ANALYSIS OF THREE GRAPH PARAMETERS FOR RANDOM TREES

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Abstract. We consider three basic graph parameters, the node-independence number, the path node-covering number, and the size of the kernel, and study their distributional behaviour for an important class of random tree models, namely the class of simply generated trees, which contains, e.g., binary trees, rooted labelled trees and planted plane trees, as special instances. We can show for simply generated tree families that the mean and the variance of each of the three parameters under consideration behave for a randomly chosen tree of size n asymptotically $\sim \mu n$ and $\sim \nu n$, where the constants μ and ν depend on the tree family and the parameter studied. Furthermore we show for all parameters, suitably normalized, convergence in distribution to a Gaussian distributed random variable.

1. Introduction

We are dealing here with three basic graph parameters for simple graphs $G = (V, E)$ (no multiple edges or loops are appearing), with vertex-set $V = V(G)$ and edge-set $E = E(G)$. The parameters *node-independence number* and *path node-covering number* are defined for undirected graphs, whereas the parameter *size of the kernel* requires directed graphs:

Node-independence number: A subset $\mathcal{S} \subseteq V(G)$ is called *independent* if no two nodes of S are joined by an edge $e \in E(G)$ of G. The *node-independence number* $N(G)$ (often denoted by $\alpha(G)$) of an undirected graph $G = (V, E)$ is given by the number of nodes in any largest independent subset $\mathcal{S}$ of nodes of G.

Path node-covering number: A set $\mathcal{Q}$ of disjoint paths in a graph G is called a node-covering of G if every node $v \in V(G)$ of G is contained in a path of $\mathcal{Q}$. The *path node-covering number* (also called path number) $P(G)$ of an undirected graph $G = (V, E)$ is given by the smallest number of paths $\mathcal{Q}$ in a node-covering of G.

Size of the kernel: A kernel $\mathcal{K}$ of a directed graph $G = (V, E)$ is a nonempty subset $\mathcal{K} \subseteq V(G)$ with the following two properties: (i) for any two vertices a, b in the kernel, $a, b \in \mathcal{K}$, the edge (a, b) does not belong to $E(G)$, i.e., $(a, b) \notin E$, (ii) for any vertex c outside the kernel, $c \notin \mathcal{K}$, there exists a vertex d in the kernel, $d \in \mathcal{K}$, such that the edge(c, d) belongs to $E(G)$, $(c, d) \in E$. One can also define a kernel as a nonempty independent and dominating set of nodes in G, see, e.g., [3]. If a kernel $\mathcal{K}$ of a directed graph G exists and if this kernel is unique, then we define the *size of the kernel* $K(G)$ as the number of vertices of $\mathcal{K}$.

The parameters under consideration are illustrated in Figure 1-3.

Whereas the parameters node-independence number and path node-covering number are well-defined for any undirected graph, it is well-known that not every directed graph has a kernel and if a directed graph has a kernel, this kernel is not necessarily unique. In game theory the existence of a kernel corresponds to a winning strategy for Nim-type games with two players. Imagine that two players A and B play the following game on a directed graph $G = (V, E)$ in which they move a token in each turn. Player A starts the game by choosing an initial vertex $v_0 \in V$. Then player B has to make the first move from v_0 to a vertex $v_1 \in V$. A move consists in taking the token from the present position v_i and placing it on a child of v_i, i.e., a vertex v_{i+1} such that $(v_i, v_{i+1}) \in E$. Then player A has to make a move from v_1 to a vertex $v_2 \in V$. Now again player B has to make a move from v_2 to a vertex $v_3 \in V$, and so on. The first player unable to move loses the game. If one plays the game on an acyclic graph then one obtains a finite game, i.e., it ends after a finite number of moves, and thus one

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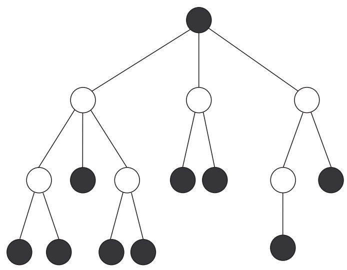

FIGURE 1. A tree T of size 16 with node-independence number $N(T) = 10$. The nodes colored black give a largest independent set of T with 10 independent nodes.

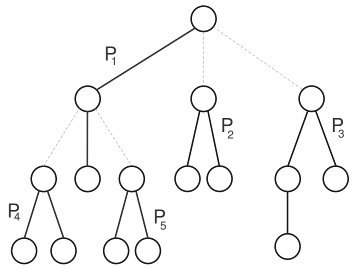

FIGURE 2. A tree T of size 16 with path node-covering number $P(T) = 5$. An example of a node-covering of T with the minimal number of 5 paths is given.

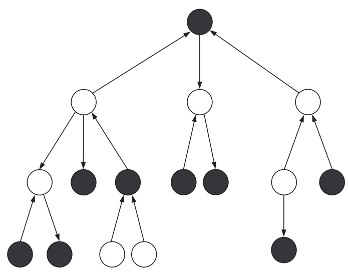

FIGURE 3. A directed tree T of size 16 with kernel-size $K(T) = 9$. The unique kernel of T is given by the 9 nodes colored black.

of the two players must have a winning strategy. For graphs with cycles one can extend the rules by saying that the game is lost for the player who replays a position previously reached, this also leads to a finite game. It is shown in [14] that any directed acyclic graph has a unique kernel, which is exactly the set of winning positions of player A: A always forces player B to leave the kernel, until B cannot play anymore. Thus, for directed acyclic graphs the parameter *size of the kernel* is well-defined and corresponds with the cardinality of the set of winning positions of the first player.

Despite the fact that all considered quantities are basic graph parameters, relatively little is known about the behaviour of the node-independence number, the path node-covering number and the size of the kernel for random tree families. For random (n, n)-trees bounds for the node-independence number of almost all (n, n)-trees are given in [4]. In [10] and [11] the expected node-independence number has been determined for trees selected at random among all trees of size n, asymptotically for $n \to \infty$, for several random tree families, as rooted labelled unordered trees (also called Cayley-trees or rooted labelled trees for short), planted plane trees (also called ordered trees) and recursive trees. For these tree families analogous results for the path node-covering number of random size-n trees are shown in [13]. The expected size of the kernel has been determined for randomly chosen trees of size n, asymptotically for $n \to \infty$, for the family of directed rooted labelled trees in [2]. However, as far as we know, the only limiting distribution result has been obtained in [15] for the node-independence number of random rooted labelled trees: it was shown for this tree family that the node-independence number of a randomly chosen tree of size n is asymptotically normal, with mean μn and variance νn, where $\nu = \frac{\mu(1-\mu-\mu^2)}{1+\mu}$ and μ is given by the solution of $te^t = 1$.

In this paper we are going to study the parameters under consideration for an important class of tree families, called simply generated tree families. Simply generated tree families were introduced in [12] and they include rooted labelled trees, planted plane trees and also binary trees as special instances. Moreover, they are strongly related to Galton-Watson branching processes, since it is well known [1] that random simply generated trees can be considered as conditioned Galton-Watson trees, obtained as the family tree of a Galton-Watson branching process conditioned on the given total size. A detailed definition of simply generated trees is given in Subsection 2.2.

In a probabilistic setting we are studying here the random variables N_n, P_n and K_n. Given a simply generated tree family $\mathcal{T}$, N_n and P_n count the node-independence number $N(T)$ and the path node-covering number $P(T)$, respectively, of a randomly chosen simply generated tree $T \in \mathcal{T}$ of size n, where the size of T is measured as usual by the number of nodes of T. Since simply generated trees can be considered as weighted trees, we assume as the model of randomness that every tree T of size n is selected with probability proportional to its weight. For the size of the kernel we are considering random directed simply generated trees, i.e., we assume that any of the 2^{n-1} orientations of the $n-1$ edges of a size-n tree can occur with the same probability. Let us denote the family of directed trees obtained from $\mathcal{T}$ with $\tilde{\mathcal{T}}$. Then K_n counts the size of the kernel of a randomly chosen directed simply generated tree $T \in \tilde{\mathcal{T}}$ of size n.

We can show that the behaviour of the quantities examined is similar for all simply generated tree families, provided that they satisfy several technical conditions, e.g., it holds that for every simply generated tree family $\mathcal{T}$ there exist constants $\mu_{\mathcal{T}}^{[N]}$ and $\sigma_{\mathcal{T}}^{[N]}$ such that, apart from degenerate cases, the node-independence number N_n is asymptotically normal, for $n \to \infty$, with mean $\mathbb{E}(N_n) \sim \mu_{\mathcal{T}}^{[N]} n$ and variance $\mathbb{V}(N_n) \sim \nu_{\mathcal{T}}^{[N]} n$. Analogous results hold for the path node-covering number P_n and the size of the kernel K_n also.

Furthermore, since the maximum matching number $M(T)$ of a tree T of size n, also called edge-independence number of T, and the node-independence number $N(T)$ are strongly connected due to the following relation (see, e.g., [4, 10]): $N(T)+M(T) = n$, one immediately obtains as a corollary also limiting distribution results for the random variable M_n, which counts the maximum matching number $M(T)$ of a randomly chosen tree T of size n of a simply generated tree family $\mathcal{T}$. It holds that the maximum matching number M_n is asymptotically normal, for $n \to \infty$, with mean $\mathbb{E}(M_n) \sim (1-\mu_{\mathcal{T}}^{[N]})n$ and variance $\mathbb{V}(M_n) \sim \nu_{\mathcal{T}}^{[N]} n$.

2. Mathematical preliminaries

2.1. **Recursive descriptions of the parameters.** Fundamental for our treatment of the parameters studied are the following recursive descriptions of the quantities for rooted trees. For the node-independence number $N(T)$ this description appeared first in [10], where the authors mention that it goes back to De Brujn. For the path node-covering number $P(T)$ it has been given in [13]. Hence, we will only briefly discuss these two cases, and give a more detailed recursive description of the size of the kernel $K(T)$ of a directed tree, extending the considerations of [2].

Node-independence number. One uses the following partition of rooted trees T, where r denotes the root of T, into two classes, type $\mathrm{I}^{[N]}$ trees and type $\mathrm{II}^{[N]}$ trees. A rooted tree T is called a type $\mathrm{I}^{[N]}$ tree, if every largest set of $N(T)$ independent nodes of T contains the root r. If at least one largest set of $N(T)$ independent nodes of T does not contain the root r, then T is called a type $\mathrm{II}^{[N]}$ tree. By removing the root r of a rooted tree T and the edges incident to r we obtain a (possibly empty) collection of rooted trees $B_1, \dots, B_j$, the so called branches, whose roots were originally joined to r. We will use the following result.

