Introduction to Popular Matchings

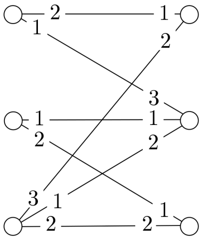
Kavitha Telikepalli

(Tata Institute of Fundamental Research, Mumbai)

Laboratoire d'Informatique de Paris-Nord (LIPN)
Université Sorbonne Paris Nord

The input

A bipartite graph where every vertex has a strict ranking of its neighbors.



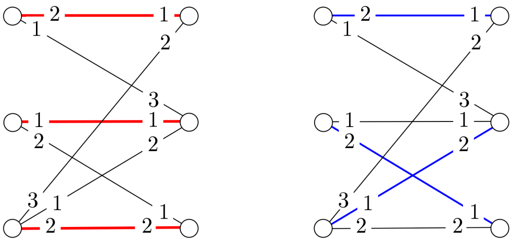
A well-studied model used in many two-sided markets:

- students to schools;
- medical residents to hospitals.

What we seek is a matching in this graph.

Matchings

A matching is a subset of edges such that at most one edge is incident to any vertex.



Recall that vertices have preferences.

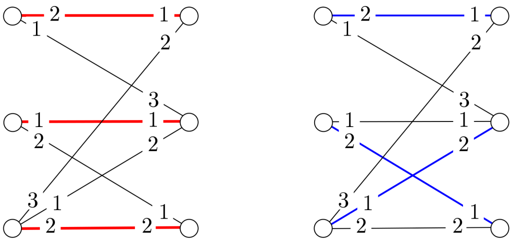
- Our problem is to find an optimal matching as per vertex preferences.

Stability

A matching M is stable if there is no edge ab such that:

$$b \succ_a M(a) \qquad \text{and} \qquad a \succ_b M(b)$$

(i.e., a and b prefer each other to their respective assignments in M)



- The red matching is stable but the blue one is not.

Stable matchings

Do stable matchings always exist? Can we find one efficiently?

- Yes [Gale and Shapley, 1962].

David Gale (1921-2008)
PROFESSOR, UC BERKELEY

Lloyd Shapley
PROFESSOR EMERITUS, UCLA

Stable matchings

https://medium.com/@UofCalifornia/how-a-matchmaking-algorithm-saved-lives-2a65ac448698

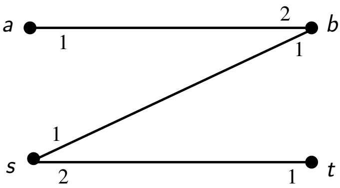
- In assigning new doctors to hospitals around the US.
- In helping kidney transplant patients find a match.

Stable matchings

Do stable matchings always exist? Can we find one efficiently?

- Yes [Gale and Shapley, 1962].

The Gale-Shapley algorithm: agents propose and jobs dispose — this is a very simple and clean algorithm.



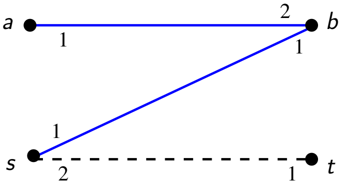
Let us run Gale-Shapley algorithm on this instance.

Stable matchings

Do stable matchings always exist? Can we find one efficiently?

- Yes [Gale and Shapley, 1962].

The Gale-Shapley algorithm: agents propose and jobs dispose — this is a very simple and clean algorithm.



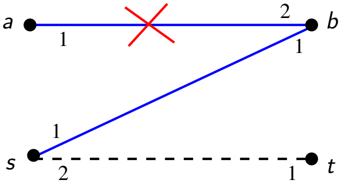
Initially both *a* and *s* propose to their top neighbor *b*.

Stable matchings

Do stable matchings always exist? Can we find one efficiently?

- Yes [Gale and Shapley, 1962].

The Gale-Shapley algorithm: agents propose and jobs dispose — this is a very simple and clean algorithm.



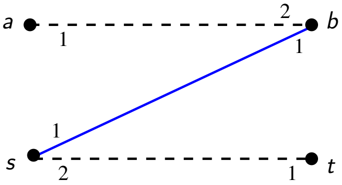
b (tentatively) accepts s's proposal and rejects a's proposal.

Stable matchings

Do stable matchings always exist? Can we find one efficiently?

- Yes [Gale and Shapley, 1962].

The Gale-Shapley algorithm: agents propose and jobs dispose — this is a very simple and clean algorithm.



a has no other neighbor to propose to; we get the matching $\{sb\}$.

Applications of stable matchings

Stable matchings are used in several problems in economics, computer science, and operations research.

To match students to schools in New York:

- How Game Theory Helped Improve New York City's High School Application Process, New York Times, December 5, 2014.

To match students to colleges in France:

- Stable Matching in Practice, Claire Mathieu. ESA 2018, Keynote talk.

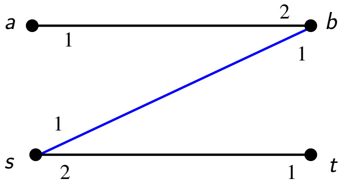
To match students to engineering colleges in India:

- Centralized admissions for engineering colleges in India, S. Baswana, P. P. Chakrabarti, S. Chandran, Y. Kanoria, and U. Patange. INFORMS Journal on Applied Analytics, 2018.

Size versus Stability

All stable matchings match the same subset of vertices [Rural Hospitals Theorem].

- The size of a stable matching could be only half the size of a maximum matching.



The maximum matching $\{ab, st\}$ is unstable.

- We seek *large* matchings in all applications.
- Forbidding blocking edges constrains the size of the matching.

Beyond stability

Drawbacks of stability:

- Size can be half the size of a maximum matching;
- Models a situation where every edge has a "veto power".

