Classification de la densité sur des graphes infinis

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Definition of cellular automata

Definition given by S. Ulam and J. von Neumann (50s)

Let $\mathcal{A}$ be a finite alphabet, a **cellular automaton** is a function $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ characterized by

- a finite neighborhood $V \subset \mathbb{Z}$,
- a local function $f : \mathcal{A}^V \to \mathcal{A}$ such that

$$F(x)_k = f((x_{k+v})_{v \in V}).$$

Example of CA

Example: $\mathcal{A} = \{0, 1\}, V = (-1, 0, 1), f(x, y, z) = \text{maj}(x, y, z)$

$$\text{where} \quad \text{maj}(x, y, z) = \begin{cases} 0 \text{ if } x + y + z \leq 1 \\ 1 \text{ if } x + y + z \geq 2 \end{cases}$$

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Motivations

CA are natural examples of discrete dynamical systems:

- CA $\Leftrightarrow$ continuous functions commuting with the shift (Hedlund, 1969)
- a very simple description generating complex behaviors, question of the classification of cellular automata (Wolfram and then Kůrka, 1997)

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And they are used to modelize various physical and biological processes.

Presentation of the problem

We fix $\mathcal{A} = \{0, 1\}$. Let $p \in [0, 1]$.
Choice of the initial configuration: for each cell, we choose independently to write a 1 with probability p and a 0 with probability $1 - p$ (distribution μ_p on $\mathcal{A}^{\mathbb{Z}}$).

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Challenge

Find a CA such that when iterating it, the configuration converges weakly to $0^{\mathbb{Z}}$ if $p < 1/2$ and to $1^{\mathbb{Z}}$ if $p > 1/2$ (synchronization of all the cells in the majority state).

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Equivalently, we search a CA F such that if the initial configuration x is chosen according to μ_p, then for any $k \in \mathbb{Z}$, the probability that $F^n(x)_k = 1$ tends to 0 if $p < 1/2$ and to 1 if $p > 1/2$. Such a CA is said to classify the density.

Some examples

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Let F be the GKL (Gács-Kurdyumov-Levin) CA of neighborhood $V = (-3, -2, -1, 0, 1, 2, 3)$ defined by

- $F(x)_n = \text{maj}(x_n, x_{n+1}, x_{n+3})$ if $x_n = 1$,
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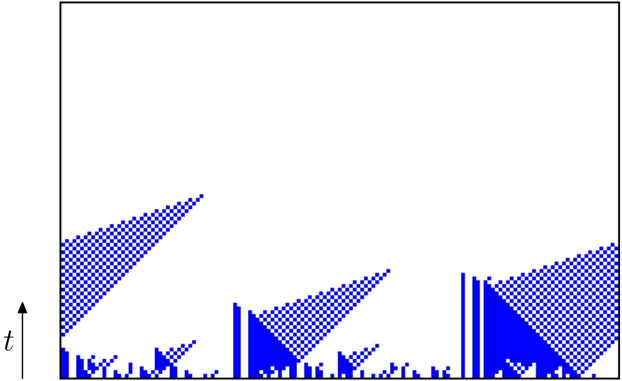
It is an open problem to know if GKL classifies the density.

Space-time diagrams



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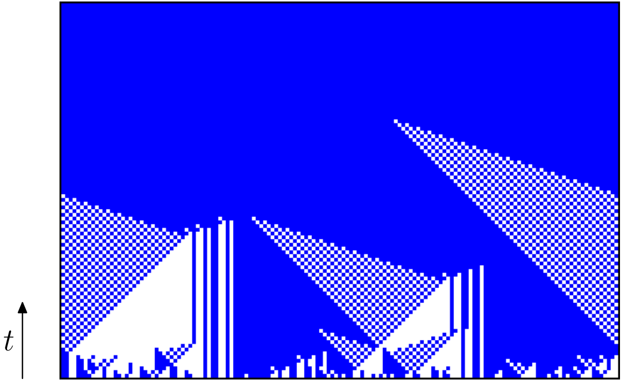
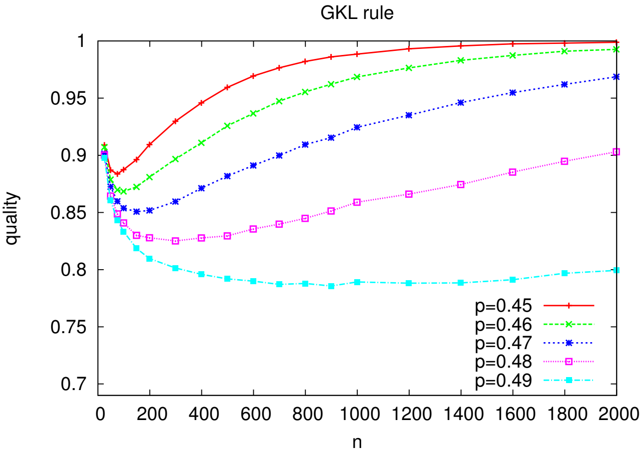


Numerical results



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- on a first tape compute the traffic CA

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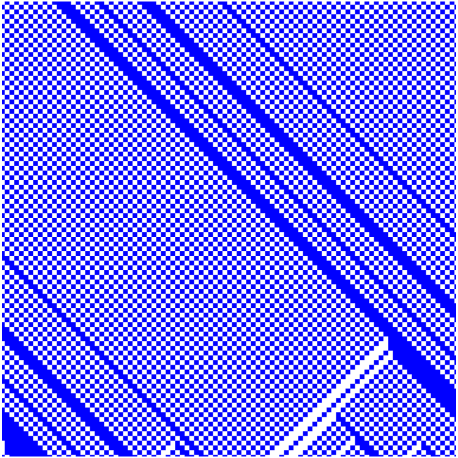
The second tape of this CA converges to the right answer.