Lemma 1 (Meir & Moon [10])**.** *If each of the branches $B_1, \dots, B_j$ of T is a type $II^{[N]}$ tree, then T is a type $I^{[N]}$ tree and $N(T) = 1 + \sum_{k=1}^{j} N(B_k)$. If at least one of the branches $B_1, \dots, B_j$ of T is a type $I^{[N]}$ tree, then T is a type $II^{[N]}$ tree and $N(T) = \sum_{k=1}^{j} N(B_k)$.*

Path node-covering number. Again one uses a partition of rooted trees T, where r denotes the root of T, into two classes, which are called here type $\mathrm{I}^{[P]}$ trees and type $\mathrm{II}^{[P]}$ trees. To describe them we use the following notation: a vertex $v \in V$ is called an interior node in a node-covering $\mathcal{Q}$ if it has degree two in $\mathcal{Q}$, otherwise v is in $\mathcal{Q}$ an endnode (degree one in $\mathcal{Q}$) or an isolated node (degree zero in $\mathcal{Q}$). A rooted tree T is called a type $\mathrm{I}^{[P]}$ tree, if the root r is an interior node in every smallest set of $P(T)$ paths of a node-covering $\mathcal{Q}$ of T. If the root r is an endnode or an isolated node in at least one smallest set of $P(T)$ paths of a node covering $\mathcal{Q}$ of T, then T is called a type $\mathrm{II}^{[P]}$ tree.

Lemma 2 (Meir & Moon, [13])**.** *If each of the branches $B_1, \dots, B_j$ of T is a type $I^{[P]}$ tree, then T is a type $II^{[P]}$ tree and $P(T) = 1 + \sum_{k=1}^{j} P(B_k)$. If exactly one of the branches $B_1, \dots, B_j$ of T is a type $II^{[P]}$ tree, then T is a type $II^{[P]}$ tree and $P(T) = \sum_{k=1}^{j} P(B_k)$. If at least two of the branches $B_1, \dots, B_j$ of T are type $II^{[P]}$ trees, then T is a type $I^{[P]}$ tree and $P(T) = -1 + \sum_{k=1}^{j} P(B_k)$.*

Size of the kernel. Here we use a natural partition of directed rooted trees T, where r denotes the root of T, into two classes, which are called type $\mathrm{I}^{[K]}$ trees and type $\mathrm{II}^{[K]}$ trees. A directed rooted tree T is called a type $\mathrm{I}^{[K]}$ tree if the root r is contained in the kernel $\mathcal{K}$ of T and it is called a type $\mathrm{II}^{[K]}$ tree if the root r is not contained in the kernel of T.

However, in order to establish a recursive description of the size of the kernel $K(T)$ of a rooted tree T with root r we introduce the auxiliary parameter $\hat{K}(T)$. Given a tree T, the parameter $\tilde{K}(T)$ is defined as $\hat{K}(T) := K(\hat{T}) - 1$, where the tree $\hat{T}$ is obtained from T by attaching an additional node w (i.e., $w \notin T$) to the root r by a directed edge $e = (r, w)$. This quantity is required here to describe the "recoloring" of a tree whose root is originally in the kernel, when we attach the root to a new node, which will itself lie in the kernel.

By removing the root r of a directed rooted tree T and the directed edges incident to r we obtain a (possibly empty) collection of directed rooted trees $B_1, \dots, B_j$, whose roots were originally joined to r. We obtain then the following characterization of type $\mathrm{I}^{[K]}$ and type $\mathrm{II}^{[K]}$ trees via the type of the branches $B_1, \dots, B_j$ and the orientation of the edges connecting r with the roots of the branches together with a recursive description of $K(T)$ using the auxiliary quantity $\hat{K}(T)$.

Lemma 3. *If at least one of the branches $B_1, \dots, B_j$ of T, let us assume B_ℓ, is a type $I^{[K]}$ tree that is attached to the root r of T by a directed edge $e = (r, root(B_\ell))$, then T is a type $II^{[K]}$ tree and*

$$K(T) = \sum_{k=1}^{j} K(B_k).$$

If every branch B_ℓ of the branches $B_1, \dots, B_j$ of T has one of the following three properties:

(i) *it is a type $II^{[K]}$ tree that is attached to the root r of T by a directed edge $e = (r, root(B_\ell))$,*
(ii) *it is a type $II^{[K]}$ tree that is attached to the root r of T by a directed edge $e = (root(B_\ell), r)$,*
(iii) *it is a type $I^{[K]}$ tree that is attached to the root r of T by a directed edge $e = (root(B_\ell), r)$,*

then T is a type $I^{[K]}$ tree.

If we denote by L the following index set:

$$L := \{\ell \,:\, 1 \le \ell \le j,\ B_\ell \text{ is a type } I^{[K]} \text{ tree that is attached to the root } r \text{ of } T \\ \text{by a directed edge } e = (r, root(B_\ell))\},$$

then we obtain for type $I^{[K]}$ trees the following recursive description of $K(T)$ and $\hat{K}(T)$:

$$K(T) = 1 + \sum_{\ell \notin L} K(B_\ell) + \sum_{\ell \in L} \hat{K}(B_\ell), \quad \hat{K}(T) = \sum_{k=1}^{j} K(B_k).$$

2.2. **Simply generated tree families.** A class $\mathcal{T}$ of simply generated trees can be defined in the following way. A sequence of non-negative real numbers $(\varphi_k)_{k\geq 0}$ with $\varphi_0 > 0$ (φ_k can be seen as the multiplicative weight of a node with out-degree k) is used to define the weight $w(T)$ of any ordered tree T by $w(T) := \prod_v \varphi_{d(v)}$, where v ranges over all vertices of T and $d(v)$ is the out-degree (the number of children) of v (in order to avoid degenerate cases we always assume that there exists a $k \geq 2$ such that $\varphi_k > 0$). The family $\mathcal{T}$ consists then of all trees T with $w(T) \neq 0$ together with their weights $w(T)$. It follows further that for a given degree-weight sequence $(\varphi_k)_{k\geq 0}$ the generating function $T(z) := \sum_{n\geq 1} T_n z^n$ of the quantity total weights $T_n := \sum_{|T|=n} w(T)$, where $|T|$ denotes the size of the tree T, satisfies the functional equation

$$T(z) = z\varphi\big(T(z)\big), \tag{1}$$

where the degree-weight generating function $\varphi(t)$ is given by $\varphi(t) = \sum_{k\geq 0} \varphi_k t^k$.

A class $\tilde{\mathcal{T}}$ of directed simply generated trees is obtained from the corresponding undirected simply generated tree family $\mathcal{T}$ if for every tree $T \in \mathcal{T}$, $2^{|T|-1}$ copies $\tilde{T}_1, \dots, \tilde{T}_{2^{|T|-1}}$ of T are equipped with all possible orientations of the $|T| - 1$ edges of T. The weights of the trees remain unchanged by the orientation: if $\tilde{T} \in \tilde{\mathcal{T}}$ denotes a directed simply generated tree and $T \in \mathcal{T}$ its corresponding undirected tree, then $w(\tilde{T}) = w(T)$. Of course, the total weights $\tilde{T}_n := \sum_{|\tilde{T}|=n} w(\tilde{T})$ are given by $\tilde{T}_n = 2^{n-1} T_n$ and the generating function $\tilde{T}(z) := \sum_{n\geq 1} \tilde{T}_n z^n$ satisfies $\tilde{T}(z) = \frac{T(2z)}{2}$ and also the functional equation

$$\tilde{T}(z) = z\varphi\big(2\tilde{T}(z)\big).$$

The asymptotic behaviour of $T(z)$ as solution of (1) is discussed in detail in [7] and we collect some of their results concerning $T(z)$ and the growth of its coefficients T_n. We consider here $T(z)$ as a complex function, $z \in \mathbb{C}$.

We will make throughout this paper always the following assumptions on $\varphi(t)$.

Assumption 1. *The degree-weight generating function $\varphi(t) = \sum_{k\geq 0} \varphi_k t^k$, with $\varphi_k \geq 0$ for all $k \geq 0$, satisfies the following three properties:*

- *(i) $\varphi(t)$ has a positive radius of convergence $R > 0$,*
- *(ii) there exists a minimal positive solution $\tau < R$ of the equation $t\varphi'(t) = \varphi(t)$,*
- *(iii) the period $p := gcd\{k : \varphi_k > 0\}$ is one: $p = 1$.*

If Assumption 1 is satisfied, then it holds that τ is the only solution of smallest modulus of the equation $t\varphi'(t) = \varphi(t)$ (if $\hat{\tau} \in \mathbb{C}$ is another solution of this equation then $|\hat{\tau}| > \tau$) and the function $T(z)$ has a unique dominant singularity $z = \rho$ with $\rho = \frac{\tau}{\varphi(\tau)} = \frac{1}{\varphi'(\tau)}$ (again, if $\hat{\rho}$ is another singularity of $T(z)$ then $|\hat{\rho}| > \rho$).

Furthermore, the local expansion of $T(z)$ around the singularity $z = \rho$ is given as follows, where κ denotes a certain constant:

$$T(z) = \tau - \sqrt{\frac{2\varphi(\tau)}{\varphi''(\tau)}}\sqrt{1 - \frac{z}{\rho}} + \kappa\big(1 - \frac{z}{\rho}\big) + \mathcal{O}\Big(\big(1 - \frac{z}{\rho}\big)^{\frac{3}{2}}\Big). \tag{2}$$

By applying singularity analysis [8] one can translate the expansion (2) into the following asymptotic expansion of the coefficients T_n:

$$T_n = \sqrt{\frac{\varphi(\tau)}{2\pi\varphi''(\tau)}}\rho^{-n} n^{-\frac{3}{2}}\big(1 + \mathcal{O}(n^{-1})\big). \tag{3}$$

We want to mention further that it is often advantageous to describe a simply generated tree family $\mathcal{T}$ by the formal recursive equation

$$\mathcal{T} = \bigcirc \times \Big(\varphi_0 \cdot \{\epsilon\} \,\dot{\cup}\, \varphi_1 \cdot \mathcal{T} \,\dot{\cup}\, \varphi_2 \cdot \mathcal{T} \times \mathcal{T} \,\dot{\cup}\, \varphi_3 \cdot \mathcal{T} \times \mathcal{T} \times \mathcal{T} \,\dot{\cup}\, \cdots\Big) = \bigcirc \times \varphi(\mathcal{T}), \tag{4}$$

with $\bigcirc$ a node, $\times$ the cartesian product, and $\varphi(\mathcal{T})$ the substituted structure (see e. g. [16]).

We also want to describe briefly the most important instances of simply generated tree families.

Binary trees. They are simply generated trees with degree-weight generating function $\varphi(t) = (1+t)^2$. The generating function $T(z)$ satisfies the quadratic equation $T(z) = z(1+T(z))^2$, which leads to the solution $T(z) = \frac{1-2z-\sqrt{1-4z}}{2z}$. Thus the number of binary trees of size n (i.e., the total weights) are for $n \geq 1$ given by the Catalan numbers: $T_n = [z^n]T(z) = \frac{1}{n+1}\binom{2n}{n}$.

