Pseudorandom Objects and Generators

Journées ALEA 2012

Lecture 1: Pseudorandom objects: examples and constructions

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Plan

- Today: examples of pseudorandom objects
  - Expander graphs
  - Error-correcting codes
- Tomorrow: applications of pseudorandom objects to computer science

Why Pseudorandom Objects?

- Because random objects are interesting!
- Can show random objects have many interesting properties
- "Probabilistic method": show existence of object satisfying some property
  - Define probability distribution D
  - Show $\Pr_{x \leftarrow D}[x \text{ does not satisfy property}] \ll 1$
- First used systematically in work of Erdös
- For example, proves existence of good expander graphs and good error-correcting codes

Pseudorandom objects

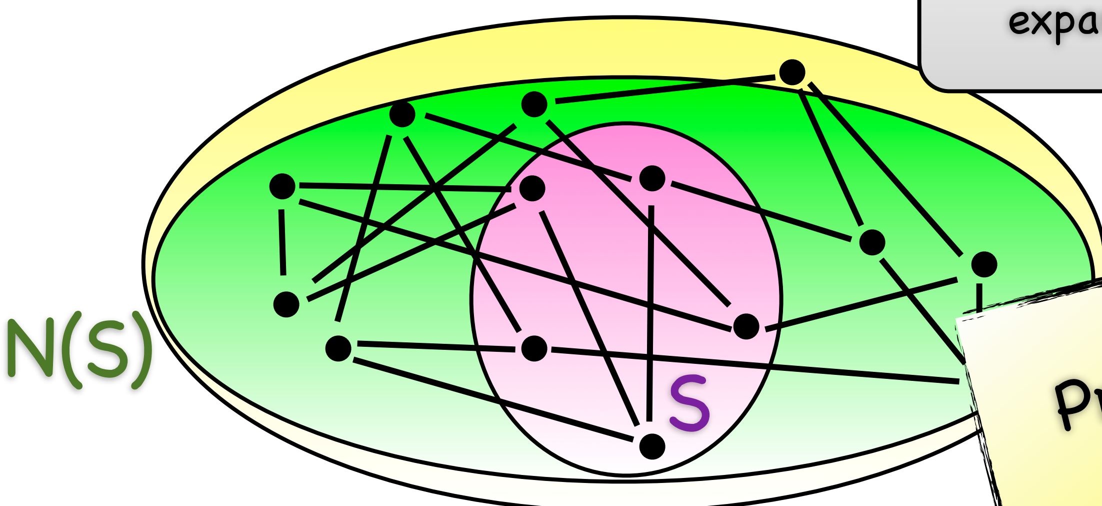
- Great, random objects have nice properties
- But: usually need explicit constructions
  - Will see applications of expanders tomorrow
- Explicit: give algorithm for constructing size n object in time $poly(n)$

Expander Graphs

Expander graphs

- Expander graphs: highly connected and sparse graphs, e.g. $|E| = O(|V|)$
- Useful: algorithms, network design, coding theory, graph theory, topology, geometry, group theory, number theory...
- Many equivalent definitions

- Def: for all sets $S \subseteq V$, where $|S| \leq |V|/2$ it holds that $|N(S)| \geq (3/2)\ |S|$
- Thm [Pinsker'73]: random graphs are expander graphs



Proof of bipartite case...