Example of space-time diagram for the traffic CA

The majority-traffic PCA

Let us define the *Maj-traf* PCA by $V = (-1, 0, 1)$ and

$$f(x,y,z) = \alpha\,\delta_{\mathrm{maj}(x,y,z)} + (1-\alpha)\,\delta_{\mathrm{traf}(x,y,z)},$$

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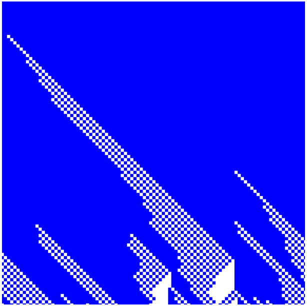

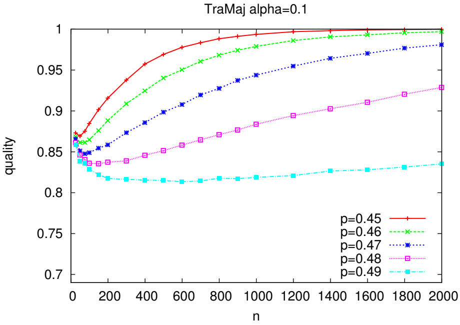
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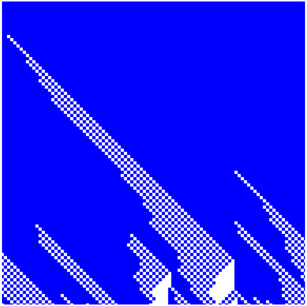
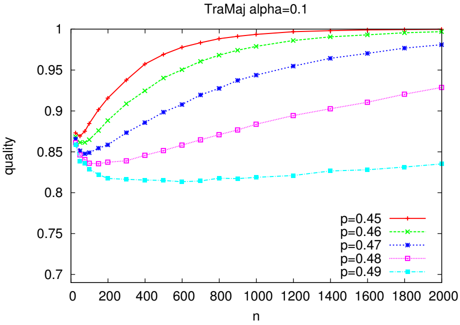
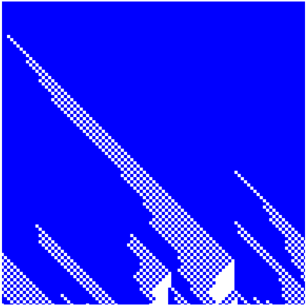
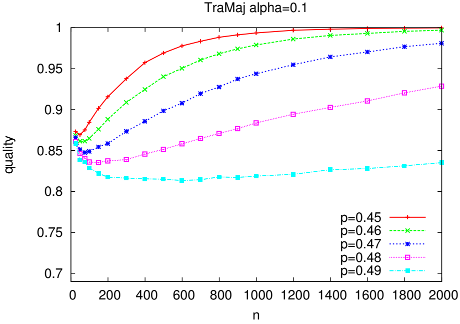




It is an open problem to know if this PCA classifies the density.





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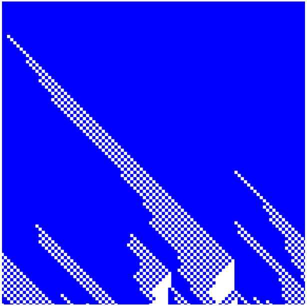

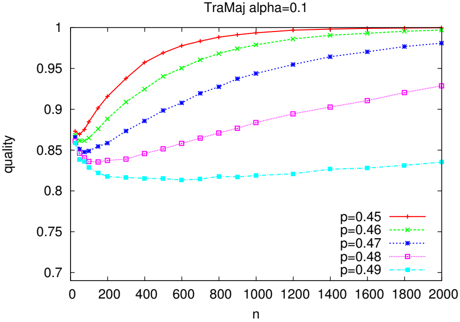

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The initial problem is in fact easier on $\mathbb{Z}^2$...

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Definition of Toom's rule

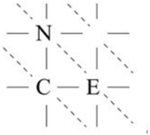
The alphabet is still $\mathcal{A} = \{0, 1\}$, the set of cells is now $\mathbb{Z}^2$.

