Asymptotic behaviour of a simple cellular automaton: Use of scale invariance.

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joint work with Mathieu Sablik

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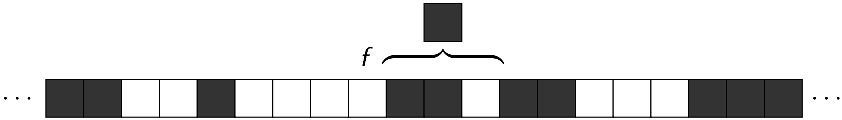
Cellular automata

- $\mathcal{A}$ a finite alphabet ;
- $\mathcal{A}^{\mathbb{Z}}$ the set of **configurations**.

A **cellular automaton** (CA) is an action $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by a **local rule** $f : \mathcal{A}^{[-r,r]} \to \mathcal{A}$ (for some $r > 0$).

Example with $\mathcal{A} = \{\blacksquare, \square\}$ and $r = 1$:



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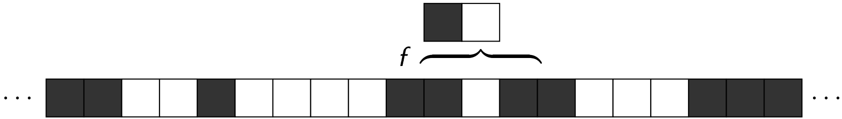

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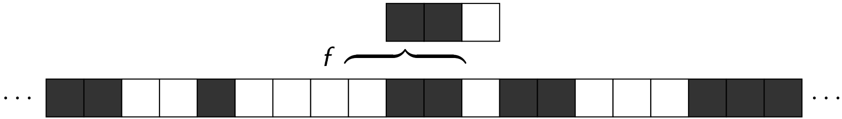

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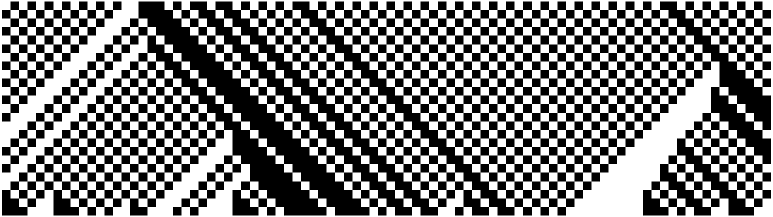

Define the **shift** action as $\sigma(a)_i = a_{i-1}$.

Initial measure

We are considering the case where the initial configuration is chosen at random.

- $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$: set of σ-invariant probability measures;
- F extends to an action $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$;
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Examples of initial measures:

- **Bernoulli measures** where each cell is drawn independently;
- **Markov measures**, which have finite memory;
- **Hidden Markov** (image of a Markov measure by a factor).

Limit measures, μ-limit set

Asymptotic behaviour can be described by the **persistent words**, whose probability to appear does not tend to 0 as $t \to \infty$.

μ-limit set

The μ-**limit set** $\Lambda_\mu(F)$ is the set of configurations containing only persistent words.

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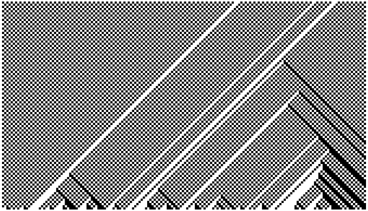
Alternatively, it can be described as the **support of the adherence values** of the sequence $(F^n\mu)$ in the appropriate topology:

$\mathcal{A}^{\mathbb{Z}}$:	$\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$:
Cantor distance	Lévy-Prohorov distance
$d_c(u,v) = \lvert\mathcal{A}\rvert^{-\min\{\lvert n\rvert; u_n \neq v_n\}}$	$d(\mu,\nu) = \sum_{u\in\mathcal{A}^*} \frac{\mu([u]_0) - \nu([u]_0)}{\mathcal{A}^{\lvert u\rvert}}$
Product topology	Weak-* convergence topology

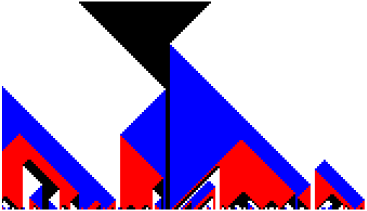
Self-organization: qualitative approach

Rule 184 (trafic rule)