Can we relax stability so as to cope with these issues? We want a set that:

- contains stability as a special case;
- shifts the focus from "veto power" to "collective decision";
- allows for matchings of size larger than stable matchings.

Beyond stability

Drawbacks of stability:

- Size can be half the size of a maximum matching;
- Models a situation where every edge has a "veto power".

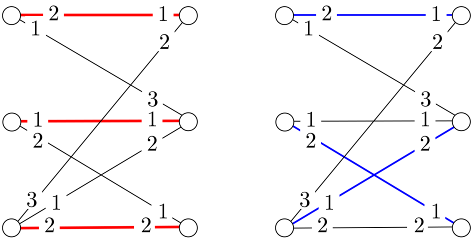
Can we relax stability so as to cope with these issues? We want a set that:

- contains stability as a special case;
- shifts the focus from "veto power" to "collective decision";
- allows for matchings of size larger than stable matchings.

$\Rightarrow$ Popular matchings

Elections between pairs of matchings

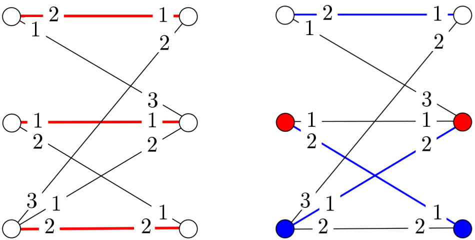
Any pair of matchings can be compared via a pairwise election.



Elections between pairs of matchings

Any pair of matchings can be compared via a pairwise election.

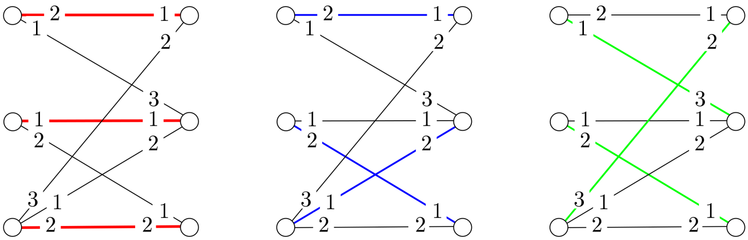
- the red vs blue election is a tie (so red $\sim$ blue).



Elections between pairs of matchings

Any pair of matchings can be compared via a pairwise election.

- the red vs blue election is a tie (so red $\sim$ blue).

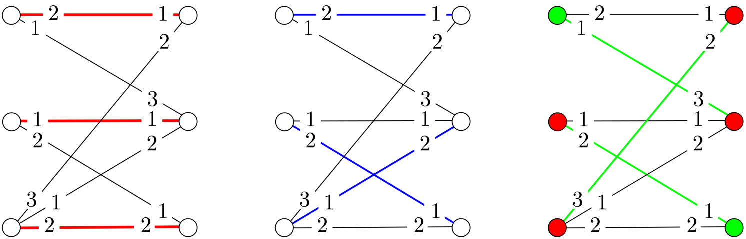


Consider the election between the red and green matchings.

Elections between pairs of matchings

Any pair of matchings can be compared via a pairwise election.

- the red vs blue election is a tie (so red $\sim$ blue).



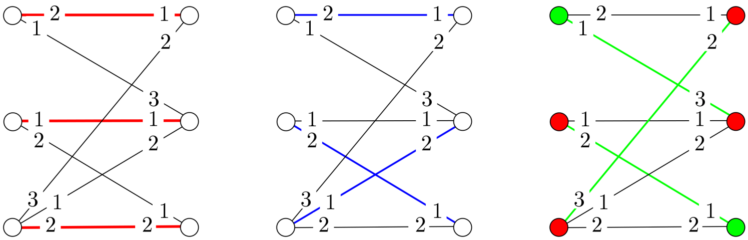
Consider the election between the red and green matchings.

- the green matching loses this election, thus red $\succ$ green.

Elections between pairs of matchings

Any pair of matchings can be compared via a pairwise election.

- the red vs blue election is a tie (so red $\sim$ blue).



Consider the election between the red and green matchings.

- the green matching loses this election, thus red $\succ$ green.

A popular matching is one that does not lose any election.

Condorcet winner

Condorcet winner: A candidate who defeats every other candidate in their head-to-head election.

	30%	30%	40%
1	a	b	c
2	b	a	a
3	c	c	b

- Here a is the Condorcet winner.
- $a \succ b$ and $a \succ c$. (a defeats b and a defeats c)

Condorcet winner

Condorcet winner: A candidate who defeats every other candidate in their head-to-head election.

	30%	30%	40%
1	a	b	c
2	b	a	a
3	c	c	b

- Here a is the Condorcet winner.
- $a \succ b$ and $a \succ c$. (a defeats b and a defeats c)

Condorcet winner

Condorcet winner: A candidate who defeats every other candidate in their head-to-head election.

	30%	30%	40%
1	a	b	c
2	b	a	a
3	c	c	b

- Here a is the Condorcet winner.
- $a \succ b$ and $a \succ c$. (a defeats b and a defeats c)

Weak Condorcet winner

A weak Condorcet winner is one that is never defeated.

- x is a weak Condorcet winner $\implies$ $x \succ y$ or $x \sim y$ for all candidates y.

Weak Condorcet winner

A weak Condorcet winner is one that is never defeated.

- x is a weak Condorcet winner $\implies x \succ y$ or $x \sim y$ for all candidates y.

However a (weak) Condorcet winner need not always exist.

	33.3%	33.3%	33.3%
1	a	b	c
2	b	c	a
3	c	a	b

- Here we have: $a \succ b \succ c \succ a$.

Weak Condorcet winner

A weak Condorcet winner is one that is never defeated.