Planted plane trees. They are simply generated trees with degree-weight generating function $\varphi(t) = \frac{1}{1-t}$. The generating function $T(z)$ satisfies the equation $T(z) = \frac{z}{1-T(z)}$, which leads to the solution $T(z) = \frac{1-\sqrt{1-4z}}{2}$. Thus the number of planted plane trees of size n (i.e., the total weights) are for $n \geq 1$ given by $T_n = [z^n]T(z) = \frac{1}{n}\binom{2n-2}{n-1}$.

Rooted labelled trees. They can be considered as simply generated trees with degree-weight generating function $\varphi(t) = e^t$. The generating function $T(z)$ satisfies the equation $T(z) = ze^{T(z)}$. The sum of the weights of all rooted labelled trees of size n is for $n \geq 1$ given by $T_n = [z^n]T(z) = \frac{n^{n-1}}{n!}$. The number of rooted labelled trees of size n is given by $n!T_n = n^{n-1}$.

3. Results

We are stating here the results of the paper concerning the behaviour of the graph parameters studied for random simply generated trees. Here $\Phi(x)$ denotes the distribution function of the standard normal distribution $\mathcal{N}(0,1)$.

Theorem 1. *Let $\mathcal{T}$ be a simply generated tree family and $\tilde{\mathcal{T}}$ the corresponding directed simply generated tree family, where the degree-weight generating function $\varphi(t)$ satisfies Assumption 1. Then the following results hold for the random variable N_n, which counts the node-independence number of a random tree of size n in the family $\mathcal{T}$, the r.v. P_n, which counts the path node-covering number of a random tree of size n in the family $\mathcal{T}$, and the r.v. K_n, which counts the size of the kernel of a random tree of size n in the family $\tilde{\mathcal{T}}$.*

(*i*) *There exist constants $\mu_{\mathcal{T}}^{[N]}$, $\mu_{\mathcal{T}}^{[P]}$, $\mu_{\mathcal{T}}^{[K]}$ and constants $\nu_{\mathcal{T}}^{[N]}$, $\nu_{\mathcal{T}}^{[P]}$, $\nu_{\mathcal{T}}^{[K]}$, which are specified in Lemma 4-6, such that the expectations $\mathbb{E}(N_n)$, $\mathbb{E}(P_n)$, $\mathbb{E}(K_n)$ and the variances $\mathbb{V}(N_n)$, $\mathbb{V}(P_n)$, $\mathbb{V}(K_n)$ are given by the following asymptotic expansions:*

$$\mathbb{E}(N_n) = \mu_{\mathcal{T}}^{[N]} n + \mathcal{O}(1), \qquad \mathbb{V}(N_n) = \nu_{\mathcal{T}}^{[N]} n + \mathcal{O}(1),$$
$$\mathbb{E}(P_n) = \mu_{\mathcal{T}}^{[P]} n + \mathcal{O}(1), \qquad \mathbb{V}(P_n) = \nu_{\mathcal{T}}^{[P]} n + \mathcal{O}(1),$$
$$\mathbb{E}(K_n) = \mu_{\mathcal{T}}^{[K]} n + \mathcal{O}(1), \qquad \mathbb{V}(K_n) = \nu_{\mathcal{T}}^{[K]} n + \mathcal{O}(1).$$

(*ii*) *If $\nu_{\mathcal{T}}^{[N]} \neq 0$ then N_n is asymptotically normal distributed, where the rate of convergence is of order $\mathcal{O}(n^{-\frac{1}{2}})$:*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{N_n - \mathbb{E}(N_n)}{\sqrt{\mathbb{V}(N_n)}} \leq x \right\} - \Phi(x) \right| = \mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big).$$

Analogous statements hold for P_n and K_n also, i.e., if $\nu_{\mathcal{T}}^{[P]} \neq 0$ then P_n is asymptotically normal distributed, and if $\nu_{\mathcal{T}}^{[K]} \neq 0$ then K_n is asymptotically normal distributed, both with rate of convergence of order $\mathcal{O}(n^{-\frac{1}{2}})$.

Remark. We conjecture that $\nu_{\mathcal{T}}^{[N]} \neq 0$, $\nu_{\mathcal{T}}^{[P]} \neq 0$ and $\nu_{\mathcal{T}}^{[K]} \neq 0$ holds for all simply generated tree families satisfying Assumption 1, which would imply that degenerate cases are not appearing.

Theorem 2.

- *For the family of binary trees all three parameters considered, N_n, P_n and K_n, are asymptotically normal distributed, where the constants specified in Theorem 1 are given as follows:*

$$\mu^{[N]} = 4 - 2\sqrt{3}, \qquad \nu^{[N]} = \frac{52}{3} - 10\sqrt{3},$$
$$\mu^{[P]} = 3 - 2\sqrt{2}, \qquad \nu^{[P]} = 17 - 12\sqrt{2},$$

	Node-independence number		Path node-covering number	
	$\mu^{[N]}$	$\nu^{[N]}$	$\mu^{[P]}$	$\nu^{[P]}$
Binary trees	0.53589838...	0.01282525...	0.17157287...	0.02943725...
Planted plane trees	0.61803398...	0.04721359...	0.35689586...	0.11161451...
Rooted labelled trees	0.56714329...	0.02568032...	0.25289897...	0.05975246...

	Size of the kernel		Maximum matching number	
	$\mu^{[K]}$	$\nu^{[K]}$	$\mu^{[M]} = 1 - \mu^{[N]}$	$\nu^{[M]} = \nu^{[N]}$
Binary trees	0.48528137...	0.03575054...	0.46410161...	0.01282525...
Planted plane trees	0.57735026...	0.08860388...	0.38196601...	0.04721359...
Rooted labelled trees	0.52041864...	0.05547192...	0.43285670...	0.02568032...

TABLE 1. Numerical results for the parameters studied for the most important simply generated tree families.

$$\mu^{[K]} = 6\sqrt{2} - 8, \qquad \nu^{[K]} = \frac{563}{4} - \frac{199\sqrt{2}}{2}.$$

- *For the family of planted plane trees all three parameters considered, N_n, P_n and K_n, are asymptotically normal distributed, where the constants specified in Theorem 1 are given as follows:*

$$\mu^{[N]} = \frac{\sqrt{5}}{2} - \frac{1}{2}, \qquad \nu^{[N]} = \frac{\sqrt{5}}{5} - \frac{2}{5},$$

$$\mu^{[P]} = 4\lambda(1-\lambda), \qquad \nu^{[P]} = \frac{-128\lambda^2 + 78\lambda - 1}{49},$$

where λ is the smallest positive solution of $1 - 12t + 20t^2 - 8t^3 = 0$,

$$\mu^{[K]} = \frac{\sqrt{3}}{3}, \qquad \nu^{[K]} = \frac{2\sqrt{3}}{9} - \frac{8}{27}.$$

- *For the family of rooted (or unrooted) labelled trees all three parameters considered, N_n, P_n and K_n, are asymptotically normal distributed, where the constants specified in Theorem 1 are given as follows:*

$$\mu^{[N]} = \lambda, \qquad \nu^{[N]} = \frac{\lambda(1-\lambda-\lambda^2)}{(1+\lambda)^2},$$

where λ is the solution of $te^t = 1$,

$$\mu^{[P]} = \frac{1 - 3\lambda + \lambda^2}{2-\lambda}, \qquad \nu^{[N]} = \frac{(\lambda-1)(\lambda^6 - 10\lambda^5 + 42\lambda^4 - 93\lambda^3 + 111\lambda^2 - 62\lambda + 7)}{(\lambda-2)^2(\lambda^2 - 3\lambda + 3)^2},$$

where λ is the solution of $e^t(t-2) + e(1-t) = 0$,

$$\mu^{[K]} = \frac{2\lambda}{1+\lambda}, \qquad \nu^{[K]} = \frac{2\lambda(1-\lambda-\lambda^2-\lambda^3)}{(1+\lambda)^6},$$

where λ is the solution of $te^t = \frac{1}{2}$.

Numerical results for the constants appearing in Theorem 2 are given in Table 1.

For random labelled trees the limiting distribution result of the node-independence number together with a computation of the constants $\mu^{[N]}$ and $\nu^{[N]}$ already appeared in [15]. For labelled trees and planted plane trees the constants $\mu^{[N]}$ have been computed in [10] and [11], whereas the constants $\mu^{[P]}$ have been computed in [13]. Finally for labelled trees the constant $\mu^{[K]}$ already appears in [2].

Remark. The results hold for unrooted labelled trees also, since all parameters studied do not depend on the actual root of a tree.

4. Proof of Theorem 1 for the node-independence number

4.1. **A system of functional equations.** We will study the random variable N_n, which counts the node-independence number of a random tree of size n in a simply generated tree family $\mathcal{T}$ that satisfies Assumption 1. To do this we use the recursive description of the node-independence number $N(T)$ described in Subsection 2.1, which leads to distributional recurrences for N_n conditioned on the type $\mathrm{I}^{[N]}$ and type $\mathrm{II}^{[N]}$ trees.

Given a simply generated tree family $\mathcal{T}$ we denote by $\mathcal{F}$ the subfamily of type $\mathrm{I}^{[N]}$ trees and by $\mathcal{G}$ the subfamily of type $\mathrm{II}^{[N]}$ trees in $\mathcal{T}$. Furthermore we denote by F_n the total weight of size-n trees of type $\mathrm{I}^{[N]}$, $F_n := \sum_{|T|=n, T\in\mathcal{F}} w(T)$, and by G_n the total weight of size-n trees of type $\mathrm{II}^{[N]}$, $G_n := \sum_{|T|=n, T\in\mathcal{G}} w(T)$. Of course it holds $F_n + G_n = T_n$, where T_n gives the total weight of size-n trees in $\mathcal{T}$, see Subsection 2.2.

