right side of a negative half-edge. An external strand is called right (R) if it is on the right side of a positive half-edge or on the left side of a negative half-edge.

(L - blue, straight (S) - green, R - red)
There exists an infinite number of melon-free graphs of a given degree.

Nevertheless, some configurations can be repeated without increasing the degree.
A (two-)dipole is a subgraph formed by a couple of vertices connected by two parallel edges which has a face of length two, which, if the graph is rooted, does not pass through the root.
In a Feynman graph, a **chain** is as a sequence of dipoles $d_1 \ldots, d_p$ such that for each $1 \leq i < p$, $d_i$ and $d_{i+1}$ are connected by two edges involving two half-edges on the same side of $d_i$ and two half-edges on the same side of $d_{i+1}$.
A chain is called **unbroken** if all the $p$ dipoles are of the same type.

A **proper chain** is a chain of at least two dipoles.

A proper chain is called **maximal** if it cannot be extended into a larger proper chain.
Chains, chain-vertices and their strand configurations

\[
\begin{align*}
&\Rightarrow L \quad \Rightarrow S_0 \\
&\Rightarrow R \quad \Rightarrow S_e \\
&\Rightarrow B
\end{align*}
\]

strand configurations:

\[
\begin{align*}
L & \Leftrightarrow \quad S_0 \\
R & \Leftrightarrow \quad S_e \\
B & \Leftrightarrow
\end{align*}
\]
Let $G$ be a rooted melon-free MO-graph. The scheme of $G$ is the graph obtained by simultaneously replacing any maximal proper chain of $G$ by a chain-vertex.
A reduced scheme is a rooted melon-free MO-graph with chain-vertices and with no proper chain.

By construction, the scheme of a rooted melon-free MO-graph (with no chain-vertices) is a reduced scheme.

Every rooted melon-free MO-graph is uniquely obtained as a reduced scheme where each chain-vertex is consistently substituted by a chain of at least two dipoles.
Proposition

Let $G$ be an MO-graph with chain-vertices and let $G'$ be an MO-graph with chain-vertices obtained from $G$ by consistently substituting a chain-vertex by a chain of dipoles. Then the degrees of $G$ and $G'$ are the same.

Proof. Carefully counting the number of faces, vertices and connected components and using the formula:

$$2\omega = 6c + 3V - 2F.$$
Finiteness of the set of reduced schemes of a given degree

Theorem
For each \( \omega \in \frac{1}{2}\mathbb{Z}_+ \), the set of reduced schemes of degree \( \omega \) is finite.

Proof.
1. For each reduced scheme of degree \( \omega \), the sum \( N(G) \) of the numbers of dipoles and chain-vertices satisfies \( N(G) \leq 7\omega - 1 \).

2. For \( k \geq 1 \) and \( \omega \in \frac{1}{2}\mathbb{Z}_+ \), there is a constant \( n_{k,\omega} \) s.t. any connected unrooted MO-graph of degree \( \omega \) with at most \( k \) dipoles has at most \( n_{k,\omega} \) vertices.
Proof - dipole and chain-vertex reductions

- removal of a chain-vertex (of any type)
- removal of a dipole of type L, R and S.

2 types of chain-vertices (and dipoles):
1. separating
2. non-separating

(if the number of connected components is conserved or not after removal)
the generating function of melonic graphs:

\[ T(z) = 1 + z \left( T(z) \right)^4. \]
Generating functions of our objects

\( u \) marks half the number of vertices
(i.e., for \( p \in \frac{1}{2} \mathbb{Z}_+ \), \( u^p \) weight given to a MO Feynman graph with \( 2p \) vertices)

generating function for:

- unbroken chains of type L (or R)

\[
\frac{u^2}{1 - u} = u^2 + u^3 + \ldots
\]

- even straight chains

\[
\frac{u^2}{1 - u^2} = \frac{u^2}{1 - u} = u^2 + u^4 + u^6 + \ldots
\]

- odd straight chains

\[
\frac{u^3}{1 - u^2} = \frac{u^3}{1 - u} = u^3 + u^5 + u^6 + \ldots
\]

e etc.
putting together the generating functions of all contributions
\[ \implies G_{S}^{(\omega)}(u) \] - the generating function of rooted melon-free
MO-graphs of reduced scheme $S$ of degree $\omega$,

\[ G_{S}^{(\omega)}(u) = u^p \frac{u^{2a}}{(1 - u)^a} \frac{u^{2s_{e}}}{(1 - u^2)^{s_{e}}} \frac{u^{3s_{o}}}{(1 - u^2)^{s_{o}}} \frac{6^b u^{2b}}{(1 - 3u)^b (1 - u)^b}. \]