- x is a weak Condorcet winner $\implies x \succ y$ or $x \sim y$ for all candidates y.

However a (weak) Condorcet winner need not always exist.

	33.3%	33.3%	33.3%
1	a	b	c
2	b	c	a
3	c	a	b

- Here we have: $a \succ b \succ c \succ a$.

Weak Condorcet winner

A weak Condorcet winner is one that is never defeated.

- x is a weak Condorcet winner $\implies$ $x \succ y$ or $x \sim y$ for all candidates y.

However a (weak) Condorcet winner need not always exist.

	33.3%	33.3%	33.3%
1	a	b	c
2	b	c	a
3	c	a	b

- Here we have: $a \succ b \succ c \succ a$.

Weak Condorcet winner

A weak Condorcet winner is one that is never defeated.

- x is a weak Condorcet winner $\implies x \succ y$ or $x \sim y$ for all candidates y.

However a (weak) Condorcet winner need not always exist.

	33.3%	33.3%	33.3%
1	a	b	c
2	b	c	a
3	c	a	b

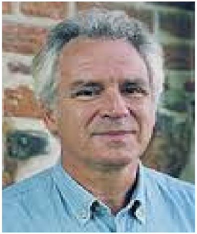
- Here we have: $a \succ b \succ c \succ a$.

Weak Condorcet winner in our setting

Matching M is a weak Condorcet winner $\equiv M \succ N$ or $M \sim N$ for all matchings N.

- Do weak Condorcet winners always exist in our setting?

Every stable matching is a weak Condorcet winner [Gärdenfors, 1975].



Comparing a stable matching S with any matching N:

- u prefers N to $S \implies N(u)$ has to prefer S to N;

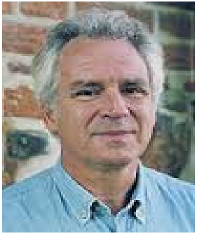
(otherwise the edge between u and $N(u)$ blocks S)

Weak Condorcet winner in our setting

Matching M is a weak Condorcet winner $\equiv M \succ N$ or $M \sim N$ for all matchings N.

- Do weak Condorcet winners always exist in our setting?

Every stable matching is a weak Condorcet winner [Gärdenfors, 1975].



Comparing a stable matching S with any matching N:

- u prefers N to $S \implies N(u)$ has to prefer S to N;

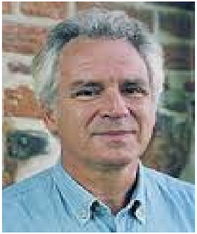
  (otherwise the edge between u and $N(u)$ blocks S)
- so the number of votes for $N \leq$ the number of votes for S.

Weak Condorcet winner in our setting

Matching M is a weak Condorcet winner $\equiv M \succ N$ or $M \sim N$ for all matchings N.

- Do weak Condorcet winners always exist in our setting?

Every stable matching is a weak Condorcet winner [Gärdenfors, 1975].



Comparing a stable matching S with any matching N:

- u prefers N to $S \implies N(u)$ has to prefer S to N;

  (otherwise the edge between u and $N(u)$ blocks S)
- so the number of votes for $N \leq$ the number of votes for S.

Matchings that are weak Condorcet winners = Popular matchings.

Popular matchings

Properties of popular matchings:

- contains stability as a special case;
- shifts the focus from "veto power" to "collective decision"; ✓
- allows for matchings of size larger than stable matchings.

Popular matchings

Properties of popular matchings:

- contains stability as a special case; ✓
- shifts the focus from "veto power" to "collective decision"; ✓
- allows for matchings of size larger than stable matchings.

Every stable matching is popular [Gärdenfors, 1975].

Popular matchings

Properties of popular matchings:

- contains stability as a special case; ✓
- shifts the focus from "veto power" to "collective decision"; ✓
- allows for matchings of size larger than stable matchings. ✓

Every stable matching is popular [Gärdenfors, 1975].

- Stable matchings are min-size popular matchings.

Popular matchings

Properties of popular matchings:

- contains stability as a special case; ✓
- shifts the focus from "veto power" to "collective decision"; ✓
- allows for matchings of size larger than stable matchings. ✓

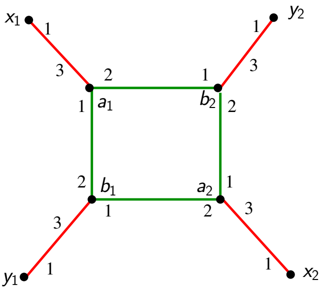
Every stable matching is popular [Gärdenfors, 1975].

- Stable matchings are min-size popular matchings.

Is there an efficient algorithm to find a <u>max-size</u> popular matching?

An interesting example

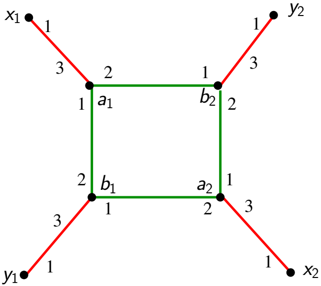
There is a popular matching of size 2 and there is also one of size 4.



- But there is no popular matching of size 3 here.

An interesting example

There is a popular matching of size 2 and there is also one of size 4.



- But there is no popular matching of size 3 here.
- So the following iterative approach — have a popular matching of size i and use this popular matching to build one of size $i+1$ — will not work.

To find a max-size popular matching

To find a max-size popular matching, can we adapt the Gale-Shapley algorithm?

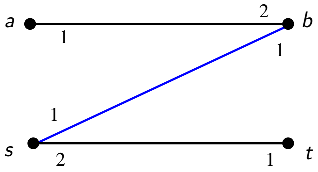
- Stability is easy to check: no edge blocks a stable matching.
- Popularity requires comparing our matching with all the matchings in G.