Definition of the CA

We denote by $\mathcal{T}$ the CA of neighborhood $V = \{(0,0), (0,1), (1,0)\}$ (north-east-center) defined by the majority rule, that is,

$$(\mathcal{T}(x))_{i,j} = \text{maj}(x_{i,j}, x_{i,j+1}, x_{i+1,j}).$$

This CA is known as **Toom's rule.**

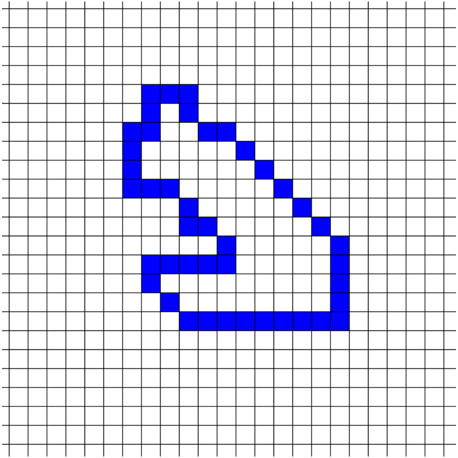


Main result

Proposition

Toom's rule classifies the density.
That is, the sequence $(\mu_p \mathcal{T}^n)_{n \geq 0}$ converges to $\delta_{0^{\mathbb{Z}^2}}$ if $p < 1/2$ and to $\delta_{1^{\mathbb{Z}^2}}$ if $p > 1/2$.

The proof in pictures



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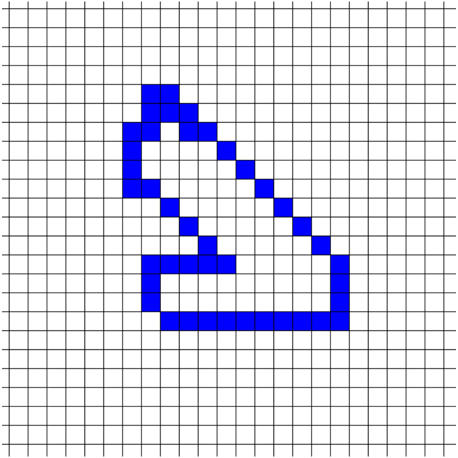

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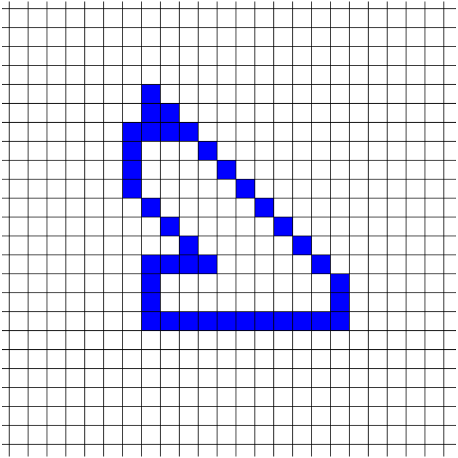

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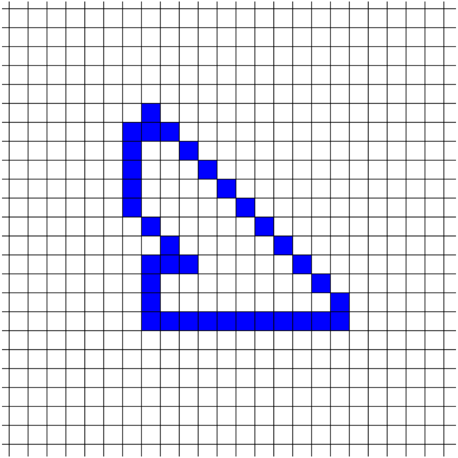

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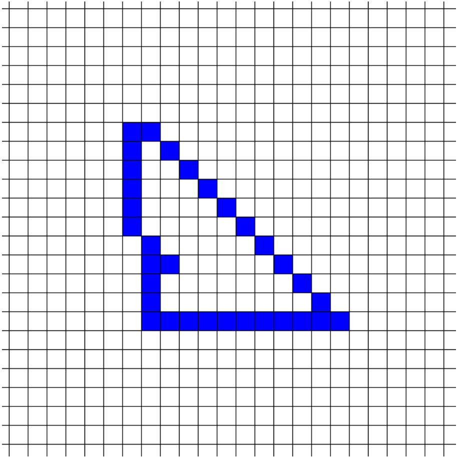

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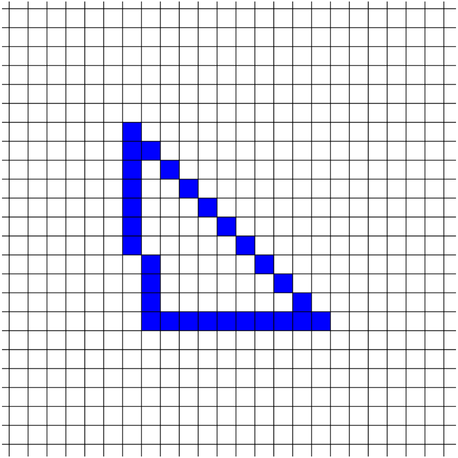