Expander graphs

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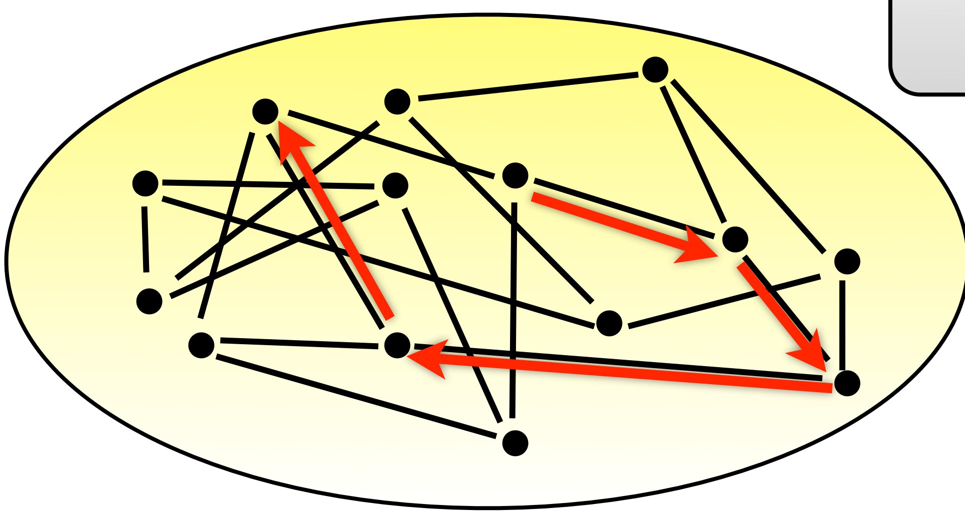
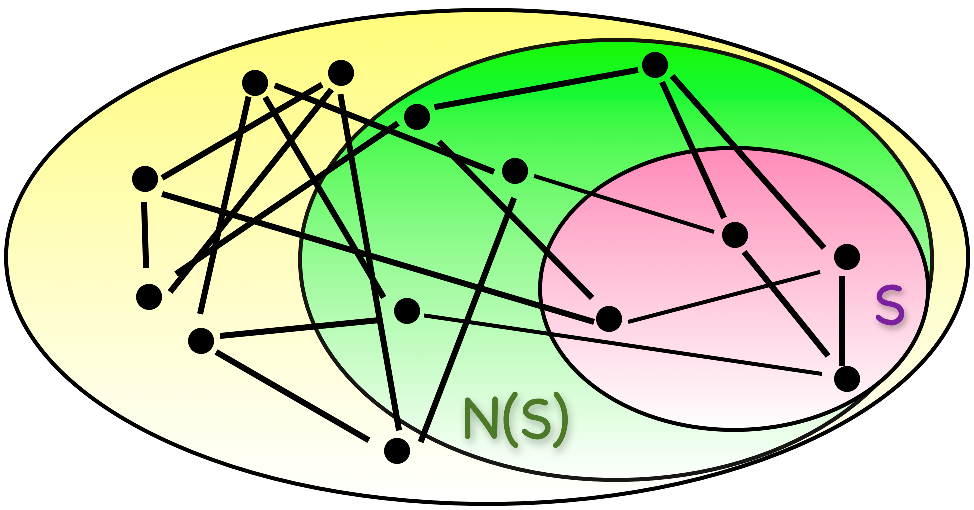
Random walk converges quickly to uniform



Defining Expanders

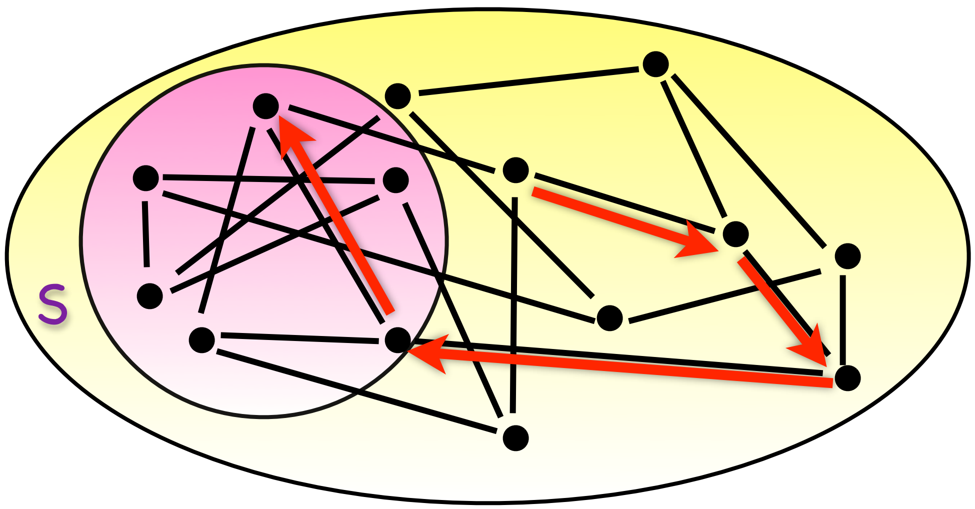


Spectral expander: G is (n, D, λ)-expander if:

- G is D-regular, $|V| = n$
- Let M = adjacency matrix of G
  - $M_{ij} = 1/D$ if $(i, j) \in G$, 0 else
  - Eigenvalues of M in $[-1, 1]$
  - Max eigenvalue = 1
- $\lambda \geq$ all other eigenvalues of M in absolute value

- Want family of (n, D, λ) graphs with $n \to \infty$, D constant, λ constant in $[0, 1[$
- Suppose G is (n, D, λ) expander, then:
  - G has vertex expansion [Alon-Milman'85, Tanner'84]:
    - For all $S \subseteq V$, $|S| \leq |V|/2$, it holds that $|N(S)| \geq 2/(\lambda^2+1)\ |S|$