To find a max-size popular matching

To find a max-size popular matching, can we adapt the Gale-Shapley algorithm?

- Stability is easy to check: no edge blocks a stable matching.
- Popularity requires comparing our matching with all the matchings in G.

Suppose G is our earlier example.



Our goal is to find the matching $\{ab, st\}$ of size 2 via the Gale-Shapley algorithm.

- This is a max-size popular matching in G.

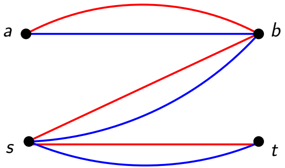
A new instance G'

A new graph G' such that $\{ab, st\}$ is the stable matching in G'?

Suppose we replace every edge uv in G by the pair of edges uv and uv in G':

- that is, by two parallel edges: one red and the other blue.

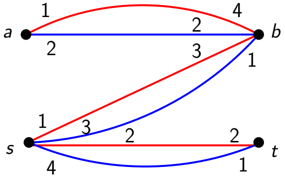
The corresponding graph G' is:



- Every vertex on the left prefers any red edge to any blue edge.
- Every vertex on the right prefers any blue edge to any red edge.

A new instance G'

So the graph G' with preferences is:



- The preference order of s in G is $b \succ t$. Its preference order in G' is:

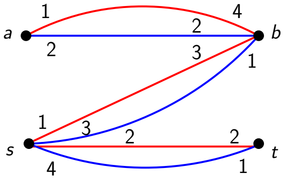
$$b \succ t \succ b \succ t.$$

- The preference order of b in G is $s \succ a$. Its preference order in G' is:

$$s \succ a \succ s \succ a.$$

A new instance G'

The graph G' with preferences is:

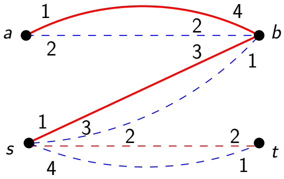


Recall the stable matching $\{sb\}$ in G.

- In the graph G', neither $\{sb\}$ nor $\{sb\}$ is stable.
  - The edge ab blocks the matching $\{sb\}$.
  - The edge st blocks the matching $\{sb\}$.

Computing a stable matching in G'

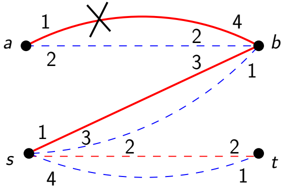
Let us run Gale-Shapley algorithm in G'.



- Both a and s propose to b along their red edges.
- b prefers s's proposal to a's proposal.

Computing a stable matching in G'

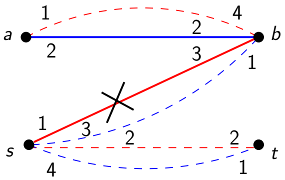
Let us run Gale-Shapley algorithm in G'.



- So b (tentatively) accepts s's proposal and rejects a's proposal.
- Then a proposes along its next favorite edge: this is ab.

Computing a stable matching in G'

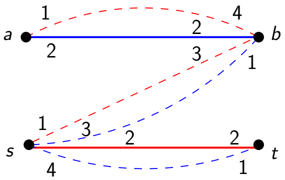
Let us run Gale-Shapley algorithm in G'.



- Observe that now b prefers a's proposal to s's proposal.
- So b (tentatively) accepts a's proposal and rejects s's proposal.

Computing a stable matching in G'

Let us run Gale-Shapley algorithm in G'.



- Then s proposes along its next most favorite edge st.
- t (tentatively) accepts s's proposal. This is the end of the algorithm.

Computing a stable matching in G'

So we get the stable matching $\{ab, st\}$ in G'.

Ignoring colors, this is the desired matching $M = \{ab, st\}$ in G.

Our algorithm in $G = (A \cup B, E)$

- Construct the red/blue graph $G' = (A \cup B, E')$.
- Run Gale-Shapley algorithm in G' to compute M'.
- Return the corresponding matching M in G.

Computing a stable matching in G'

So we get the stable matching $\{ab, st\}$ in G'.

Ignoring colors, this is the desired matching $M = \{ab, st\}$ in G.

Our algorithm in $G = (A \cup B, E)$

- Construct the red/blue graph $G' = (A \cup B, E')$.
- Run Gale-Shapley algorithm in G' to compute M'.
- Return the corresponding matching M in G.

CLAIM. M is a max-size popular matching in G.

- We use linear programming to prove the popularity of M.

Analyzing our algorithm

Every popular matching admits a simple certificate of its popularity.

- The certificate for M is given by red/blue edge colours in the matching M'.

Analyzing our algorithm

Every popular matching admits a simple certificate of its popularity.

- The certificate for M is given by red/blue edge colours in the matching M'.

Let us define an edge weight function in G. For any edge ab:

$$\mathrm{wt}_M(ab) \;=\; \mathrm{vote}_a(b, M(a)) \;+\; \mathrm{vote}_b(a, M(b)).$$

$$\text{Here } \mathrm{vote}_v(u, u') \;=\; \begin{cases} 1 & \text{if } v \text{ prefers } u \text{ to } u' \\ -1 & \text{if } v \text{ prefers } u' \text{ to } u \\ 0 & \text{otherwise.} \end{cases}$$

So $\mathrm{wt}_M(e) \in \{0, \pm 2\}$ for any edge e.

- Observation. For any edge e, $\mathrm{wt}_M(e) = 2 \iff e$ is a blocking edge to M.

An appropriate edge weight function

Let us augment G with self-loops:

- any matching $\rightsquigarrow$ a perfect matching via self-loops.