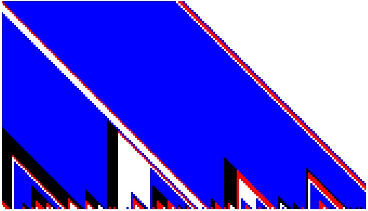


3-state cyclic CA

4-state cyclic CA



One-sided captive such that f(ab)=f(ba)

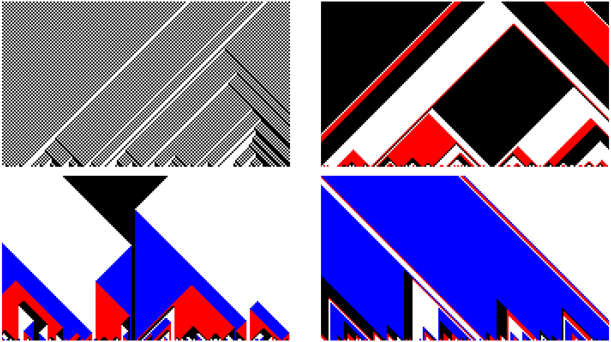


Self-organization: qualitative approach

Theorem [H., Sablik, 2011]

Let F be a CA, μ a σ-ergodic measure. Define a "set of particles" evolving at constant "speed" and such that any particle interaction is "destructive".

Then **particles appearing in $\Lambda_\mu(F)$ all have the same speed.**



Gliders automaton

Let $\mathcal{A} = \{+, 0, -\}$ and $v_- < v_+$.
The (v_-, v_+)-**gliders automaton** is the CA with two particles:

- $+$ evolving at speed v_+;
- $-$ evolving at speed v_-.

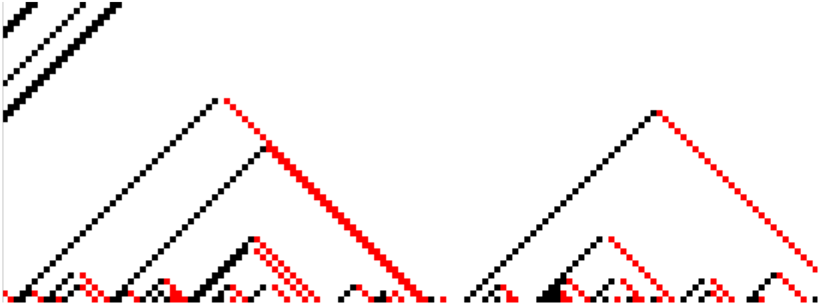


Figure: $v_- = -1, v_+ = 1$

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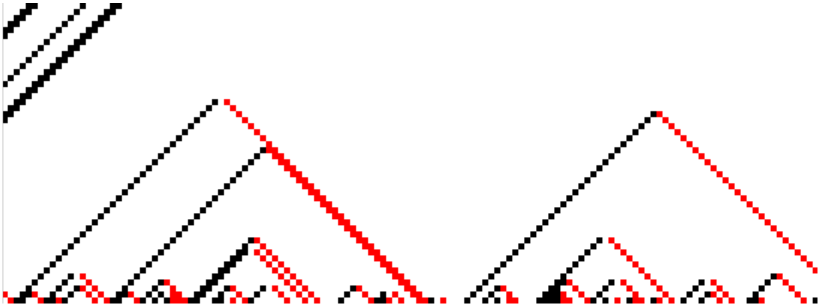

For any ergodic measure μ with $\mu([+]) = \mu([-])$, Λ_μ **contains no particle.**

Scope of our results

We will consider two families of initial measures :

- $\mathcal{B}er$ the Bernoulli measures satisfying $p_+ = p_- \neq 0$ (simple case);
- $\mathcal{HM}$ the hidden transitive, aperiodic Markov measures satisfying:
  - $\mu([+]) = \mu([-])$;
  - $\sum_{k\geq 0} \mathbb{E}(\pi_0\mu \cdot \pi_k\mu) > 0$ (asymptotic variance).