Defining Expanders



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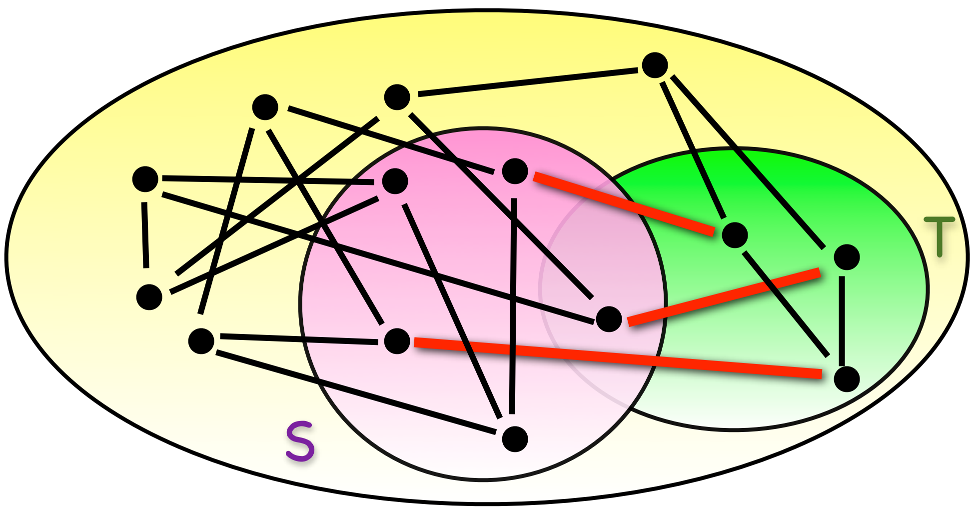
- Suppose G is (n, D, λ) expander, then:
  - Expander Chernoff bound [Gillman'93]:
    For any $S \subseteq V$, small $|S| \leq |V|/3$
    $\Pr[$ majority of random walk of length t lies in $S\ ] < 2^{-(1-\lambda)t}$

Defining Expanders



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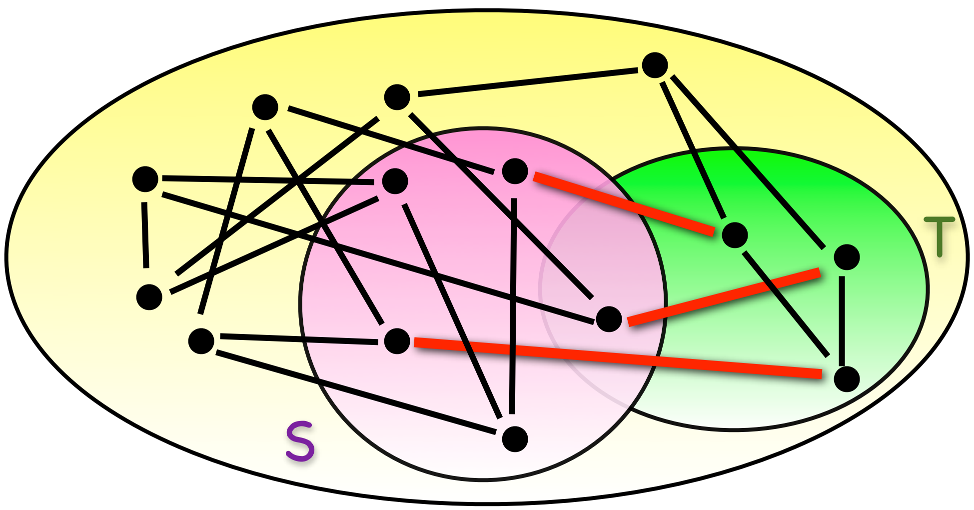
- Suppose G is (n, D, λ) expander, then:
  - Expander mixing lemma [Alon-Chung'88]:
    For all $S, T \subseteq V$, $\left|\, |E(S, T)| - |S|\,|T|\, D/n \,\right| \leq \lambda D \sqrt{(|S|\,|T|)}$

edges between S and T

expected # edges between S and T in random D-regular graph

Proof...