For any self-loop uu:

$$\text{let } \mathsf{wt}_M(uu) \;=\; \mathsf{vote}_u(u, M(u)) \;=\; \begin{cases} 0 & \text{if } M(u) = u \\ -1 & \text{otherwise.} \end{cases}$$

An appropriate edge weight function

Let us augment G with self-loops:

- any matching $\rightsquigarrow$ a perfect matching via self-loops.

For any self-loop uu:

$$\text{let } \mathsf{wt}_M(uu) \;=\; \mathsf{vote}_u(u, M(u)) \;=\; \begin{cases} 0 & \text{if } M(u) = u \\ -1 & \text{otherwise.} \end{cases}$$

Observation. For any perfect matching N:

$$\mathsf{wt}_M(N) \;=\; \#\text{ of votes for } N \;-\; \#\text{ of votes for } M.$$

- M is popular $\iff$ $\mathsf{wt}_M(N) \le 0$ for any perfect matching N.

  $\iff$ any perfect matching in G with edge weights given by wt_M has weight at most 0.

LP for max-weight perfect matching

$$\max \sum_{e} \mathrm{wt}_M(e) \cdot x_e$$

$$\begin{aligned} \sum_{e \in \delta(u) \cup \{uu\}} x_e &= 1 \;\; \forall u \in A \cup B \\ x_e &\geq 0 \;\; \forall e \in E \cup \{\text{self-loops}\}. \end{aligned}$$

M is popular $\iff$ the optimal value of this LP is at most 0.

LP for max-weight perfect matching

$$\max \sum_{e} \mathrm{wt}_M(e) \cdot x_e$$

$$\begin{aligned} \sum_{e \in \delta(u) \cup \{uu\}} x_e &= 1 \quad \forall u \in A \cup B \\ x_e &\geq 0 \quad \forall e \in E \cup \{\text{self-loops}\}. \end{aligned}$$

M is popular $\iff$ the optimal value of this LP is at most 0.

Dual LP

$$\min \sum_{u} \alpha_u$$

$$\begin{aligned} \alpha_a + \alpha_b &\geq \mathrm{wt}_M(ab) \quad \forall\ ab \in E \\ \alpha_u &\geq \mathrm{wt}_M(uu) \quad \forall\ u \in A \cup B. \end{aligned}$$

M is popular $\iff$ the optimal value of the dual LP is at most 0.

Dual certificate

Every stable matching S has a simple dual certificate: $\vec{\alpha} = \vec{0}$.

- This is because $\text{wt}_S(e) \leq 0$ for all edges e.

Does M computed by our algorithm have an easy-to-describe dual certificate?

Dual certificate

Every stable matching S has a simple dual certificate: $\vec{\alpha} = \vec{0}$.

- This is because $\mathrm{wt}_S(e) \leq 0$ for all edges e.

Does M computed by our algorithm have an easy-to-describe dual certificate?

For each vertex $a \in A$:

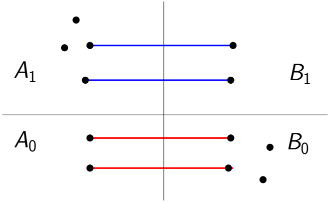
- a is matched along a red edge in M': set $\alpha_a = 1$.
- a is matched along a blue edge in M': set $\alpha_a = -1$.
- a is unmatched in M': set $\alpha_a = 0$.

For each vertex $b \in B$:

- b is matched along a red edge in M': set $\alpha_b = -1$.
- b is matched along a blue edge in M': set $\alpha_b = 1$.
- b is unmatched in M': set $\alpha_b = 0$.

Dual certificate

A useful picture:



So vertices matched along red edges are in $A_0 \cup B_0$.

And vertices matched along blue edges are in $A_1 \cup B_1$.

- Unmatched vertices of A (resp., B) are in A_1 (resp., B_0).

α-values were assigned as follows:

- $\alpha_u = 1$ for all $u \in A_0 \cup B_1$;
- $\alpha_u = -1$ for all matched $u \in A_1 \cup B_0$;
- $\alpha_u = 0$ for all unmatched u.

Dual feasibility of $\vec{\alpha}$

We need to show this vector $\vec{\alpha}$ is a feasible solution to the dual LP.

Dual LP

$$\min \sum_u \alpha_u$$

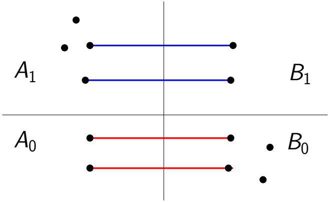
$$\begin{aligned} \alpha_a + \alpha_b &\geq \mathrm{wt}_M(ab) \quad \forall\ ab \in E \\ \alpha_u &\geq \mathrm{wt}_M(uu) \quad \forall\ u \in A \cup B. \end{aligned}$$

We will also show that $\sum_{u \in A \cup B} \alpha_u = 0$.

- This will mean the dual optimal solution is at most 0.
- This will prove M is a popular matching.

Dual feasibility of $\vec{\alpha}$

Recall that $\alpha_u \in \{0, \pm 1\}$:

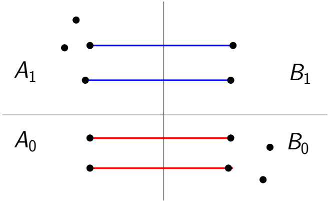


OBSERVATION. *The constraint* $\alpha_u \geq \mathrm{wt}_M(uu)$ *holds for all vertices* u.

- For a matched vertex u, we have $\alpha_u \geq -1 = \mathrm{wt}_M(uu)$.
- For an unmatched vertex u, we have $\alpha_u = 0 = \mathrm{wt}_M(uu)$.