We introduce now the bivariate generating function of the probability that N_n has value m,

$$T(z,v) := \sum_{n\ge1}\sum_{m\ge0} T_n \mathbb{P}\{N_n = m\} z^n v^m,$$

and the auxiliary generating functions for the corresponding probabilities conditioned on type $\mathrm{I}^{[N]}$ and type $\mathrm{II}^{[N]}$ trees,

$$F(z,v) = \sum_{n\ge1}\sum_{m\ge0} F_n \mathbb{P}\{N_n = m \mid T\in\mathcal{F}\} z^n v^m,$$
$$G(z,v) = \sum_{n\ge1}\sum_{m\ge0} G_n \mathbb{P}\{N_n = m \mid T\in\mathcal{G}\} z^n v^m.$$

Of course we have the relation

$$T(z,v) = F(z,v) + G(z,v).$$

The recursive description of type $\mathrm{I}^{[N]}$ and type $\mathrm{II}^{[N]}$ trees given in Subsection 2.1 can be described by the following formal equations:

$$\mathcal{F} = \bigcirc \times \Big(\varphi_0\cdot\{\epsilon\} \mathbin{\dot\cup} \varphi_1\cdot\mathcal{G} \mathbin{\dot\cup} \varphi_2\cdot\mathcal{G}\times\mathcal{G} \mathbin{\dot\cup} \varphi_3\cdot\mathcal{G}\times\mathcal{G}\times\mathcal{G} \mathbin{\dot\cup} \cdots\Big) = \bigcirc\times\varphi(\mathcal{G}), \tag{5a}$$

$$\begin{aligned}\mathcal{G} &= \bigcirc \times \Big(\varphi_1\cdot\big((\mathcal{F}\mathbin{\dot\cup}\mathcal{G})\setminus\mathcal{G}\big) \mathbin{\dot\cup} \varphi_2\cdot\big((\mathcal{F}\mathbin{\dot\cup}\mathcal{G})^2\setminus\mathcal{G}^2\big) \mathbin{\dot\cup} \varphi_3\cdot\big((\mathcal{F}\mathbin{\dot\cup}\mathcal{G})^3\setminus\mathcal{G}^3\big) \mathbin{\dot\cup} \cdots\Big) \\ &= \bigcirc\times\big(\varphi(\mathcal{F}\mathbin{\dot\cup}\mathcal{G})\setminus\varphi(\mathcal{G})\big).\end{aligned} \tag{5b}$$

The formal recursive equation (5) together with the recursive description of $N(T)$ immediately leads to the following system of functional equations for $F(z,v)$ and $G(z,v)$:

$$F(z,v) = zv\varphi(G(z,v)), \tag{6a}$$

$$G(z,v) = z\Big(\varphi\big(F(z,v)+G(z,v)\big) - \varphi(G(z,v))\Big). \tag{6b}$$

Since we can substitute $F(z,v)$ by $zv\varphi(G(z,v))$, we obtain a single functional equation for $G(z,v)$:

$$G(z,v) = z\Big(\varphi\big(zv\varphi(G(z,v))+G(z,v)\big) - \varphi(G(z,v))\Big), \tag{7}$$

and $T(z,v)$ can be expressed as follows:

$$T(z,v) = zv\varphi(G(z,v)) + G(z,v). \tag{8}$$

4.2. **Treating the functional equations.** In our distributional analysis of N_n we are interested in the asymptotic behaviour of the coefficients $[z^n]T(z,v)$ uniformly in a neighbourhood of $v=1$. We will first establish a local expansion of $G(z,v)$ around its dominant singularity (singularity of smallest modulus) $z=\rho(v)$, which holds uniformly for $|v-1|\le\eta$ with $\eta>0$. This also leads to a local expansion of $T(z,v)$ around $z=\rho(v)$ due to (8). To establish a suitable local expansion of $G(z,v)$ from the functional equation (8) we will apply a slightly modified version of a general theorem of Drmota for such a kind of functional equations. Here, we basically state this theorem as given in [6], which is suitable for our study of N_n here $f_z(z,y,v) := \frac{\partial}{\partial z}f(z,y,v)$ always denotes the partial derivative of the function $f(z,y,v)$ w.r.t. z; analogous for the other variables.

Theorem 3 (see Drmota [5],[6]). *Suppose that $f(z,y,v)=\sum_{n,m} f_{n,m}(v)z^n y^m$ is an analytic function in z, y around 0 and v around 0 such that $f(0,y,v)=0$, that $f(z,0,v)\neq 0$, and that all coefficients $f_{n,m}(1)$ of $f(z,y,1)$ are real and non-negative. Then the unique solution $y=y(z,v)=\sum_n y_n(v)z^n$ of the functional equation*

$$y=f(z,y,v),$$

with $y(0,v)=0$, is analytic around 0 and has non-negative coefficients $y_n(1)$ for $v=1$. Furthermore, if the region of convergence of $f(z,y,v)$ is large enough such that there exist non-negative solutions $z=\rho$ and $y=\tau_1$ of the system of equations

$$y=f(z,y,1), \tag{9a}$$

$$1=f_y(z,y,1), \tag{9b}$$

with $f_z(\rho,\tau_1,1)\neq 0$ and $f_{yy}(\rho,\tau_1,1)\neq 0$, and $f(z,y,v)$ is analytic around $z=\rho$, $y=\tau_1$ and $v=1$, then there exist functions $\rho(v)$, $p(z,v)$, $q(z,v)$ which are analytic around $z=\rho(v)$ and $v=1$ such that $y(z,v)$ is analytic for $|z|<\rho(v)$ and $|v-1|\leq\epsilon$ (for some $\epsilon>0$) and has a representation of the form

$$y(z,v)=p(z,v)-q(z,v)\sqrt{1-\frac{z}{\rho(v)}}, \tag{10}$$

locally around $z=\rho$ and $v=1$. We have $\tau_1(v):=p(\rho(v),v)=y(\rho(v),v)$ and

$$q(\rho(v),v)=\sqrt{\frac{2\rho(v)f_z(\rho(v),\tau_1(v),v)}{f_{yy}\Big(\rho(v),\tau_1(v),v\Big)}}. \tag{11}$$

Moreover, (11) *provides a local analytic continuation of $y(z,v)$ for $\arg(z-\rho(v))\neq 0$. Furthermore, if $y_n(1)>0$, for all $n>n_0$ with an arbitrary $n_0\in\mathbb{N}$, then $z=\rho(v)$ is the only dominant singularity of $y(z,v)$ for $|v-1|\leq\epsilon$.*

Remark 1. We have to assume that the function $f(z,y,v)$ exists in a neighbourhood of $z=\rho$, $y=\tau_1$ and $v=1$, in order ensure that $y(z,v)$ is analytic for $|z|<\rho(v)$ and $|v-1|\leq\epsilon$. Without our new assumptions one may construct pathological examples of the form $f(z,y,v)=a(z,y)\cdot b(v)$, satisfying all other assumptions, where $b(1)$ is finite and $b(v)$ having i.e. a branching point at $v=1$, which would violate the analyticity of $y(z,v)$ at $v=1$. The extra assumption is usually fulfilled in the context of combinatorial enumeration, and therefore not a severe restriction, since it can easily be checked for concrete problems. Note that the assumption certainly implies the stated result, but we expect that weaker assumptions may also be sufficient.

Remark 2. The proof of Theorem 3 is fully analogous to the original proof of Drmota [5], page 113, Proposition 1, except the fact, that we a priori assume that $f(z,y,v)$ is analytic around $z=\rho$, $y=\tau_1$ and $v=1$, which justifies the argument of Drmota [5], page 115, regarding the Weierstrass preperation theorem. Moreover, this also proves that the argument regarding the dominant singularity in the proof of Lemma 1 of Drmota on page 116 is correct, when assuming the analyticity of $f(z,y,v)$ around $z=\rho$, $y=\tau_1$ and $v=1$.

We will apply Theorem 3 to the functional equation (7), which is written now in the following form:

$$G=f(z,G,v):=z\Big(\varphi\big(zv\varphi(G)+G\big)-\varphi(G)\Big). \tag{12}$$

We easily verify that $f(0,G,v)=0$ and that

$$f(z,0,v)=z\sum_{k\geq 1}\varphi_k z^k v^k \varphi_0^k\neq 0, \tag{13}$$

due to Assumption 1. We obtain by Taylor-series expansion,

$$f(z,G,v)=z\Big(\varphi\big(zv\varphi(G)+G\big)-\varphi(G)\Big)=z\Big(\sum_{j\geq 0}\varphi^{(j)}(G)\frac{(\varphi(G)zv)^j}{j!}-\varphi(G)\Big)=\sum_{j\geq 1}\varphi^{(j)}(G)\frac{\varphi(G)^j v^j z^{j+1}}{j!}. \tag{14}$$

Hence, the coefficients $[z^nG^m]f(z,G,1)$ are real and non-negative, since $\varphi(G)$ and thus also its j-th derivatives $\varphi^{(j)}(G)$ have itself real and non-negative coefficients.

Following (9) we consider the system of equations

$$G = f(z,G,1) = z\Big(\varphi\big(z\varphi(G)+G\big) - \varphi(G)\Big) \tag{15a}$$

$$1 = f_G(z,G,1) = z\Big(\varphi'\big(z\varphi(G)+G\big)\big(z\varphi'(G)+1\big) - \varphi'(G)\Big). \tag{15b}$$

We are searching for values $z=\rho$ and $G=\tau_1$, which solve the system of equations (15). We set

$$H := G + z\varphi(G), \quad \tau := \tau_1 + \rho\varphi(\tau_1), \tag{16}$$

and immediately obtain from (15a)

$$H = z\varphi(H), \quad \tau = \rho\varphi(\tau). \tag{17}$$

Due to Assumption 1, condition (ii), there exist a minimal positive solution of (17) within the radius of convergence of $\varphi(t)$.We obtain exactly the constants τ and ρ described in Subsection 2.2 (in particular $z=\rho$ is the unique dominant singularity of the generating function $T(z)$ also). Hence τ_1 is given as the minimal positive solution of equation(16), and it follows by our assumptions on $\varphi(t)$ that τ_1 exists with $0<\tau_1<\tau$. By equation 14 $f(z,y,v)$ clearly exists in a neighbourhood of $z=\rho$, $G=\tau_1$ and $v=1$.

Now the core statement of Theorem 3 guarantees that in a neighbourhood of $v=1$, $|v-1|\le\epsilon$, there exist analytic functions $\rho(v)$, $p(z,v)$ and $q(z,v)$ such that $G(z,v)$ is analytic for $|z|<\rho(v)$ and it has the following local representation around $z=\rho(v)$:

$$G(z,v) = p(z,v) - q(z,v)\sqrt{1-\frac{z}{\rho(v)}}. \tag{18}$$

For our purpose the following local expansion of $G(z,v)$ around the dominant singularity $z=\rho(v)$, which holds uniformly for $|v-1|\le\epsilon$ and is an immediate consequence of (18), is sufficient:

$$G(z,v) = \tau_1(v) - q(\rho(v),v)\sqrt{1-\frac{z}{\rho(v)}} + \kappa(v)\big(1-\frac{z}{\rho(v)}\big) + \mathcal{O}\big((1-\frac{z}{\rho(v)})^{\frac{3}{2}}\big), \tag{19}$$

where $\tau_1(v) := p(\rho(v),v) = G(\rho(v),v)$ and, due to (11):

$$q(\rho(v),v) = \sqrt{\frac{2\rho(v)f_z(\rho(v),\tau_1(v),v)}{f_{GG}\Big(\rho(v),\tau_1(v),v\Big)}}. \tag{20}$$

The function $\kappa(v)$ is not specified, since the actual function is of no importance here.