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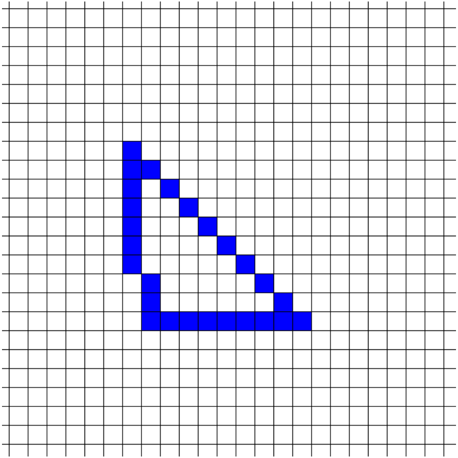

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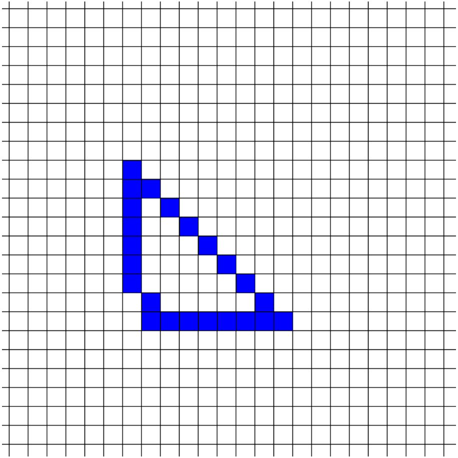

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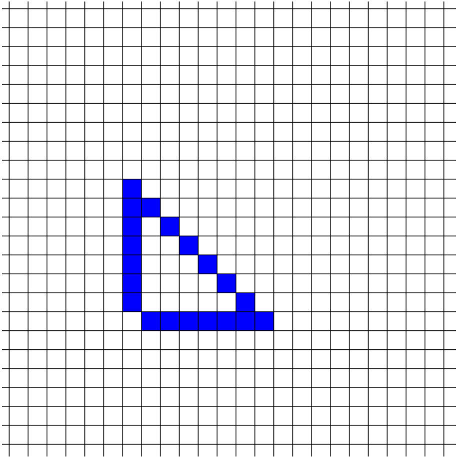

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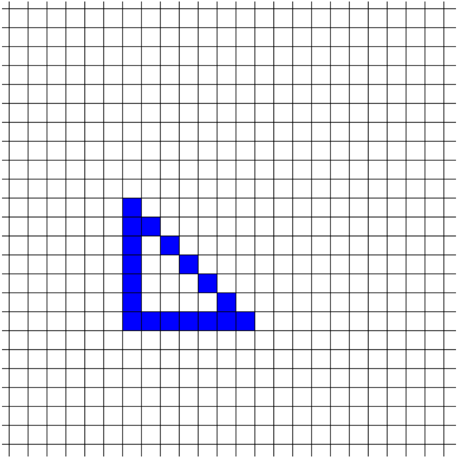

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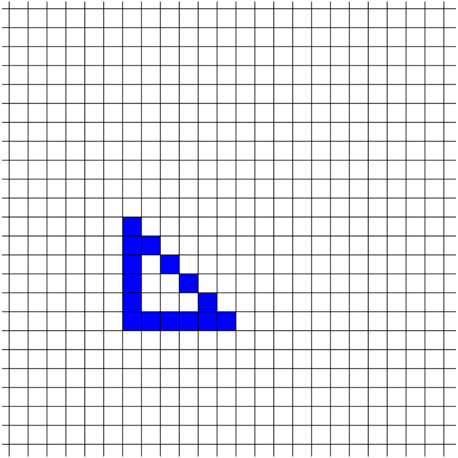

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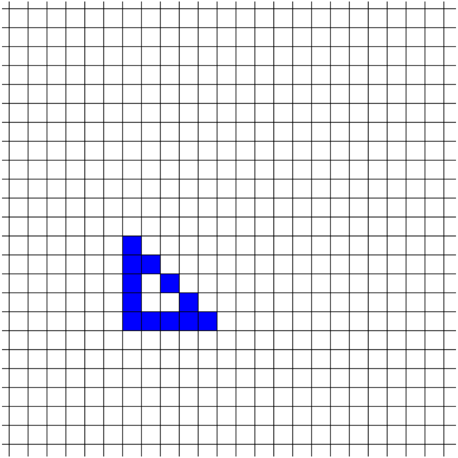

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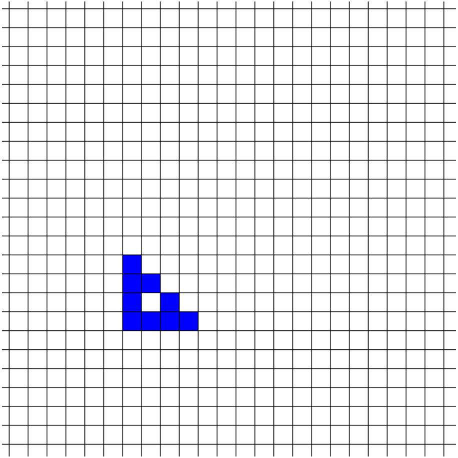