Dual feasibility of $\vec{\alpha}$

Recall that $\alpha_u \in \{0, \pm 1\}$:



Observation. *The constraint $\alpha_u \geq \mathrm{wt}_M(uu)$ holds for all vertices u.*

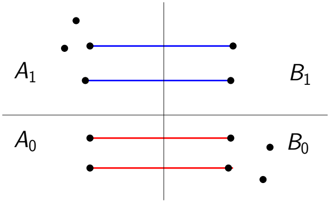
- For a matched vertex u, we have $\alpha_u \geq -1 = \mathrm{wt}_M(uu)$.
- For an unmatched vertex u, we have $\alpha_u = 0 = \mathrm{wt}_M(uu)$.

Lemma. *The constraint $\alpha_a + \alpha_b \geq \mathrm{wt}_M(ab)$ holds for all $ab \in E$.*

- We will use the stability of M' in the instance G' to prove the lemma.

Conclusion. So $\vec{\alpha}$ is dual-feasible.

Optimal value of the dual LP



Every edge in M' is a red edge or a blue edge.

- So $\alpha_a + \alpha_b = 0$ for all $ab \in M$.
- Since $\alpha_u = 0$ for all unmatched vertices, $\sum_{u \in A \cup B} \alpha_u = 0$.

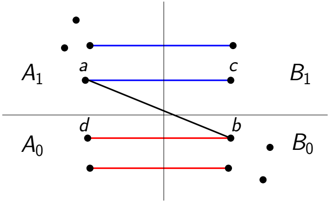
Thus the optimal value of the dual LP is at most 0.

- Hence M is a popular matching.

Proof of the lemma

To show $\alpha_a + \alpha_b \geq \text{wt}_M(ab)$ holds for all $ab \in E$.

Case 1. Suppose $\alpha_a = \alpha_b = -1$.



- So $ac \in M'$ and $bd \in M'$ for some neighbors c and d of a and b, respectively.
- Observe that (i) a prefers c to b and (ii) b prefers d to a.
  - This is because a never proposed along ab.
  - Furthermore, b rejected a's proposal along ab.

Thus $\text{wt}_M(ab) = -2$, hence $\alpha_a + \alpha_b = -2 = \text{wt}_M(ab)$.

Proof of the lemma

Case 2. Suppose $\alpha_a = \alpha_b = 1$.

- Since $\text{wt}_M(ab) \in \{0, \pm 2\}$, we have $\alpha_a + \alpha_b = 2 \geq \text{wt}_M(ab)$.

Case 3. Suppose $\alpha_a = 1$ and $\alpha_b = -1$.

- This means ac and bd are in M' for some neighbors c and d.
- M' is stable in $G' \Rightarrow ab$ does not block M'.

Thus $\text{wt}_M(ab) \leq 0$, hence $\alpha_a + \alpha_b = 0 \geq \text{wt}_M(ab)$.

Case 4. Suppose $\alpha_a = -1$ and $\alpha_b = 1$.

- This means ac and bd are in M' for some neighbors c and d.
- M' is stable in $G' \Rightarrow ab$ does not block M'.

Thus $\text{wt}_M(ab) \leq 0$, hence $\alpha_a + \alpha_b = 0 \geq \text{wt}_M(ab)$.

Proof of the lemma

Case 5. Suppose $\alpha_a = 0$.

Since M' is stable in G', ab does not block M'.

- This means $bd \in M'$ for some neighbor d that b prefers to a.

Thus $\alpha_b = 1$, hence $\alpha_a + \alpha_b = 1 \geq 0 = \text{wt}_M(ab)$.

An analogous analysis holds when $\alpha_b = 0$.

- Then $\alpha_a = 0$ and $\alpha_b = 1$, so $\alpha_a + \alpha_b = 1 \geq 0 = \text{wt}_M(ab)$.

This finishes the proof of the lemma. □

Proof of the lemma

Case 5. Suppose $\alpha_a = 0$.

Since M' is stable in G', ab does not block M'.

- This means $bd \in M'$ for some neighbor d that b prefers to a.

Thus $\alpha_b = 1$, hence $\alpha_a + \alpha_b = 1 \geq 0 = \text{wt}_M(ab)$.

An analogous analysis holds when $\alpha_b = 0$.

- Then $\alpha_a = 0$ and $\alpha_b = 1$, so $\alpha_a + \alpha_b = 1 \geq 0 = \text{wt}_M(ab)$.

This finishes the proof of the lemma. □

A useful observation

For any edge ab incident to an unmatched vertex (either a or b is unmatched):

- we have $\alpha_a + \alpha_b = 1 > 0 = \text{wt}_M(ab)$, thus the edge ab is *slack*.

The dual LP and slack edges

$$\min \sum_u \alpha_u$$

$$\begin{aligned} \alpha_a + \alpha_b &\geq \mathrm{wt}_M(ab) \quad \forall\, ab \in E \\ \alpha_u &\geq \mathrm{wt}_M(uu) \quad \forall\, u \in A \cup B. \end{aligned}$$

Recall that $\vec{\alpha}$ is an optimal solution to the dual LP.

<u>COMPLEMENTARY SLACKNESS</u>

Any matching N with a slack edge is not an optimal solution to the primal LP;

- in other words, $\mathrm{wt}_M(N) < 0$ (equivalently, M defeats N).

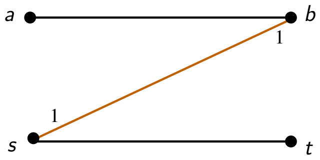
Thus any matching larger than M is unpopular.

- So M is a <u>max-size</u> popular matching. □

Thus there is a linear time algorithm to find a max-size popular matching.

Lower bound on $|M|$

CLAIM. There is no length 3 augmenting path wrt M in G.



- $a - b - s - t$ is an augmenting path wrt $M \implies$ either ab or st blocks M'
(a contradiction to M''s stability in G')

Hence any augmenting path in $M \oplus M_{\text{max}}$ has length ≥ 5.