Defining Expanders



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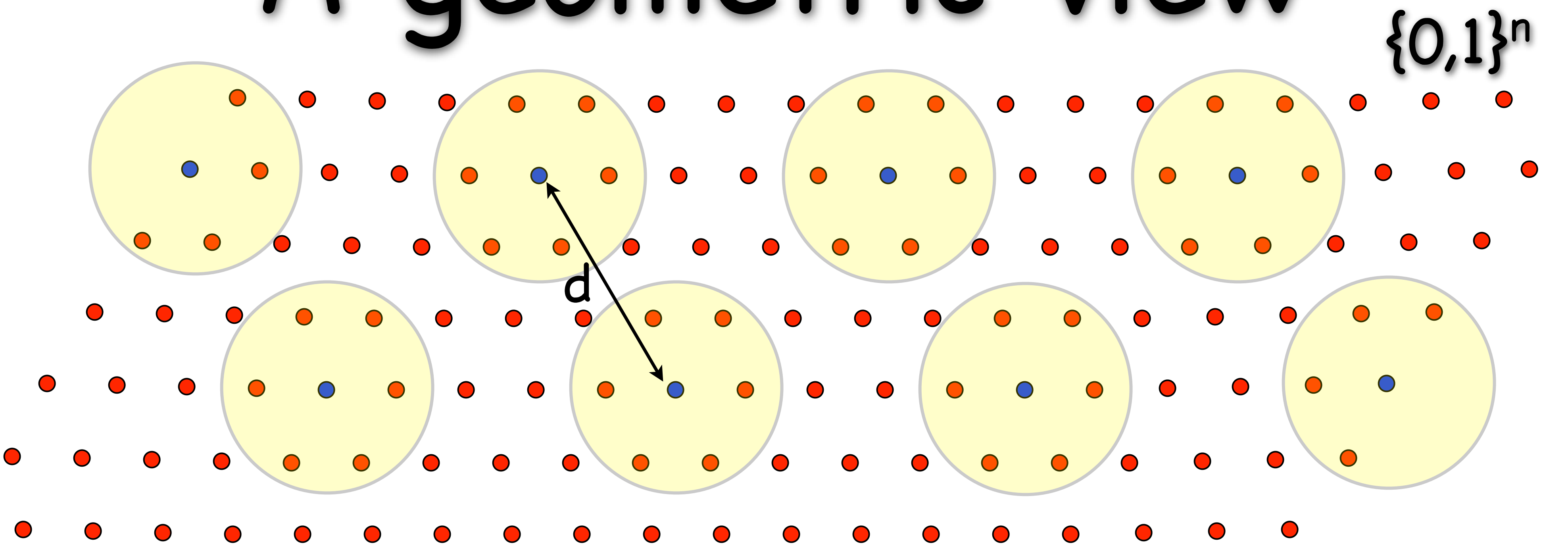
- Building expander graphs?
- $V = (\mathbb{Z}/N\mathbb{Z})^2$ E: (x, y) connected to:
  $(x, y + 2x), (x, y + 2x + 1), (x, y - 2x), (x, y - 2x - 1)$
  $(x + 2y, y), (x + 2y + 1, y), (x - 2y, y), (x - 2y - 1, y)$
- Theorem [Gabber-Galil'81]: above is $(N^2, 8, 0.89)$-expander
- Theorem [Lubotzky-Philips-Sarnak'88, Margulis'88]: constructions of "Ramanujan graphs" where $\lambda = (2/D)\sqrt{(D-1)}$ (optimal [Alon'86])
- Theorem [Reingold-Vadhan-Wigderson'01]: combinatorial constructions of expander graphs

Error correcting codes

Error correcting codes

- Alice and Bob communicate over noisy channel
- Encode messages to handle errors
- [n, k, d] code:
  - Codeword length n: bits transmitted across channel
  - Message length k: bits before encoding
  - Distance d = 2 * (maximum # of errors tolerated)
- Given n, maximize k and d

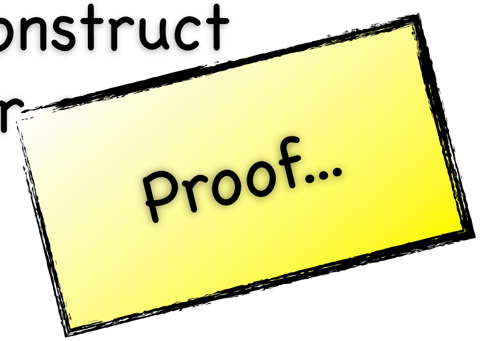
A geometric view



- Code: subset of $\{0,1\}^n$, codeword length n
- Message length $k = \log(\#\text{ codewords})$
- Distance d = minimal distance between any two codewords
- Linear code: code forms subspace of $\{0,1\}^n \simeq GF(2)^n$
  - Suffices to define basis of subspace $v_1 \ldots v_k$

Gilbert-Varshamov Bound

- Theorem [G'52]: for all n and ε, random code is a $[n, \varepsilon^2 n, n(1/2-\varepsilon)]$ code
- Theorem [V'57]: for all n and ε, random linear code is a $[n, \varepsilon^2 n, n(1/2-\varepsilon)]$ linear code
- No known explicit codes with such good parameters
- Theorem [Alon-Goldreich-Håstad-Peralta'92]: for all ε and infinitely many n, can construct explicitly $[n, 2\varepsilon \sqrt{n}, n(1/2-\varepsilon)]$ linear



Summary

- Pseudorandom objects: non-random objects that have some properties of random objects:
  - Expander graphs: connectivity
  - Error-correcting codes: large distance
- Common tools:
  - Extremal combinatorics
  - Linear Algebra
  - Group theory, representation theory
  - Finite fields, polynomials over finite fields
- Open questions: better constructions
  - Combinatorial construction of optimal expanders?
  - Binary linear codes matching Gilbert-Varshamov bound?
- Tomorrow: applications to computer science

Fin