Since $T(z,v)$ is obtained from $G(z,v)$ via (8), we obtain that, in a neighbourhood of $v=1$, $T(z,v)$ is analytic for $|z|<\rho(v)$ and $z=\rho(v)$ is a dominant singularity of $T(z,v)$. Using (19) and a Taylor-series expansion of $\varphi(t)$ we obtain the following local expansion of $T(z,v)$ around $z=\rho(v)$, which holds uniformly for $|v-1|\le\epsilon$:

$$T(z,v) = \tau_1(v)+v\rho(v)\varphi(\tau_1(v)) - q(\rho(v),v)(1+\varphi'(\tau_1(v)))\sqrt{1-\frac{z}{\rho(v)}} + \tilde{\kappa}(v)\big(1-\frac{z}{\rho(v)}\big) + \mathcal{O}\big((1-\frac{z}{\rho(v)})^{\frac{3}{2}}\big), \tag{21}$$

with a certain function $\tilde{\kappa}(v)$.

Up to now, we did not consider, whether $z=\rho(v)$ is the unique dominant singularity of $T(z,v)$ for $|v-1|\le\epsilon$. However, the last statement of Theorem 3 gives a condition, which guarantees that $z=\rho(v)$ is the unique dominant singularity of $G(z,v)$, which implies due to (16) that this also holds for $T(z,v)$: we only have to show that G_n, i.e., the total weight of size-n trees of type $\mathrm{II}^{[N]}$, is positive for trees large enough, $n>n_0$. This is shown in Subsection 7.1 for families satisfying Assumption 1 and thus it is guaranteed for those simply generated tree families that $z=\rho(v)$ is indeed the unique dominant singularity of $T(z,v)$.

4.3. **Extracting coefficients.** The appearing local expansion (21) is amenable for singularity analysis and we will apply the transfer lemmata of Flajolet and Odlyzko [8], which allows to transfer the local behaviour of a generating function around its dominant singularity to the asymptotic behaviour of its coefficients. The asymptotic expansion for the coefficients $[z^n]T(z,v) = \sum_{m\geq 0} T_n \mathbb{P}\{N_n = m\} v^m$ can be translated immediately into an expansion of the moment generating function (= Laplace transform) $\mathbb{E}\big(e^{N_n s}\big)$ of the random variable N_n considered via

$$\mathbb{E}\big(e^{N_n s}\big) = \sum_{m\geq 0} \mathbb{P}\{X_n = m\} e^{sm} = \frac{[z^n]T(z,e^s)}{[z^n]T(z,1)} = \frac{[z^n]T(z,e^s)}{T_n}. \tag{22}$$

Then we can apply the continuity theorem of the Laplace transform to obtain the convergence in distribution of N_n to a Gaussian distributed random variable. In the instances appearing here we can apply directly the so called quasi-power theorem due to Hwang. It gives a powerful method not only to prove the Gaussian limit law but also to determine the rate of convergence. The theorem as given in [9] is stated below.

Theorem 4 (Quasi-power theorem, H. K. Hwang, [9])**.** *Let $(X_n)_{n\geq 1}$ by a sequence of integer random variables. Assume that the moment generating functions $M_n(s) := \mathbb{E}(e^{X_n s})$ of X_n are analytic in a disc $|s| \leq \rho$, for some $\rho > 0$, and satisfy there an expansion of the form*

$$M_n(s) = e^{\phi(n)U(s)+V(s)}\Big(1 + \mathcal{O}(\kappa_n^{-1})\Big), \tag{23}$$

where $\phi(n), \kappa_n \to \infty$ and $U(s), V(s)$ are independent of n and analytic for $|s| \leq \rho$. Assume also the variability condition $U''(0) \neq 0$. Under these assumptions, the mean and variance of X_n satisfy

$$\mathbb{E}(X_n) = U'(0)\phi(n) + V'(0) + \mathcal{O}(\kappa_n^{-1}), \qquad \mathbb{V}(X_n) = U''(0)\phi(n) + V''(0) + \mathcal{O}(\kappa_n^{-1}). \tag{24}$$

The distribution of X_n is asymptotically Gaussian and the speed of convergence to the Gaussian limit is $\mathcal{O}(\kappa_n^{-1} + \phi(n)^{-1/2})$:

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} \leq x \right\} - \Phi(x) \right| = \mathcal{O}\Big(\frac{1}{\kappa_n} + \frac{1}{\sqrt{\phi(n)}}\Big), \tag{25}$$

where $\Phi(x)$ denotes the distribution function of the Gaussian normal distribution $\mathcal{N}(0,1)$.

We will proceed now with showing Theorem 1 following the steps outlined above. Starting from the local expansion (21) we obtain by an application of singularity analysis immediately

$$[z^n]T(z,v) = \frac{h(\rho(v),v)\big(1+\varphi'(\tau_1(v))\big)}{2\sqrt{\pi}} \cdot \frac{1}{\rho(v)^n n^{\frac{3}{2}}} \Big(1 + \mathcal{O}(\frac{1}{n})\Big). \tag{26}$$

Together with the asymptotic expansion (3) for T_n we get from (26) via (22) the following expansion of the moment generating function of N_n uniformly in a complex neighbourhood of $s = 0$:

$$\begin{aligned}\mathbb{E}\big(e^{N_n s}\big) &= \frac{[z^n]T(z,e^s)}{T_n} = h(\rho(e^s),e^s)\big(1+\varphi'(\tau_1(e^s))\big)\sqrt{\frac{\varphi''(\tau)}{2\varphi(\tau)}}\frac{\rho^n}{\rho(e^s)^n}\Big(1+\mathcal{O}(\frac{1}{n})\Big) \\ &= \exp\big(nU(s)+V(s)\big)\Big(1+\mathcal{O}(\frac{1}{n})\Big).\end{aligned} \tag{27}$$

The functions $U(s)$ and $V(s)$, which are analytic around $s = 0$, are given as follows:

$$U(s) = \log\big(\frac{\rho}{\rho(e^s)}\big), \tag{28}$$

and

$$V(s) = \log\big(h(\rho(e^s),e^s)\big) + \log\big(1+\varphi'(\tau_1(e^s))\big) + \frac{1}{2}\log(\varphi''(\tau)) - \log(\varphi(\tau)). \tag{29}$$

The corresponding part of Theorem 1 concerning the Gaussian limit distribution of N_n follows now by an application of Theorem 4, provided that $U''(0) \neq 0$ holds. Furthermore we obtain from Theorem 4 the following asymptotic expansion of the expectation $\mathbb{E}(N_n)$ and $\mathbb{V}(N_n)$:

$$\mathbb{E}(N_n) = \mu_{\mathcal{T}}^{[N]} n + \mathcal{O}(1), \qquad \mathbb{V}(N_n) = \nu_{\mathcal{T}}^{[N]} n + \mathcal{O}(1), \tag{30}$$

with
$$\mu_{\mathcal{T}}^{[N]} := U'(0), \qquad \nu_{\mathcal{T}}^{[N]} := U''(0). \tag{31}$$
It remains to compute the leading constants $\mu_{\mathcal{T}}^{[N]} = U'(0)$ and $\nu_{\mathcal{T}}^{[N]} = U''(0)$. Using (28) we obtain
$$U'(0) = -\frac{\rho'(1)}{\rho} \tag{32}$$
and
$$U''(0) = \frac{\rho'(1)^2}{\rho^2} - \frac{\rho'(1)}{\rho} - \frac{\rho''(1)}{\rho} = U'(0)^2 + U'(0) - \frac{\rho''(1)}{\rho}. \tag{33}$$
To compute $\rho'(v)$ and $\rho''(v)$ we use the fact that the solution of the following system of equations is given by $z = \rho(v)$ and $G = \tau_1(v)$, which is a consequence of the implicit function theorem:
$$G = f(z, G, v), \tag{34a}$$
$$1 = f_G(z, G, v), \tag{34b}$$
with $f(z, G, v)$ given by (12). Implicit differentiation of (34) gives then the following result.

Lemma 4. *The constants $\mu_{\mathcal{T}}^{[N]}$ and $\nu_{\mathcal{T}}^{[N]}$ appearing in Theorem 1 are given as follows:*
$$\mu_{\mathcal{T}}^{[N]} = -\frac{\rho'(1)}{\rho}, \qquad \nu_{\mathcal{T}}^{[N]} = \frac{\rho'(1)^2}{\rho^2} - \frac{\rho'(1)}{\rho} - \frac{\rho''(1)}{\rho},$$
where ρ, $\rho'(1)$ and $\rho''(1)$ are specified below and the constants τ and τ_1 appearing there are given by the minimal positive real solutions of the functional equations stated. It holds then with
$$f(z, G, v) = z\Big(\varphi\big(zv\varphi(G) + G\big) - \varphi(G)\Big)$$
the following:
$$\tau\varphi'(\tau) = \varphi(\tau), \qquad \rho = \frac{\tau}{\varphi(\tau)}, \qquad \tau_1 + \rho\varphi(\tau_1) = \tau,$$
$$\rho'(1) = -\frac{f_v}{f_z}, \qquad \tau_1'(1) = \frac{-f_{zG} \cdot \rho'(1) - f_{vG}}{f_{GG}},$$
$$\rho''(1) = -\frac{1}{f_z}\big(f_{zz} \cdot \rho'(1)^2 + 2f_{zG} \cdot \rho'(1)r'(1) + f_{GG} \cdot r'(1)^2 + 2f_{vz} \cdot \rho'(1) + 2f_{vG} \cdot r'(1) + f_{vv}\big),$$
where we use the abbreviations $f_d := f_d(\rho, \tau_1, 1)$ and $f_{de} := f_{de}(\rho, \tau_1, 1)$, for $d, e \in \{z, G, v\}$.

The results of Theorem 2 concerning the node-independence number N_n follow by establishing the computations of Lemma 4 for $\varphi(t) = (1 + t)^2$: binary trees, $\varphi(t) = \frac{1}{1-t}$: planted plane trees and $\varphi(t) = e^t$: rooted labelled trees.

5. Proof of Theorem 1 for the path node-covering number

The study of the random variable P_n, which counts the path-node covering number of a random tree of size n in a simply generated tree family $\mathcal{T}$ that satisfies Assumption 1, can be carried out completely analogous to N_n, the corresponding random variable for the node-independence number, as done in Section 4. Thus we will be more brief and carry out the main steps only.

Here we denote by $\mathcal{F}$ the subfamily of type $\mathrm{I}^{[P]}$ trees and by $\mathcal{G}$ the subfamily of type $\mathrm{II}^{[P]}$ trees in a simply generated tree family $\mathcal{T}$. We also denote by F_n the total weight of size-n trees of type $\mathrm{I}^{[P]}$, $F_n := \sum_{|T|=n, T\in\mathcal{F}} w(T)$, and by G_n the total weight of size-n trees of type $\mathrm{II}^{[P]}$, $G_n := \sum_{|T|=n, T\in\mathcal{G}} w(T)$. Thus $F_n + G_n = T_n$, where T_n gives the total weight of size-n trees in $\mathcal{T}$.