- Thus $|M| \geq \frac{2}{3} \cdot |M_{\text{max}}|$.
- There are simple examples where $|M| = 2$ and $|M_{\text{max}}| = 3$.

Maximum matchings

Applications where the size of the matching is more important than vertex preferences:

- matching medical students to hospitals for residency;
- matching doctors to hospitals in a pandemic;
- assigning accommodation to sailors.

Here $\{\text{admissible solutions}\} = \{\text{maximum matchings}\}$.

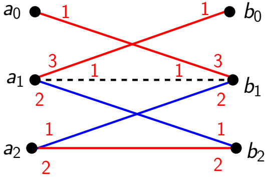
The goal is to find a *best* maximum matching as per vertex preferences.

- How about a maximum matching with the *minimum* number of blocking edges?
  - Finding such a matching is NP-hard [Biro, Manlove, and Mittal, 2010].

- How about a maximum matching that is popular?

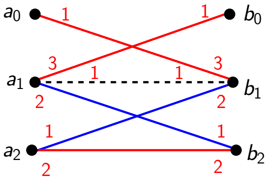
Maximum matchings and popularity

It can be the case that no maximum matching is popular.



Maximum matchings and popularity

It can be the case that no maximum matching is popular.

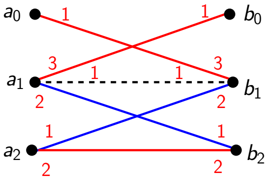


How about a maximum matching M that is popular *among* maximum matchings?

- So M is a maximum matching.
- Furthermore, $M \succ N$ or $M \sim N$ for all maximum matchings N.

Maximum matchings and popularity

It can be the case that no maximum matching is popular.



How about a maximum matching M that is popular *among* maximum matchings?

- So M is a maximum matching.
- Furthermore, $M \succ N$ or $M \sim N$ for all maximum matchings N.

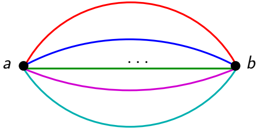
Does such a "popular maximum matching" always exist in G?

- Furthermore, is it easy to find one?

More colorful graphs

Suppose we use n colors, where $|A| = n$. Call the resulting graph G^*.

- Every edge ab in G has n parallel copies in G^*: $ab, ab, \ldots, ab, \ldots, ab, ab$.



For any vertex on the left:

$$red \succ blue \succ \cdots \succ green \succ \cdots magenta \succ cyan.$$

For any vertex on the right:

$$cyan \succ magenta \succ \cdots \succ green \succ \cdots \succ blue \succ red.$$

Within any color class, every vertex maintains its original preference order $\succ$.

More colorful graphs

An extension of our algorithm

- Construct the colorful graph $G^* = (A \cup B, E^*)$.
- Run Gale-Shapley algorithm in G^* to compute M^*.
- Return the corresponding matching M in G.

- Claim 1. M is a maximum matching in G.
- Claim 2. $M \succ N$ or $M \sim N$ for every maximum matching N in G.

Claims 1 and 2 $\Rightarrow$ M is a popular maximum matching.

- Moreover, such a matching can be computed easily.

The LP method

Recall the following edge weight function wt_M in G. For any edge ab:

$$\mathsf{wt}_M(ab) \;=\; \mathsf{vote}_a(b, M(a)) \;+\; \mathsf{vote}_b(a, M(b)).$$

$$\text{Here } \mathsf{vote}_v(u, u') \;=\; \begin{cases} 1 & \text{if } v \text{ prefers } u \text{ to } u' \\ -1 & \text{if } v \text{ prefers } u' \text{ to } u \\ 0 & \text{otherwise.} \end{cases}$$

So $\mathsf{wt}_M(e) \in \{0, \pm 2\}$ for any edge e.

- Let M be a maximum matching in G.
- Observation. $\mathsf{wt}_M(N) \le 0$ for all maximum matchings N

  $\Rightarrow M$ is a popular maximum matching in G.

The LP method

LP for max-weight maximum matching in G:

$$\max \sum_{e} \mathsf{wt}_M(e) \cdot x_e$$

$$\begin{aligned} \sum_{e \in \delta(u)} x_e &\leq 1 \quad \forall u \in A \cup B \\ \sum_{a \in A} \sum_{e \in \delta(a)} x_e &= k \quad \text{and} \quad x_e \geq 0 \;\; \forall e \in E. \end{aligned}$$

Here k is the size of a maximum matching in G.

Optimal value of this LP is at most $0 \Rightarrow \mathsf{wt}_M(N) \leq 0$ for all maximum matchings N

$\Rightarrow M$ is a popular maximum matching in G.

The dual LP

Dual LP

$$\min \quad k \cdot z \; + \; \sum_u \alpha_u$$

$$\begin{aligned} \alpha_a + \alpha_b + z \quad &\geq \quad \mathrm{wt}_M(ab) \quad \forall\; ab \in E \\ \alpha_u \quad &\geq \quad 0 \qquad\qquad\quad \forall\; u \in A \cup B. \end{aligned}$$

Our goal is to show that the optimal value of the dual LP is at most 0.

- Thus our goal is to show a dual feasible solution $(\vec{\alpha}, z)$ such that

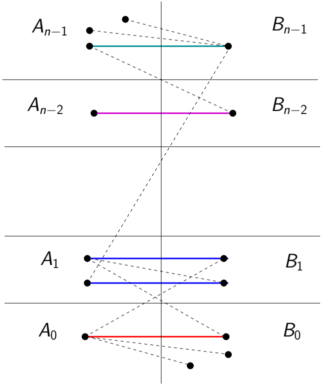
$$k \cdot z + \sum_u \alpha_u = 0.$$

- Recall the colorful graph G^*:
  - let color 0, color 1, ..., color $n-1$ denote the n colors (here $n = |A|$).