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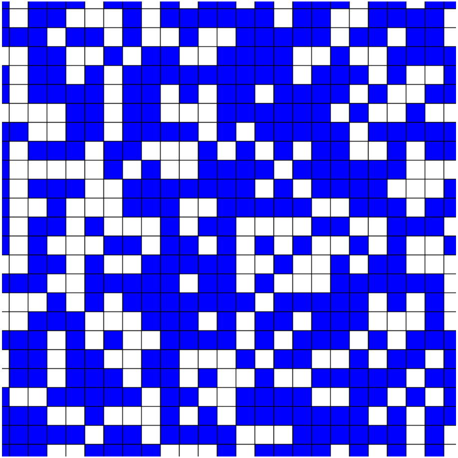

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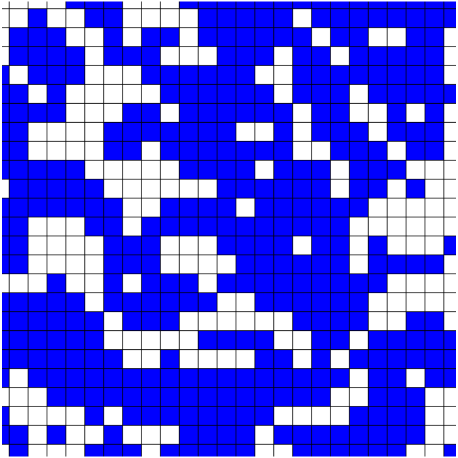

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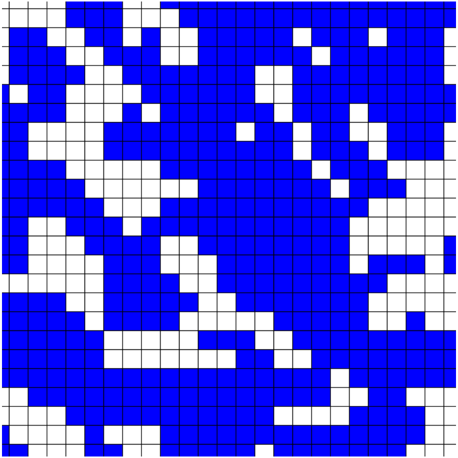

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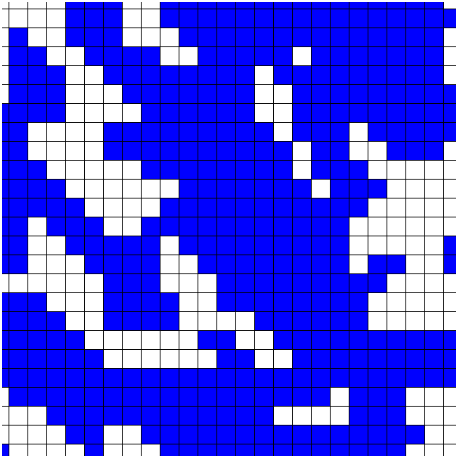

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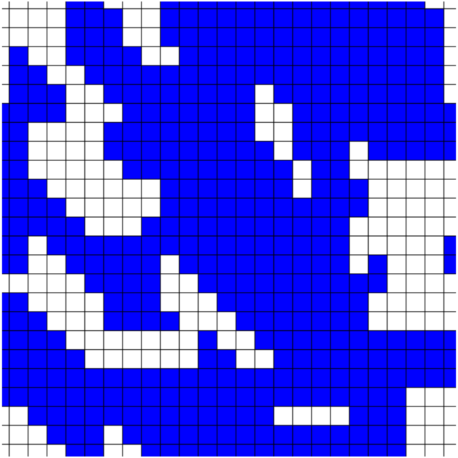

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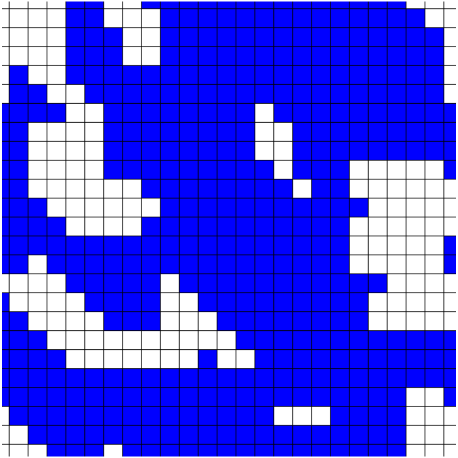

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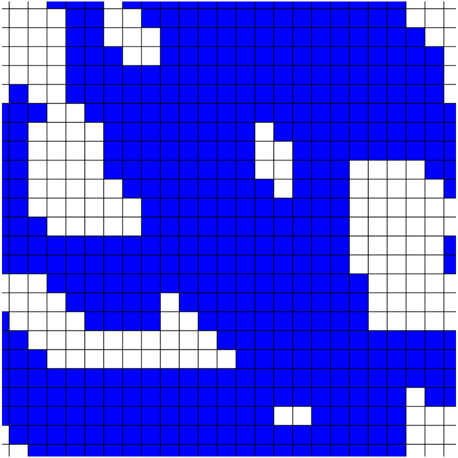