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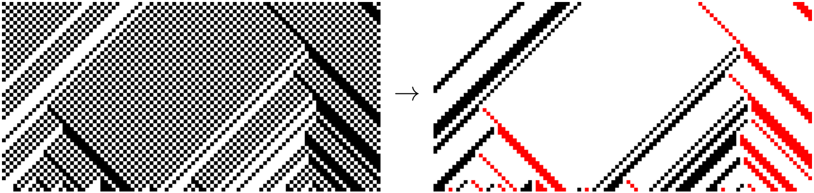


Factor Φ : □□ $\to$ ■ ■■ $\to$ ■ □■, ■□ $\to$ □.

Then we have $\Phi : \mathcal{B}er \to \mathcal{HM}$.

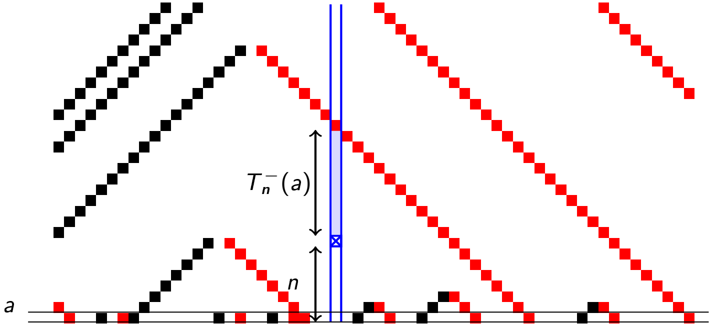
Entry times

Entry times

Let $v_- < v_+ \in \mathbb{Z}$ and $a \in \mathcal{A}^{\mathbb{Z}}$. If $v_- \neq 0$, we define:

$$T_n^-(a) = \min\{k \in \mathbb{N} \mid F^{k+n}(a)_{[0,|v_-|-1]} \text{ contains a particle } -1\}$$

respectively $T_n^+(a)$. This is the **entry time** of a into the set $\{b \in \mathcal{A}^{\mathbb{Z}} \mid b_{[0,|v_-|-1]} \text{ contains a particle } -1\}$ after time n at position 0.



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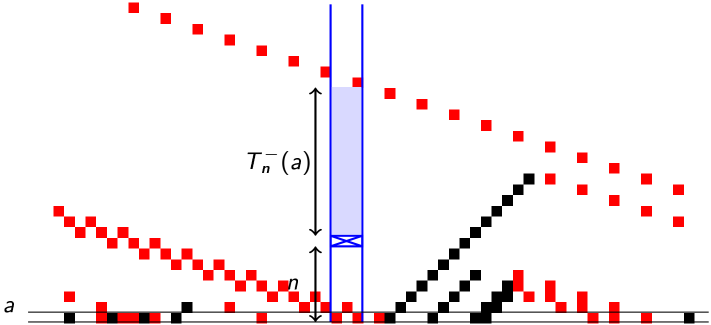


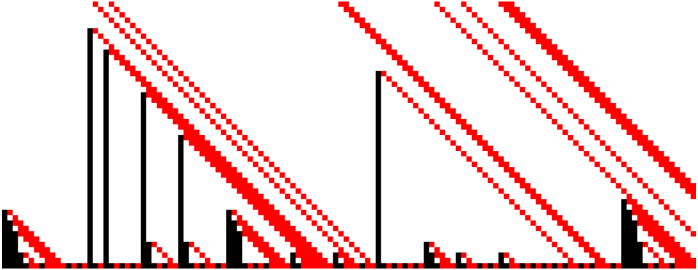
State of the art

Theorem (Denunzio, Formenti, Kurka, 2011)

Consider the $(-1,0)$-gliders automaton with an initial Bernoulli measure of parameters $p_+ = p_- = 1/2$. Then:

$$\mathbb{P}\left(\frac{T_n^-(a)}{n} \leq x\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi} \arctan \sqrt{2x}.$$

- Purely combinatorial approach.



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Conjecture (op.cit.)