$b$ - the number of broken chain-vertices
$a$ - the number of unbroken chain-vertices of type L or R
$s_{e}$ - the number of even straight chain-vertices,
$s_{o}$ - the number of odd straight chain-vertices.
$F_{S}^{(\omega)}(z)$ - the generating function of graphs of reduced scheme $S$

$$F_{S}^{(\omega)}(z) = T(z) \frac{6^b U(z)^{p+2c+s_o}}{(1 - U(z))^{c-s}(1 - U(z)^2)^s(1 - 3U(z))^b},$$

$U(z) := zT(z)^4 = T(z) - 1$

$F^{(\omega)}(z)$ - the generating function of rooted MO-graphs of degree $\omega$

$$F^{(\omega)}(z) = \sum_{S \in S_{\omega}} F_{S}^{(\omega)}(z).$$

$S_{\omega}$ - the (finite) set of reduced schemes of degree $\omega$. 
Singularity order - dominant schemes

\( T(z) \) has its main singularity at

\[ z_0 := 3^3/2^8, \]

\( T(z_0) = 4/3, \) and \( 1 - 3U(z) \sim_{z \to z_0} 2^{3/2}3^{-1/2}(1 - z/z_0)^{1/2}. \)

\[ \Rightarrow (1 - 3U(z))^{-b} \sim_{z \to z_0} (1 - z/z_0)^{-b/2} \]

\[ \Rightarrow \text{the dominant terms are those for which } b \text{ is maximized.} \]

the larger \( b \), the larger the singularity order

A reduced scheme \( S \) of degree \( \omega \in \frac{1}{2}\mathbb{Z}_+ \) is called \textbf{dominant} if it maximizes (over reduced schemes of degree \( \omega \)) the number \( b \) of broken chain-vertices.
The double scaling limit of the MO tensor model


The dominant configurations in the double scaling limit are the dominant schemes

The successive coefficient functions \( Z_g(\lambda) \) as well diverge at the same critical value of the coupling \( \lambda = \lambda_c \)

contributions from higher degree are enhanced as \( \lambda \to \lambda_c \)

\[ \kappa^{-1} := N^{\frac{1}{2}} (1 - \lambda/\lambda_c) \]

the partition function expansion:

\[ Z = \sum_{\omega} N^{3-\omega} f_\omega \]

double scaling limit: \( N \to \infty, \lambda \to \lambda_c \) while holding fixed \( \kappa \)

contribution from all degree tensor graphs

similar behaviour to the matrix model double scaling limit
The quartic $O(N)^3$-invariant tensor model
The quartic $O(N)^3$-tensor model

1. model introduced in


- The tensor $\phi_{abc}$ is invariant under the action of $O(N)^3$:

$$\phi_{abc} \rightarrow \phi'_{a'b'c'} = \sum_{a,b,c=1}^{N} O_{a'a}^1 O_{b'b}^2 O_{c'c}^3 \phi_{abc} \quad O^i \in O(N)$$

- quartic invariants:

$$I_t(\phi) = \sum_{a,a',b,b',c,c'} \phi_{abc} \phi_{ab'c'} \phi_{a'bc'} \phi_{a'b'c'}$$

$$I_{p,1}(\phi) = \sum_{a,a',b,b',c,c'} \phi_{abc} \phi_{a'bc} \phi_{ab'c'} \phi_{a'b'c'}$$

2. the model above was extended to the 1–dimensional case:


I. Klebanov, F. Popov, G. Tarnopolsky, TASI Lectures (2017)
The action of the quartic $O(N)^3$-invariant tensor model:

$$S_{CTKT}(\phi) = -\frac{N^2}{2} \phi^2 + N^{5/2} \frac{\lambda_1}{4} I_t(\phi) + N^2 \frac{\lambda_2}{4} \left( I_{p,1}(\phi) + I_{p,2}(\phi) + I_{p,3}(\phi) \right)$$
An example of Feynman graph of the model

\[ n_t = 3 \]
\[ n_p = 3 \]
\[ F_1 = 1 \]
\[ F_2 = 3 \]
\[ F_3 = 1 \]
\[ \Rightarrow \omega = 17 \]
The free energy admits a large $N$ expansion

$$F_N(\lambda_1, \lambda_2) = \ln Z_N(\lambda_1, \lambda_2) = \sum_{G \in \tilde{G}} N^{3-\omega(G)} A(G).$$