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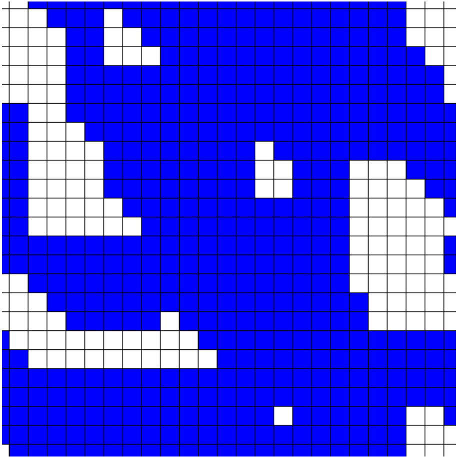

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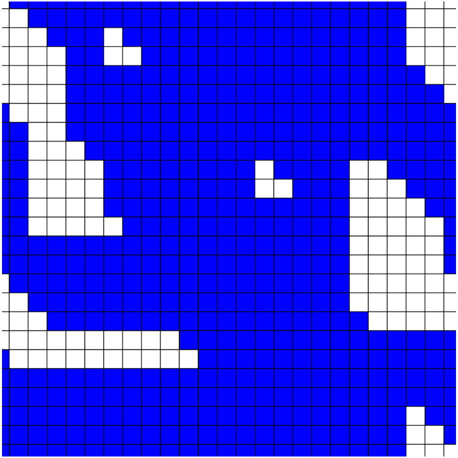

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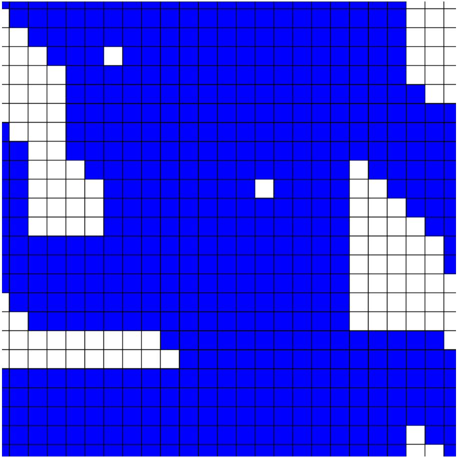

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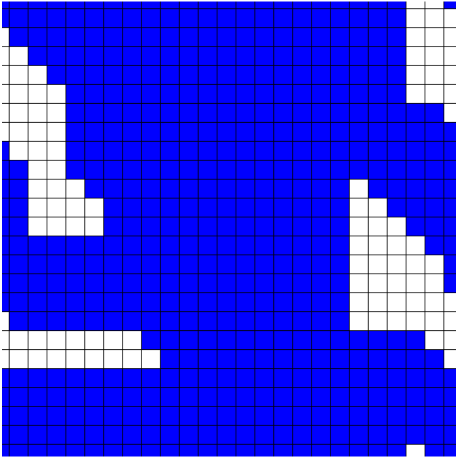

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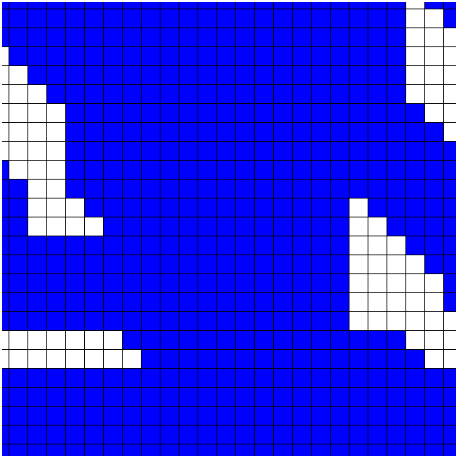

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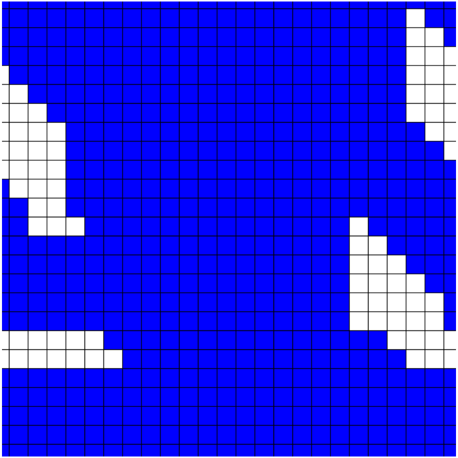

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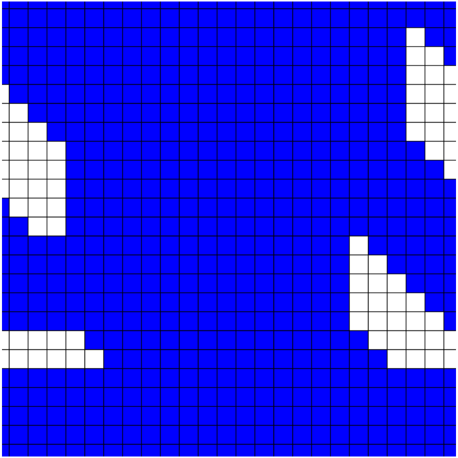