We will introduce the bivariate generating function
$$T(z, v) := \sum_{n\geq 1}\sum_{m\geq 0} T_n \mathbb{P}\{P_n = m\} z^n v^m,$$
and the auxiliary generating functions for the type $\mathrm{I}^{[P]}$ and type $\mathrm{II}^{[P]}$ trees:
$$F(z, v) = \sum_{n\geq 1}\sum_{m\geq 0} F_n \mathbb{P}\{P_n = m \mid T \in \mathcal{F}\} z^n v^m,$$

$$G(z,v)=\sum_{n\ge1}\sum_{m\ge0}G_n\mathbb{P}\{P_n=m\mid T\in\mathcal{G}\}z^nv^m,$$

which satisfy $T(z,v)=F(z,v)+G(z,v)$.

The recursive description of type $\mathrm{I}^{[P]}$ and type $\mathrm{II}^{[P]}$ trees given in Subsection 2.1 can be described by the following formal equations:

$$\begin{aligned}\mathcal{G}&=\bigcirc\times\Big(\varphi_0\cdot\{\epsilon\}\mathbin{\dot\cup}\varphi_1\cdot\mathcal{F}\mathbin{\dot\cup}\varphi_2\cdot\mathcal{F}\times\mathcal{F}\mathbin{\dot\cup}\varphi_3\cdot\mathcal{F}\times\mathcal{F}\times\mathcal{F}\mathbin{\dot\cup}\cdots\Big)\\&\quad\mathbin{\dot\cup}\ \bigcirc\times\Big(\varphi_1\cdot\mathcal{G}\mathbin{\dot\cup}2\varphi_2\cdot\mathcal{G}\times\mathcal{F}\mathbin{\dot\cup}3\varphi_3\cdot\mathcal{G}\times\mathcal{F}\times\mathcal{F}\mathbin{\dot\cup}\cdots\Big)\\&=\bigcirc\times\varphi(\mathcal{F})\mathbin{\dot\cup}\ \bigcirc\times\mathcal{G}\times\varphi'(\mathcal{F}),\end{aligned}\tag{35a}$$

$$\begin{aligned}\mathcal{F}&=\bigcirc\times\Big(\varphi_2\cdot\big((\mathcal{F}\mathbin{\dot\cup}\mathcal{G})^2\setminus(\mathcal{F}^2\mathbin{\dot\cup}2\cdot\mathcal{G}\times\mathcal{F})\big)\mathbin{\dot\cup}\varphi_3\cdot\big((\mathcal{F}\mathbin{\dot\cup}\mathcal{G})^3\setminus(\mathcal{F}^3\mathbin{\dot\cup}3\cdot\mathcal{G}\times\mathcal{F}^2)\big)\mathbin{\dot\cup}\cdots\Big)\\&=\bigcirc\times\big(\varphi(\mathcal{F}\mathbin{\dot\cup}\mathcal{G})\setminus(\varphi(F)\mathbin{\dot\cup}\mathcal{G}\times\varphi'(\mathcal{F}))\big).\end{aligned}\tag{35b}$$

The formal recursive equation (35) together with the recursive description of $P(T)$ leads then to the following system of functional equations for $F(z,v)$ and $G(z,v)$:

$$G(z,v)=zv\varphi(F(z,v))+zG(z,v)\varphi'(F(z,v)),\tag{36a}$$

$$F(z,v)=\frac{z}{v}\Big(\varphi\big(F(z,v)+G(z,v)\big)-\varphi(F(z,v))-G(z,v)\varphi'(F(z,v))\Big).\tag{36b}$$

We can express $G(z,v)$ as function of $F(z,v)$:

$$G(z,v)=\frac{zv\varphi(F(z,v))}{1-z\varphi'(F(z,v))},$$

which leads to the following single functional equation for $F(z,v)$:

$$F(z,v)=\frac{z}{v}\left[\varphi\left(F(z,v)+\frac{zv\varphi(F(z,v))}{1-z\varphi'(F(z,v))}\right)-\varphi(F(z,v))-\frac{zv\varphi(F(z,v))\varphi'(F(z,v))}{1-z\varphi'(F(z,v))}\right].\tag{37}$$

$T(z,v)$ can be expressed as follows:

$$T(z,v)=F(z,v)+\frac{zv\varphi(F(z,v))}{1-z\varphi'(F(z,v))}.\tag{38}$$

In order to show a local expansion of $F(z,v)$ (and as a consequence of $T(z,v)$) around the dominant singularity $z=\rho(v)$ we will apply Theorem 3 to the functional equation

$$F=f(z,F,v)=\frac{z}{v}\left[\varphi\left(F+\frac{zv\varphi(F)}{1-z\varphi'(F)}\right)-\varphi(F)-\frac{zv\varphi(F)\varphi'(F)}{1-z\varphi'(F)}\right].\tag{39}$$

One can easily verify that the assumptions on $f(z,F,v)$ required in Theorem 3 are satisfied. Now we turn to the system of equations

$$F=f(z,F,1)=z\left[\varphi\left(F+\frac{z\varphi(F)}{1-z\varphi'(F)}\right)-\varphi(F)-\frac{z\varphi(F)\varphi'(F)}{1-z\varphi'(F)}\right],\tag{40a}$$

$$\begin{aligned}1=f_F(z,F,1)=z\Bigg[&\varphi'\left(F+\frac{z\varphi(F)}{1-z\varphi'(F)}\right)\cdot\left(1+\frac{z\varphi'(F)}{1-z\varphi'(F)}+\frac{z^2\varphi(F)\varphi''(F)}{1-z\varphi'(F)}\right)\\&-\varphi'(F)-\frac{z(\varphi'^2(F)+\varphi(F)\varphi''(F))(1-z\varphi'(F))+z^2\varphi(F)\varphi'(F)\varphi''(F)}{(1-z\varphi'(F))^2}\Bigg],\end{aligned}\tag{40b}$$

and search for $z=\rho$ and $F=\tau_1$ satisfying the system (40). If we set

$$H:=F+\frac{z\varphi(F)}{1-z\varphi'(F)},\quad \tau:=\tau_1+\frac{\rho\varphi(\tau_1)}{1-\rho\varphi'(\tau_1)},\tag{41}$$

then we obtain the functional equation

$$H=z\varphi(H),\quad \tau=\rho\varphi(\tau),$$

Thus τ and ρ are again the values appearing in Subsection 2.2 and τ_1, with $0<\tau_1<\tau$, is given as the solution of equation (41).

Theorem 3 guarantees then the following local expansion of $F(z,v)$ around the dominant singularity $z=\rho(v)$, uniformly for $|v-1|\le\epsilon$:

$$F(z,v)=\tau_1(v)-q(\rho(v),v)\sqrt{1-\frac{z}{\rho(v)}}+\kappa(v)\big(1-\frac{z}{\rho(v)}\big)+\mathcal{O}\big((1-\frac{z}{\rho(v)})^{\frac32}\big), \tag{42}$$

where $\kappa(v)$ is a certain function, $\tau_1(v)=F(\rho(v),v)$ and $q(\rho(v),v)$ is given as follows:

$$q(\rho(v),v)=\sqrt{\frac{2\rho(v)f_z(\rho(v),\tau_1(v),v)}{f_{FF}\Big(\rho(v),\tau_1(v).v\Big)}}.$$

Equation (42) together with (38) leads then to the following local expansion of $T(z,v)$ around its dominant singularity $z=\rho(v)$:

$$T(z,v)=\alpha(v)-\beta(v)\sqrt{1-\frac{z}{\rho(v)}}+\tilde{\kappa}(v)\big(1-\frac{z}{\rho(v)}\big)+\mathcal{O}\Big(\big(1-\frac{z}{\rho(v)}\big)^{\frac32}\Big), \tag{43}$$

where $\alpha(v)$, $\beta(v)$ and $\tilde{\kappa}(v)$ are certain functions analytic around $v=1$, which are not specified here since the actual functions are not required in the sequel.

Again we can prove that $z=\rho(v)$ is the unique dominant singularity of $F(z,v)$ and $T(z,v)$ for $|v-1|\le\epsilon$, by showing that F_n, i.e., the total weight of size-n trees of type $\mathrm{I}^{[P]}$, is positive for trees large enough, $n>n_0$. This is carried out in Subsection 7.2 for families satisfying Assumption 1.

An application of singularity analysis shows then the following expansion of the moment generating function of P_n uniformly in a complex neighbourhood of $s=0$:

$$\begin{aligned}\mathbb{E}\big(e^{P_n s}\big)&=\frac{[z^n]T(z,e^s)}{T_n}=\beta(e^s)\sqrt{\frac{\varphi''(\tau)}{2\varphi(\tau)}}\frac{\rho^n}{\rho(e^s)^n}\Big(1+\mathcal{O}(\frac1n)\Big)\\&=\exp\big(nU(s)+V(s)\big)\Big(1+\mathcal{O}(\frac1n)\Big),\end{aligned} \tag{44}$$

where the functions $U(s)$ and $V(s)$ are analytic around $s=0$ and $U(s)$ is given as follows ($V(s)$ will not be required in the sequel):

$$U(s)=\log\big(\frac{\rho}{\rho(e^s)}\big). \tag{45}$$

The corresponding part of Theorem 1 concerning the Gaussian limit distribution of P_n follows now by an application of Theorem 4, provided that $U''(0)\neq 0$ holds.

The leading constants $\mu_T^{[P]}=U'(0)$ of the expectation and $\nu_T^{[P]}=U''(0)$ of the variance of P_n are specified as follows:

Lemma 5. *The constants $\mu_T^{[P]}$ and $\nu_T^{[P]}$ appearing in Theorem 1 are given as follows:*

$$\mu_T^{[P]}=-\frac{\rho'(1)}{\rho},\qquad \nu_T^{[P]}=\frac{\rho'(1)^2}{\rho^2}-\frac{\rho'(1)}{\rho}-\frac{\rho''(1)}{\rho},$$

where ρ, $\rho'(1)$ and $\rho''(1)$ are specified below and the constants τ and τ_1 appearing there are given by the minimal positive real solutions of the functional equations stated. It holds then with

$$f(z,F,v)=\frac{z}{v}\left[\varphi\left(F+\frac{zv\varphi(F)}{1-z\varphi'(F)}\right)-\varphi(F)-\frac{zv\varphi(F)\varphi'(F)}{1-z\varphi'(F)}\right]$$

the following:

$$\begin{aligned}&\tau\varphi'(\tau)=\varphi(\tau),\qquad \rho=\frac{\tau}{\varphi(\tau)},\qquad \tau_1+\frac{\rho\varphi(\tau_1)}{1-\rho\varphi'(\tau_1)}=\tau,\\&\rho'(1)=-\frac{f_v}{f_z},\qquad \tau_1'(1)=\frac{-f_{zF}\cdot\rho'(1)-f_{vF}}{f_{FF}},\\&\rho''(1)=-\frac{1}{f_z}\big(f_{zz}\cdot\rho'(1)^2+2f_{zF}\cdot\rho'(1)r'(1)+f_{FF}\cdot r'(1)^2+2f_{vz}\cdot\rho'(1)+2f_{vF}\cdot r'(1)+f_{vv}\big),\end{aligned}$$

where we use the abbreviations $f_d:=f_d(\rho,\tau_1,1)$ and $f_{de}:=f_{de}(\rho,\tau_1,1)$, for $d,e\in\{z,F,v\}$.