(2)

where the degree is:

$$\omega(G) = 3 + \frac{3}{2} n_t(G) + 2 n_p(G) - F(G)$$

(3)
Two types of LO graphs

\[ \omega(G) = 3 + \frac{3}{2} n_t(G) + 2n_p(G) - F(G) \]

**Dominant graphs:** \( \omega(G) = 0 \)

two types of interaction → two types of melonic graphs:

**Type I:**

**Type II:**

"melon-tadpoles" graphs
Recall that a scheme (of degree $\omega$) is a "blueprint" that tells us how to obtain graphs of the same degree $\omega$.

Recall the general idea: Identify operations that leave the degree invariant and use them to repackage all the graphs that differ only by the applications of these operations.

Melonic moves are such graphic operations.
Dipoles

Definition
A dipole is a 4-point graph obtained by cutting an edge in an elementary melon.
\[ D_i = \text{Diagram 1} + \text{Diagram 2} \]
Definition

Chains are the 4-point functions obtained by connecting an arbitrary number of dipoles.

\[ C_i = \sum_{k \geq 2} D_i \cdots D_i \]

(k dipoles)
Definition

The scheme $S$ of a 2-point graph $G$ is obtained by:

1. Removing all melonic 2-point subgraphs in $G$
2. Replacing all maximal chains with chain-vertices and all dipoles with dipole-vertex of the same color.

Figure: An example of scheme
Theorem
(Bonzom-Nador-Tanasa (2022))
The set of schemes of a given degree is finite in the quartic $O(N)^3$-invariant tensor model.
The generating function associated to a dominant schemes is

\[ G_\omega(t, \mu) = (3t^{1/2})^{2\omega}(1 + 6t)^{2\omega - 1} B(t, \mu)^{4\omega - 1} \]

\[ = (3t^{1/2})^{2\omega}(1 + 6t)^{2\omega - 1} \frac{6^{4\omega - 1} U^{8\omega - 2}}{((1 - U)(1 - 3U))^{4\omega - 1}} \]

where \( B \) is the generation functions of broken chains and \( U \) is the generation function of dipoles.

Summing over the different trees (in bijection with the dominant schemes):

\[ G_\omega(t, \mu) = \text{Cat}_{2\omega - 1} M(t, \mu) G_\omega(t, \mu) \]

where \( M \) is the generation functions of melonic graphs.
Double scaling parameter

Near critical point

\[ G_{\text{dom}}^\omega(t, \mu) \sim N^{3-\omega} M_c(\mu) \text{Cat}_{2\omega-1} 9^\omega t_c^\omega \left(1 + 6t_c\right)^{2\omega-1} \]

\[ \times \left( \frac{1}{(1 - \frac{4}{3} t_c(\mu) \mu M_c(\mu)) K(\mu) \sqrt{1 - \frac{t}{t_c(\mu)}}} \right)^{4\omega-1} \]

- The double scaling parameter \( \kappa(\mu) \) is the quantity to hold fixed when sending \( N \to +\infty, t \to t_c(\mu) \).
- Dominant schemes of all degree \( \omega \) contribute in the double scaling limit.

One has

\[ \kappa(\mu)^{-1} = \frac{1}{3} \frac{1}{t_c(\mu)^{\frac{1}{2}} (1 + 6t_c(\mu))} \left( \left(1 - \frac{4}{3} t_c(\mu) \mu M_c(\mu)\right) K(\mu) \right)^2 \left(1 - \frac{t}{t_c(\mu)}\right)^{N^{\frac{1}{2}}} \]  (4)
2-point function in the double scaling limit

\[ G^{DS}_2(\mu) = N^{-3} \sum_{\omega \in \mathbb{N}/2} G^\omega_{\text{dom}}(\mu) \]

\[ = M_c(\mu) \left( 1 + N^{-\frac{1}{4}} \sqrt{3} \frac{t_c(\mu)^{\frac{1}{4}}}{(1 + 6t_c(\mu))^{\frac{1}{2}}} \frac{1 - \sqrt{1 - 4\kappa(\mu)}}{2\kappa(\mu)^{\frac{1}{2}}} \right) \]

convergent for \( \kappa(\mu) \leq \frac{1}{4} \).

tensor double scaling limit is summable
(different behaviour with respect to the celebrated matrix models case)
The prismatic tensor model