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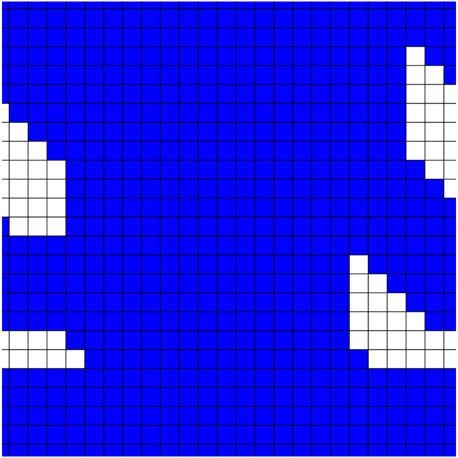

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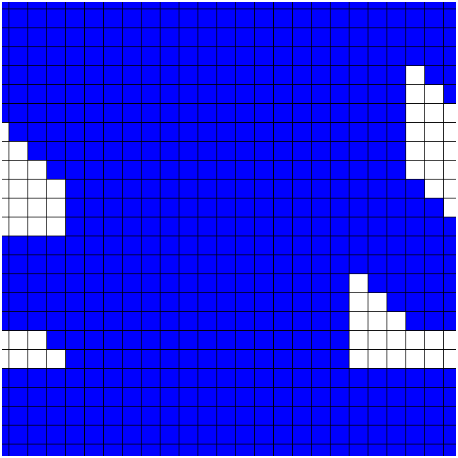

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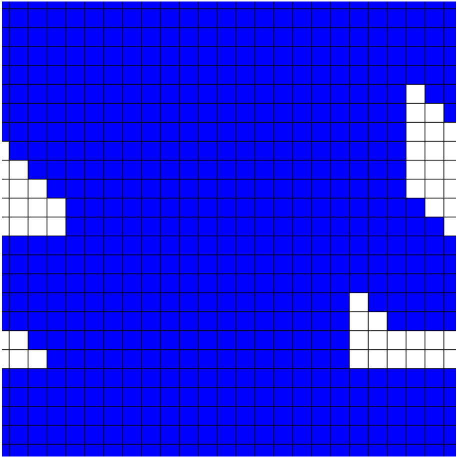

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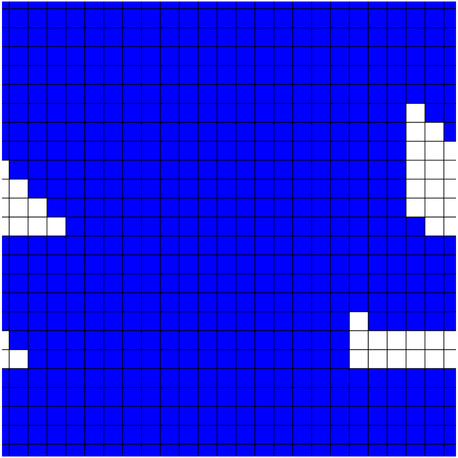

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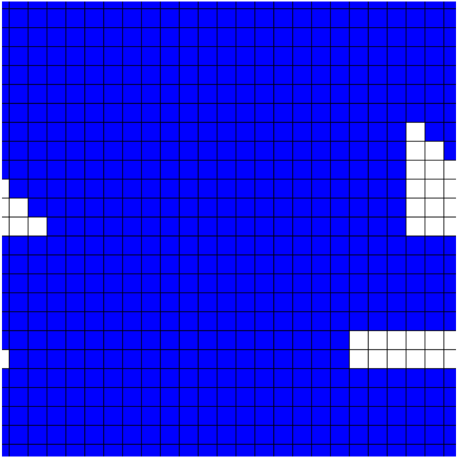

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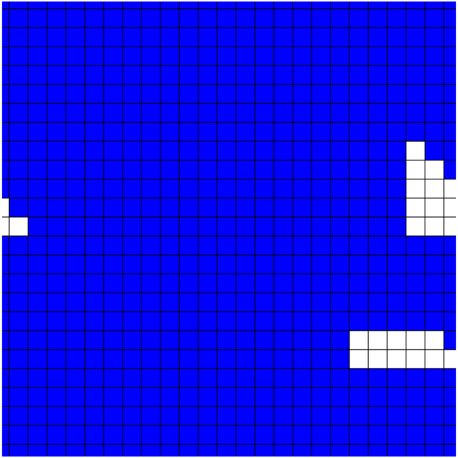

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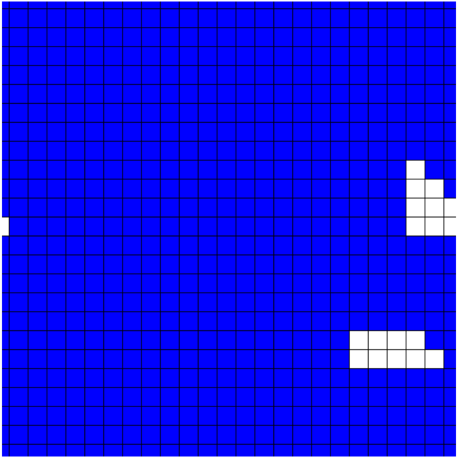