If instead we have $\mu \in \mathcal{B}er$,

$$\mathbb{P}\left(\frac{T_n^-(a)}{n} \leq x\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi} \arctan \sqrt{2px}.$$

Entry times

Theorem 1

- (v_-, v_+)-gliders automaton with $v_- \in \mathbb{Z}^-, v_+ \in \mathbb{Z}^+$.
- Initial measure in $\mathcal{HM}$

Then for almost all initial configuration a,

$$\mathbb{P}\left(\frac{T_n^-(a)}{n} \leq x\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi} \arctan\left(\sqrt{\frac{-v_- x}{v_+ - v_- + v_+ x}}\right)$$

and symmetrically for T_n^+.

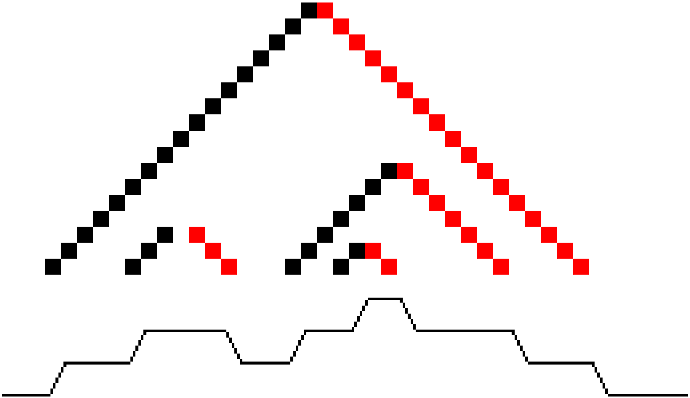
Remarks

- Independent of $\mu([+]), \mu([-])$.
- This **disproves** the conjecture.

Quantitative approach

For $a \in \mathcal{A}^{\mathbb{Z}}$, we define the process M_a that:

- goes up when it meets a $+$;
- goes down when it meets a $-$.



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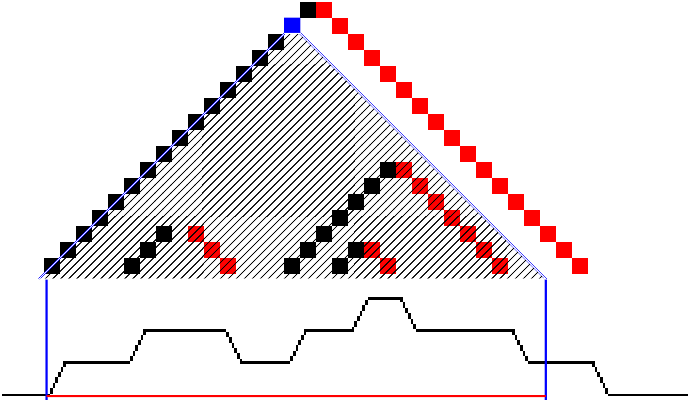
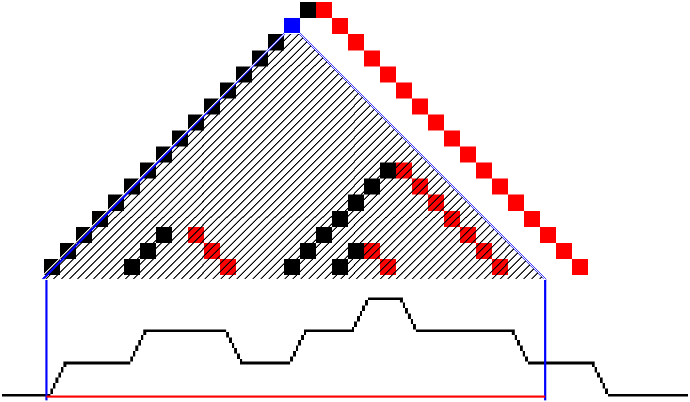


Quantitative approach

$$F_n(a)_0 = + \Leftrightarrow M_a \text{ on } [-n, n] \text{ admits a } \textbf{minimum} \text{ in } -n.$$

and symmetrically for $-$.