$O(N)^3$-invariance, 6th order interaction

Definition of the model

model introduced in


$O(N)^3$ invariance

$T_{i_1i_2i_3} = O_{i_1j_1}^{(1)} O_{i_1j_1}^{(2)} O_{i_1j_1}^{(3)} T_{j_1j_2j_3}$

The action

$$S(T) = -\frac{1}{2} \sum_{i,j,k} T_{ijk} T_{ijk} + \frac{tN^{-3}}{6} \sum_{a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3} T_{a_1b_1c_1} T_{a_1b_2c_2} T_{a_2b_1c_2} T_{a_3b_3c_1} T_{a_3b_2c_3} T_{a_2b_3c_3}$$

(generalization of $S_{CTKT}$)
Intermediate field method

the prismatic interaction term rewrites

\[ \int \frac{[d\chi]}{(2\pi)^{N^3/2}} e^{-\frac{1}{2} \sum_{i,j,k=1}^{N} \chi_{ijk} \chi_{ijk} + \sqrt{\frac{2tN-\alpha}{6}} \tilde{I}_t(T,\chi)}, \]  

(5)

where

\[ \tilde{I}_t(T,\chi) = \sum_{a_1,a_2,b_1,b_2,c_1,c_2=1}^{N} T_{a_1 b_1 c_1} T_{a_1 b_2 c_2} T_{a_2 b_1 c_2} \chi_{a_2 b_2 c_1} \chi_{a_4 b_4 c_4}. \]  

(6)

tetrahedric representation (of the prismatic model)
Melonic insertions in the tetrahedric representation

vacuum elementary melon:

2 types of melonic insertions:
Leading order graphs in the tetrahedric representation

elementary melon of the tetrahedric representation
$\rightarrow$ *elementary triple tadpole*
Melonic moves in the prismatic representation

insertion on a $T$ propagator
$\rightarrow$ insertion of a 2-point double tadpole

insertion on $\chi$ propagator $\rightarrow$ insertion at the level of a prismatic vertex (split a vertex into 2 vertices)

(where no intermediate field approach was used)
Examples of LO graphs in the prismatic representation
Implementation of the double scaling limit mechanism


use of the tetrahedric representation

much more tedious than for $S_{CTKT}$:

- 5 types of dipoles
- a bunch of types of chains
- much more involved structure of the schemes

double scaling parameter

$$\kappa(t, N) = \frac{I(t_c) L(t_c)}{4 N M^2 T_c K^2 (1 - \frac{t}{t_c})}$$
2-point function in the double scaling limit

\[ G_{2,DS}(t, N) = M_{T,c} + \sum_{\omega > 0} N^{-\omega} G_{\omega,\text{dom}} \]

\[ = M_{T,c} + M_{T,c} N^{-\frac{1}{2}} \left( \frac{L(t_c) \kappa(t, N)}{I(t_c)} \right)^{1/2} \sum_{\omega \in \mathbb{N}^*} \text{Cat}_{\omega-1} \kappa(t, N)^{\omega} \]

\[ = M_{T,c} \left( 1 + N^{-\frac{1}{2}} \left( \frac{L(t_c)}{I(t_c)} \right)^{1/2} \frac{1 - \sqrt{1 - 4 \kappa(t, N)}}{2 \kappa(t, N)^{1/2}} \right) \]

(7)
contributions of all degrees, and not just from the vanishing degree (the higher it is the degree of the graph, the greater it is the contribution from the respective degree)

in the limit $\kappa \to 0$ the large $N$ limit is recovered.

the double scaling limit series is convergent (difference wrt matrix models)
Implementation of this approach for multi-matrix models

double/triple-scaling limit mechanism

- $U(N)^2 \times O(D)$, tetrahedric interaction, multi-matrix models
  

  D. Benedetti et. al., Annales IHP D Comb., Phys. and their Interactions 2022)

- generalized interactions (all invariant quartic interactions) for multi-matrix models
  
purely **combinatorial techniques** can be used to study **physical mechanisms**, such as the double scaling limit for various tensor and multi-matrix models.
A very good book on all these topics

Je vous remercie pour votre attention !

Vă mulțumesc pentru atenție!
Comparison with the colored case

The dominant schemes differ:

for the colored model, for degree $\omega \in \mathbb{Z}_+$, the dominant schemes are associated to rooted binary trees with $\omega + 1$ leaves (and $\omega - 1$ inner nodes), where the root-leaf is occupied by a root-melon, while the $\omega$ non-root leaves are occupied by the unique scheme of degree 1.