A partition of the vertex set $A \cup B$

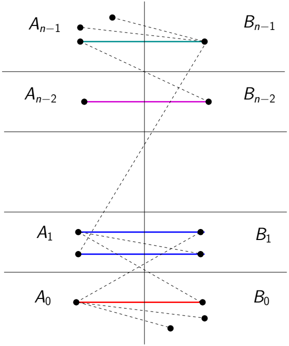
For $0 \leq i \leq n-1$, let $A_i = \{a \in A : a$ is matched along a color i edge in $M^*\}$.

For $0 \leq i \leq n-1$, let $B_i = \{b \in B : b$ is matched along a color i edge in $M^*\}$.



Unmatched vertices of A are in A_{n-1} and unmatched vertices of B are in B_0.

A partition of the vertex set $A \cup B$



The following properties hold due to the stability of M^* in G^*:

(1) For any i, the matching M restricted to $A_i \cup B_i$ is stable.

(2) For any edge ab where $a \in A_{i+1}$ and $b \in B_i$: $\text{wt}_M(ab) = -2$.

(3) G has no edge in $A_i \times B_j$ where $i \geq j + 2$.

(4) There is no augmenting path with respect to M.

A dual certificate

Property (4) implies that M is a maximum matching in G.

For $0 \le i \le n-1$:

- $a \in A_i \Rightarrow$ set $\alpha_a = 2(n-1) - 2i$;
- $b \in B_i \Rightarrow$ set $\alpha_b = 2i$.
  - so $\alpha_u = 0$ for any $u \in A_{n-1} \cup B_0$.

Set $z = -2(n-1)$.

Properties (1)-(3) allow us to prove the dual-feasibility of $\vec{\alpha}$.

$\alpha_a + \alpha_b + z \ = \ 2(n-1) - 2i + 2i - 2(n-1) \ = \ 0$ for each $ab \in M$.

(because $a \in A_i$ and $b \in B_i$ for some $i \in \{0, \ldots, n-1\}$)

- Hence $k \cdot z + \sum_u \alpha_u \ = \ \sum_{ab \in M}(\alpha_a + \alpha_b + z) \ = \ 0$.

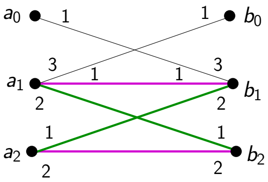
  (since $\alpha_u = 0$ for unmatched u)

Popular maximum matchings

Interestingly, *every* popular maximum matching occurs as a stable matching in the colorful graph G^*.

- So popular maximum matchings are very well-structured.

Max-size popular matchings



There are two max-size popular matchings here: purple and green.

- Only the green matching occurs as a stable matching in the red/blue graph G'.

Optimal solutions and popularity

Similar to popular maximum matchings, we can define popular optimal matchings.

Popular optimal matchings

- Suppose there is a utility function $f : E \to \mathbb{Q}$.
- It is only max-utility matchings that are relevant.

Does there exist a max-utility matching that is popular among max-utility matchings?

- If so, is it easy to find one?

Optimal solutions and popularity

Similar to popular maximum matchings, we can define popular optimal matchings.

Popular optimal matchings

- Suppose there is a utility function $f : E \to \mathbb{Q}$.
- It is only max-utility matchings that are relevant.

Does there exist a max-utility matching that is popular among max-utility matchings?

- If so, is it easy to find one?
  - The answer to both questions is "yes".

Characterizing max-utility matchings

LP for max-utility matching in $G = (A \cup B, E)$

$$\max \sum_e f(e) \cdot x_e$$

$$\begin{aligned} \sum_{e \in \delta(u)} x_e &\leq 1 \quad \forall u \in A \cup B \\ x_e &\geq 0 \quad \forall e \in E. \end{aligned}$$

The polytope of max-utility matchings is a face of the matching polytope.

Thus M is a max-utility matching $\iff M \subseteq E_0$ for some $E_0 \subseteq E$ and

- M matches all vertices in C for some $C \subseteq A \cup B$.

We want a C-perfect matching M in $G_0 = (A \cup B, E_0)$ such that:

- $M \succ N$ or $M \sim N$ for all C-perfect matchings N in G_0.

Popular C-perfect matchings

This problem can be reduced to the stable matching problem in a colorful graph $G_0^\dagger$.

- The colors of any edge ab in $G_0^\dagger$ depend on whether $a \in C$ and $b \in C$.
  - For any ab in E_0, there is always one green copy ab.
  - Every ab in E_0 where $b \in C$ has $|C \cap B|$ more copies: $ab, ab, \ldots$.
  - Every ab in E_0 where $a \in C$ has $|C \cap A|$ more copies: $ab, \ldots, ab$.

For any vertex in A:

$$\textit{red} \succ \textit{blue} \succ \cdots \succ \textit{green} \succ \textit{magenta} \succ \cdots \succ \textit{cyan}.$$

For any vertex in B:

$$\textit{cyan} \succ \cdots \succ \textit{magenta} \succ \textit{green} \succ \cdots \succ \textit{blue} \succ \textit{red}.$$

Within any color class, every vertex maintains its original preference order $\succ$.

The Gale-Shapley algorithm in $G_0^\dagger$ solves the popular C-perfect matching problem.

References

1. T. Kavitha.
   A size-popularity tradeoff in the stable marriage problem. SIAM Journal on Computing, 2014.
2. T. Kavitha.
   Maximum matchings and popularity. In ICALP 2021 (to appear in SIDMA).
3. T. Kavitha.
   Popular Solutions for Optimal Matchings. To appear in WG 2024.

Thank you! Any questions?