The results of Theorem 2 concerning the path node-covering number P_n follow again by establishing the computations of Lemma 5 for $\varphi(t) = (1+t)^2$: binary trees, $\varphi(t) = \frac{1}{1-t}$: planted plane trees and $\varphi(t) = e^t$: rooted labelled trees.

6. Proof of Theorem 1 for the size of the kernel

Now we study the random variable K_n, which counts the size of the kernel of a random tree of size n in a directed simply generated tree family $\tilde{\mathcal{T}}$ that satisfies Assumption 1. In principle one uses the same steps as in Section 4 and Section 5, but things are slightly more involved, since, unlike the corresponding computations for N_n and P_n, one cannot reduce the system of functional equations appearing to a single equation in an easy way.

Given a directed simply generated tree family $\tilde{\mathcal{T}}$ we denote by $\mathcal{F}$ the subfamily of type $\mathrm{I}^{[K]}$ trees and by $\mathcal{G}$ the subfamily of type $\mathrm{II}^{[K]}$ trees in $\tilde{\mathcal{T}}$. Furthermore we denote by F_n the total weight of size-n trees of type $\mathrm{I}^{[K]}$, $F_n := \sum_{|\tilde{T}|=n, \tilde{T}\in\mathcal{F}} w(\tilde{T})$, and by G_n the total weight of size-n trees of type $\mathrm{II}^{[K]}$, $G_n := \sum_{|\tilde{T}|=n, \tilde{T}\in\mathcal{G}} w(\tilde{T})$. Of course it holds $F_n + G_n = \tilde{T}_n$, where $\tilde{T}_n$ gives the total weight of size-n trees in $\tilde{\mathcal{T}}$, see Subsection 2.2.

We introduce now the bivariate generating function of the probability that K_n has value m,

$$T(z,v) := \sum_{n\ge1}\sum_{m\ge0} \tilde{T}_n \mathbb{P}\{K_n = m\} z^n v^m,$$

and the following auxiliary generating functions:

$$\begin{aligned}
F(z,v) &= \sum_{n\ge1}\sum_{m\ge0} F_n \mathbb{P}\{K_n = m \mid \tilde{T}\in\mathcal{F}\} z^n v^m,\\
G(z,v) &= \sum_{n\ge1}\sum_{m\ge0} G_n \mathbb{P}\{K_n = m \mid \tilde{T}\in\mathcal{G}\} z^n v^m,\\
H(z,v) &= \sum_{n\ge1}\sum_{m\ge0} F_n \mathbb{P}\{\hat{K}_n = m \mid \tilde{T}\in\mathcal{F}\} z^n v^m.
\end{aligned}$$

Here the random variable $\hat{K}_n$ is given by subtracting 1 from the size of the kernel of a tree $\hat{T}$, which is obtained from a randomly chosen size-n tree in $\tilde{T}$ by attaching the root of $\tilde{T}$ to an additional node $w \notin \tilde{T}$ by a directed edge $e = (w, root(\tilde{T}))$.

Of course, the following equation holds by definition:

$$T(z,v) = F(z,v) + G(z,v).$$

The recursive description of type $\mathrm{I}^{[K]}$ and type $\mathrm{II}^{[K]}$ trees given in Subsection 2.1 can be described by the following formal equations:

$$\mathcal{G} = \bigcirc \times \Big(\varphi\big(\mathcal{F}^{\uparrow} \,\dot\cup\, \mathcal{F}^{\downarrow} \,\dot\cup\, \mathcal{G}^{\uparrow} \,\dot\cup\, \mathcal{G}^{\downarrow}\big) \setminus \varphi\big(\mathcal{F}^{\uparrow} \,\dot\cup\, \mathcal{G}^{\uparrow} \,\dot\cup\, \mathcal{G}^{\downarrow}\big)\Big), \tag{46a}$$

$$\mathcal{F} = \bigcirc \times \varphi\big(\mathcal{F}^{\uparrow} \,\dot\cup\, \mathcal{G}^{\uparrow} \,\dot\cup\, \mathcal{G}^{\downarrow}\big). \tag{46b}$$

Here $\mathcal{F}^{\downarrow}$ denotes that $\mathcal{F}$ is attached to $\bigcirc$ by a directed edge $(\bigcirc, root(\mathcal{F}))$ and $\mathcal{F}^{\uparrow}$ denotes that $\mathcal{F}$ is attached to $\bigcirc$ by a directed edge $(root(\mathcal{F}), \bigcirc)$, etc.

The formal recursive equation (46) together with the recursive description of $K(T)$ leads then to the following system of functional equations for $F(z,v)$, $G(z,v)$ and $H(z,v)$:

$$F(z,v) = zv\varphi\big(2G(z,v) + H(z,v)\big), \tag{47a}$$

$$G(z,v) = z\Big(\varphi\big(2G(z,v) + 2F(z,v)\big) - \varphi\big(2G(z,v) + F(z,v)\big)\Big), \tag{47b}$$

$$H(z,v) = z\varphi\big(2G(z,v) + F(z,v)\big). \tag{47c}$$

To treat the system of functional equations (47) we will apply an extension of Theorem 3, which follows from [5] and deals with the situation appearing here. Before stating this theorem we introduce some notation. The theorem deals with systems of functional equations

$$y_1 = f_1(z, y_1, \ldots, y_N, v), \quad \ldots, \quad y_N = f_N(z, y_1, \ldots, y_N, v)$$

for the unknown functions $y_1(z,v),\dots,y_N(z,v)$. The system will be annotated in vectorial form as follows: $\vec{y}=\vec{f}(z,\vec{y},v)$ with $\vec{y}=(y_1,\dots,y_N)$ and $\vec{f}=(f_1,\dots,f_N)$. The identity matrix of size N is denoted by E_N. Furthermore $\vec{f}_{\vec{y}}$ denotes the $N\times N$-matrix of partial derivatives of $\vec{f}$ w.r.t. $\vec{y}$:

$$\vec{f}_{\vec{y}}=\begin{pmatrix}\frac{\partial}{\partial y_1}f_1 & \dots & \frac{\partial}{\partial y_N}f_1\\ \vdots & & \vdots\\ \frac{\partial}{\partial y_1}f_N & \dots & \frac{\partial}{\partial y_N}f_N\end{pmatrix}.$$

We also define the notion of the dependency graph $G_{\vec{f}}=(V,E)$ of a system of functional equations $\vec{y}=\vec{f}(z,\vec{y},v)$: the set of vertices $V=\{y_1,\dots,y_N\}$ is given by the set of unknown functions and an ordered pair (y_i,y_j) is contained in the edge set E if and only if $f_i(z,\vec{y},v)$ really depends on y_j.

Theorem 5 (see Drmota, [5])**.** *Suppose that $\vec{f}(z,\vec{y},v)=(f_1(z,\vec{y},v),\dots,f_N(z,\vec{y},v))$, with $\vec{y}=(y_1,\dots,y_N)$ are analytic functions around $z=0$, $\vec{y}=(y_1,\dots,y_N)=\vec{0}$ and $v=0$, such that all Taylor coefficients of $f_1(z,y,1),\dots,f_N(z,y,1)$ are real and non-negative, that $\vec{f}(0,\vec{y},v)=\vec{0}$, that $\vec{f}(z,\vec{0},v)\neq\vec{0}$, $\vec{f}_z(z,\vec{y},v)\neq\vec{0}$, and that there exists j with $\vec{f}_{y_jy_j}(z,\vec{y},v)\neq\vec{0}$. Furthermore we assume that the region of convergence of $\vec{f}(z,\vec{y},v)$ is large enough such that there exist non-negative solutions $z=\rho$ and $\vec{y}=(\tau_1,\dots,\tau_N)$ of the system of equations*

$$\vec{y}=\vec{f}(z,\vec{y},1), \tag{48a}$$

$$0=\det\big(E_N-\vec{f}_{\vec{y}}(z,\vec{y},1)\big), \tag{48b}$$

inside it. Let

$$\vec{y}=\vec{y}(z,v)=(y_1(z,v),\dots,y_N(z,v))$$

denote the analytic solutions of the system

$$\vec{y}=\vec{f}(z,\vec{y},v), \tag{49}$$

with $\vec{y}(0,v)=\vec{0}$. If the dependency graph $G_{\vec{f}}=(V,E)$ of the system (49) in the unknown functions $y_1(z,v),\dots,y_N(z,v)$ is strongly connected, and $\vec{f}(z,\vec{y},1)$ is analytic around $z=\rho$ and $\vec{y}=(\tau_1,\dots,\tau_N)$ and $v=1$, then there exist functions $\rho(v)$, $p_1(z,v),\dots,p_N(z,v)$, $q_1(z,v),\dots,q_N(z,v)$ which are analytic around $z=\rho(v)$ and $v=1$ such that $y_1(z,v),\dots,y_N(z,v)$ are analytic for $|z|<\rho(v)$ and $|v-1|\le\epsilon$ (for some $\epsilon>0$) and have a representation of the form

$$y_1(z,v)=p_1(z,v)-q_1(z,v)\sqrt{1-\frac{z}{\rho(v)}},\quad\dots,\quad y_N(z,v)=p_N(z,v)-q_N(z,v)\sqrt{1-\frac{z}{\rho(v)}} \tag{50}$$

locally around $z=\rho(v)$ and $v=1$. Moreover, (50) provides a local analytic continuation of $y_1(z,v)$, $\dots$, $y_N(z,v)$ for $\arg(z-\rho(v))\neq 0$. Furthermore, if the coefficients $[z^n]y_j(z,1)>0$ (for all $1\le j\le N$), for all $n>n_0$ with an arbitrary $n_0\in\mathbb{N}$, then $z=\rho(v)$ is the only dominant singularity of the functions $y_1(z,v),\dots,y_n(z,v)$ for $|v-1|\le\epsilon$.

It further holds that $z=\rho(v)$ and $(y_1,\dots,y_N)=(\tau_1(v),\dots,\tau_N(v))$ with $\tau_j(v):=y_j(\rho(v),v)$, for $1\le j\le N$ are the solutions of the extended system

$$\vec{y}=\vec{f}(z,\vec{y},v), \tag{51a}$$

$$0=\det\big(E_N-\vec{f}_{\vec{y}}(z,\vec{y},v)\big). \tag{51b}$$

Remark 3. As in the proof of Theorem 3 we need to ensure that the functions $y_j(z,v)$ exist locally around $v=1$. The extra condition of $\vec{f}(z,\vec{y},1)$ being analytic around $z=\rho$ and $\vec{y}=(\tau_1,\dots,\tau_N)$ and $v=1$ is sufficient to ensure this. We refer the reader again to the work of Drmota [5].