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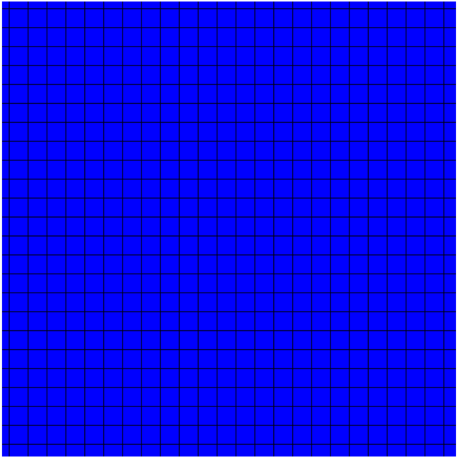
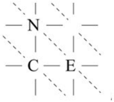


Steps of the proof

Add NW-SE diagonals to the grid, and consider the triangular lattice obtained.



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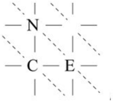

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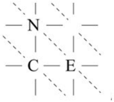

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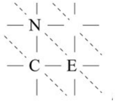

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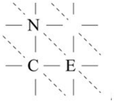


- If $p > 1/2$, there exists a.s. no infinite 0-cluster (classical result of percolation theory)
- Two different 0-clusters cannot merge
- Any finite 0-cluster disappears in finite time and always stays in its enveloping rectangle
- A given point belongs a.s. to the enveloping rectangle of an at most finite number of 0-clusters (by the exponential decay of the size of 0-clusters)

Continuous time version

On $\mathbb{Z}^2$, one can slightly modify Toom's rule in order to make the corresponding continuous time process classify the density.

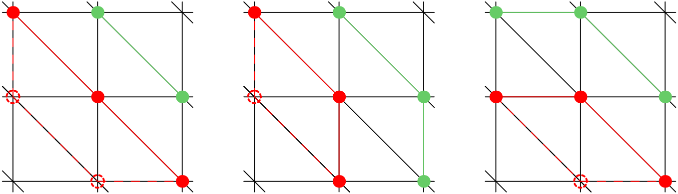


Table of contents

A solution on T_3

Let T_3 be the group $\langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle$.
The Cayley graph of T_3 is the infinite 3-regular tree.

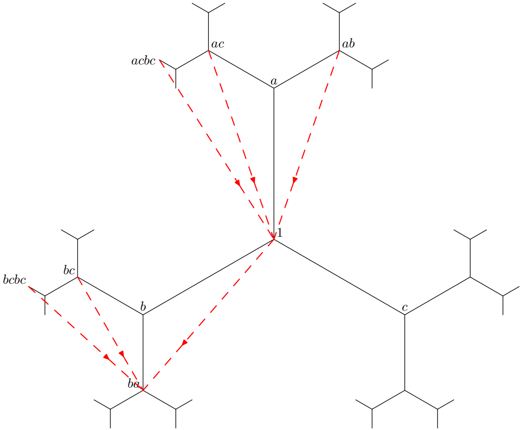
Proposition

The CA $F : \mathcal{A}^{T_3} \to \mathcal{A}^{T_3}$ defined by:

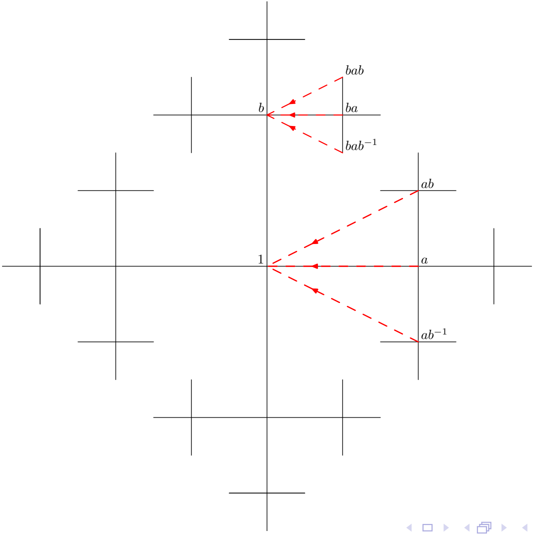
$$F(x)_g = \mathrm{maj}(x_{gab}, x_{gac}, x_{gacbc})$$

for any $x \in \mathcal{A}^{T_3}, g \in T_3$, classifies the density.

A solution on T_3



A solution on T_4



Conclusion

- On $\mathbb{Z}^d, d \geq 2$, and on $T_n, n \geq 3$, there are CA (or Probabilistic CA, or Interacting Particle Systems) that classifies the density. In the examples we have found, the neighborhoods are asymetric. Are there symetric rules that classify the density ?
- On $\mathbb{Z}$, the problem is still open. Link with the positive rate "conjecture".

Some references

- **A. Bušić, N. Fatès, J. Mairesse, and I. M.** *Density Classification on Infinite Lattices and Trees.* To appear in LATIN 2012.
- **J.T. Cox and R. Durrett.** *Nonlinear voter models.* Random walks, Brownian motion and interacting particle systems, R. Durrett and H. Kesten editors, 1991.
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- **G. Grimmett.** *Percolation.* Second ed. Springer-Verlag, Berlin, 1999.