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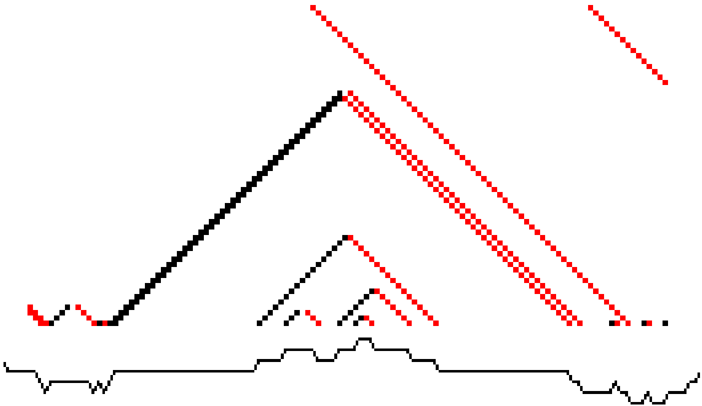

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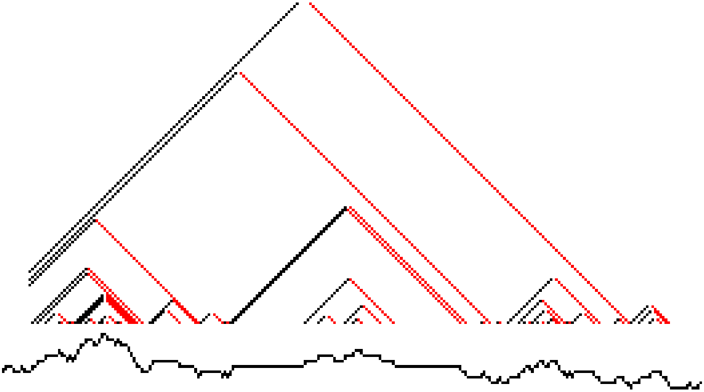

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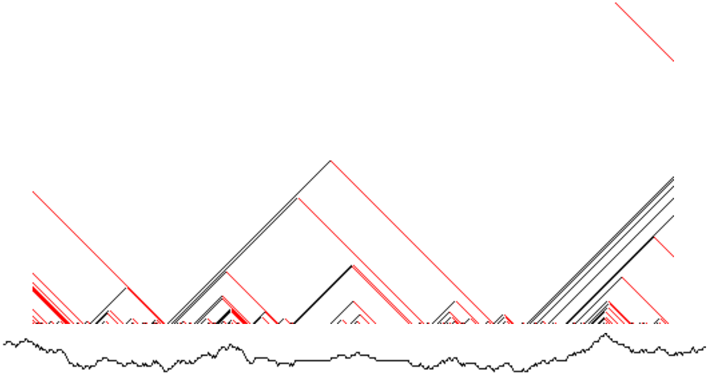

Using scale invariance, we approximate the process by a **Brownian motion**.

Density of particles

Densities

For a configuration $a \in \mathcal{A}^{\mathbb{Z}}$, the **density of particles** - (resp. +) is:

$$d_-(a) = \limsup_{n\to\infty} \frac{\#\{i \in [-n, n] \mid a_i = -\}}{2n+1}$$

Theorem 2

For an initial measure μ, we have:

- If $\mu \in \mathcal{B}er$:

$$\text{For almost all } a, d_-(F^n(a)) = \Theta\left(n^{-\frac{1}{2}}\right)$$

- If $\mu \in \mathcal{HM}$:

$$\text{For almost all } a, \forall\varepsilon > 0, d_-(F^n(a)) = O\left(n^{-\frac{1}{4}+\varepsilon}\right)$$

and similarly for d_+.

Rate of convergence

Theorem 3

Consider any (v_-, v_+)-gliders automaton with $v_- < v_+$.

- $\mu \in \mathcal{HM}$ the initial measure,
- λ the limit measure (weighing only the particleless configuration),
- d be the Lévy-Prohorov distance defined earlier.

Then:

$$\forall \varepsilon > 0,\ d(F^n\mu, \lambda) = O\left(n^{-\frac{1}{8}+\varepsilon}\right)$$

If furthermore $\mu \in \mathcal{B}er$, we have:

$$d(F^n\mu, \lambda) = \Omega\left(n^{-\frac{1}{2}}\right)$$

Perspectives

- With the same approach, better understanding of the limit diagram.
- Extending this kind of results to more particles and different interactions, and eventually to whole classes of automata (e.g. **captive automata**)