We will now apply Theorem 5 to the system (47), which is written in the following form:

$$F=f(z,F,G,H,v):=vz\varphi\big(2G+H\big), \tag{52a}$$

$$G=g(z,F,G,H,v):=z\Big(\varphi\big(2G+2F\big)-\varphi\big(2G+F\big)\Big), \tag{52b}$$

$$H=h(z,F,G,H,v):=z\varphi\big(2G+F\big). \tag{52c}$$

It is easily verified that the dependency graph of the system (52) is strongly connected and that Theorem 5 is applicable, since the assumptions made on f, g and h are satisfied. We only have to guarantee that for $v = 1$ the system of equations corresponding to (48) has positive real solutions $z = \tilde{\rho}$ and $F = \tau_1$, $G = \tau_2$, $H = \tau_3$ within the radius of convergence of f, g and h. Since we have by definition $F(z, 1) = H(z, 1)$, which implies $\tau_1 = \tau_3$, the task can be reduced to find a positive solution $z = \tilde{\rho}$, $F = \tau_1$ and $G = \tau_2$ of the following system:

$$F = z\varphi(2G + F), \tag{53a}$$

$$G = z\big(\varphi(2G + 2F) - \varphi(2G + F)\big), \tag{53b}$$

$$\begin{aligned} 0 &= \det \begin{pmatrix} 1-z\varphi'(2G+F) & -2z\varphi'(2G+F) \\ -z\big(2\varphi'(2G+2F)-\varphi'(2G+F)\big) & 1-z\big(2\varphi'(2G+2F)-2\varphi'(2G+F)\big) \end{pmatrix} \\ &= -\big(1 - 2z\varphi'(2G + 2F)\big)\big(1 + z\varphi'(2G + F)\big). \end{aligned} \tag{53c}$$

If we set $\tilde{\tau} := \tau_1 + \tau_2$ one easily obtains from (53) that $\tilde{\tau}$ is given as minimal positive solution of the following equation:

$$2\tilde{\tau}\varphi'(2\tilde{\tau}) = \varphi(2\tilde{\tau}), \tag{54}$$

and $\tilde{\rho}$ is given by

$$\tilde{\rho} = \frac{\tilde{\tau}}{\varphi(2\tilde{\tau})} = \frac{1}{2\varphi'(2\tilde{\tau})}. \tag{55}$$

Thus we obtain that $2\tilde{\tau} = \tau$ and $2\tilde{\rho} = \rho$, where the constants τ and ρ are defined in Subsection 2.2. Thus by Assumption 1 it holds that $\tilde{\tau}$, and as a consequence $\tilde{\rho}$, exist. Furthermore we obtain from (53) that τ_1 and τ_2 are given as follows, where τ_1, with $0 < \tau_1 < \tilde{\tau}$ is the minimal positive solution of the equation stated:

$$\tau_1 = \tilde{\rho}\varphi(2\tilde{\tau} - \tau_1), \qquad \tau_2 = \tilde{\tau} - \tau_1. \tag{56}$$

Now Theorem 5 guarantees that in a neighbourhood of $v = 1$, $|v-1| \le \epsilon$, there exist analytic functions $\tilde{\rho}(v)$, $p_1(z, v)$, $p_2(z, v)$, $p_3(z, v)$ and $q_1(z, v)$, $q_2(z, v)$, $q_3(z, v)$ such that $F(z, v)$, $G(z, v)$ and $H(z, v)$ are analytic for $|z| < \tilde{\rho}(v)$ and they have the following local representations around $z = \tilde{\rho}(v)$:

$$\begin{aligned} &F(z, v) = p_1(z, v) - q_1(z, v)\sqrt{1 - \frac{z}{\tilde{\rho}(v)}}, \qquad G(z, v) = p_2(z, v) - q_2(z, v)\sqrt{1 - \frac{z}{\tilde{\rho}(v)}}, \\ &H(z, v) = p_3(z, v) - q_3(z, v)\sqrt{1 - \frac{z}{\tilde{\rho}(v)}}. \end{aligned} \tag{57}$$

Since $T(z, v) = F(z, v) + G(z, v)$ we obtain that in a neighbourhood of $v = 1$, $T(z, v)$ is analytic for $|z| < \tilde{\rho}(v)$ and $z = \tilde{\rho}(v)$ is a dominant singularity of $T(z, v)$. Furthermore we obtain from (57) that $T(z, v)$ has the following local expansion around $z = \tilde{\rho}(v)$, which holds uniformly for $|v - 1| \le \epsilon$:

$$T(z, v) = \alpha(v) - \beta(v)\sqrt{1 - \frac{z}{\tilde{\rho}(v)}} + \kappa(v)\big(1 - \frac{z}{\tilde{\rho}(v)}\big) + \mathcal{O}\Big(\big(1 - \frac{z}{\tilde{\rho}(v)}\big)^{\frac{3}{2}}\Big), \tag{58}$$

with certain analytic functions $\alpha(v)$, $\beta(v)$ and $\kappa(v)$, which are not specified, since the actual functions are not required further.

In order to prove that $z = \tilde{\rho}(v)$ is the unique dominant singularity of $T(z, v)$ in a neighbourhood of $v = 1$ we will show in Subsection 7.3 that, for tree families satisfying Assumption 1, F_n and G_n, i.e., the total weight of size-n trees of type $\mathrm{I}^{[K]}$ and of type $\mathrm{II}^{[K]}$, are positive for trees large enough, $n > n_0$.

An application of singularity analysis to equation (58) shows then the following expansion of the moment generating function of K_n uniformly in a complex neighbourhood of $s = 0$:

$$\begin{aligned} \mathbb{E}\big(e^{K_n s}\big) &= \frac{[z^n]T(z, e^s)}{\tilde{T}_n} = \beta(e^s)\sqrt{\frac{2\varphi''(2\tilde{\tau})}{\varphi(2\tilde{\tau})}}\,\frac{\tilde{\rho}^n}{\tilde{\rho}(e^s)^n}\Big(1 + \mathcal{O}(\frac{1}{n})\Big) \\ &= \exp\big(nU(s) + V(s)\big)\Big(1 + \mathcal{O}(\frac{1}{n})\Big), \end{aligned} \tag{59}$$

where the functions $U(s)$ and $V(s)$ are analytic around $s=0$ and $U(s)$ is given as follows ($V(s)$ will not be required in the sequel):

$$U(s) = \log\big(\frac{\tilde{\rho}}{\tilde{\rho}(e^s)}\big). \tag{60}$$

The corresponding part of Theorem 1 concerning the Gaussian limit distribution of K_n follows now by an application of Theorem 4, provided that $U''(0) \neq 0$ holds.

In order to determine the leading constants $\mu_{\mathcal{T}}^{[K]} = U'(0)$ of the expectation and $\nu_{\mathcal{T}}^{[K]} = U''(0)$ of the variance of K_n we use the fact that $z = \tilde{\rho}(v)$, $F = \tau_1(v) := F(\tilde{\rho}(v), v)$, $G = \tau_2(v) := G(\tilde{\rho}(v), v)$ and $H = \tau_3(v) := H(\tilde{\rho}(v), v)$ are solutions of the following system of equations:

$$F = f(z, F, G, H, v), \tag{61a}$$
$$G = g(z, F, G, H, v), \tag{61b}$$
$$H = h(z, F, G, H, v), \tag{61c}$$
$$0 = W(z, F, G, H, v) := \det\begin{pmatrix} 1-f_F & -f_G & -f_H \\ -g_F & 1-g_G & -g_H \\ -h_F & -h_G & 1-h_H \end{pmatrix}, \tag{61d}$$

where the functions f, g and h are given by equation (52). The required values $\tilde{\rho}'(1)$ and $\tilde{\rho}''(1)$ can then be obtained from (61) by implicit differentiation. For the sake of completeness we collect in the following lemma these somewhat lengthy results concerning the constants $\mu_{\mathcal{T}}^{[K]}$ and $\nu_{\mathcal{T}}^{[K]}$ appearing in Theorem 1 for arbitrary degree-weight generating functions $\varphi(t)$ satisfying Assumption 1.

Lemma 6. *The constants $\mu_{\mathcal{T}}^{[K]}$ and $\nu_{\mathcal{T}}^{[K]}$ appearing in Theorem 1 are given as follows:*

$$\mu_{\mathcal{T}}^{[K]} = -\frac{\tilde{\rho}'(1)}{\tilde{\rho}}, \qquad \nu_{\mathcal{T}}^{[K]} = \frac{\tilde{\rho}'(1)^2}{\tilde{\rho}^2} - \frac{\tilde{\rho}'(1)}{\tilde{\rho}} - \frac{\tilde{\rho}''(1)}{\tilde{\rho}},$$

where $\tilde{\rho}$, $\tilde{\rho}'(1)$ and $\tilde{\rho}''(1)$ are specified below and the constants $\tilde{\tau}$ and τ_1 appearing there are given by the minimal positive real solutions of the functional equations stated. It holds then with

$$\begin{aligned}
f(z,F,G,H,v) &= vz\varphi\big(2G+H\big),\\
g(z,F,G,H,v) &= z\Big(\varphi\big(2G+2F\big) - \varphi\big(2G+F\big)\Big),\\
h(z,F,G,H,v) &= z\varphi\big(2G+F\big),\\
W(z,F,G,H,v) &= \det\begin{pmatrix} 1-f_F(z,F,G,H,v) & -f_G(z,F,G,H,v) & -f_H(z,F,G,H,v) \\ -g_F(z,F,G,H,v) & 1-g_G(z,F,G,H,v) & -g_H(z,F,G,H,v) \\ -h_F(z,F,G,H,v) & -h_G(z,F,G,H,v) & 1-h_H(z,F,G,H,v) \end{pmatrix}
\end{aligned}$$

the following:

$$2\tilde{\tau}\varphi'(2\tilde{\tau}) = \varphi(2\tilde{\tau}), \qquad \tilde{\rho} = \frac{\tilde{\tau}}{\varphi(2\tilde{\tau})}, \qquad \tau_1 = \tilde{\rho}\varphi(2\tilde{\tau}-\tau_1), \quad \tau_2 = \tilde{\tau} - \tau_1, \quad \tau_3 = \tau_1,$$
$$\tilde{\rho}'(1) = -\frac{\det(\vec{d}_1, \vec{d}_2, \vec{x}_v)}{\det(\vec{d}_1, \vec{d}_2, \vec{x}_z)}, \qquad \tilde{\rho}''(1) = -\frac{\det(\vec{d}_1, \vec{d}_2, \vec{s})}{\det(\vec{d}_1, \vec{d}_2, \vec{x}_z)},$$

where we use the abbreviations

$$\vec{d}_1 = \begin{pmatrix} 1-f_F \\ -g_F \\ -h_F \end{pmatrix}, \quad \vec{d}_2 = \begin{pmatrix} f_G \\ 1-g_G \\ -h_G \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \quad \vec{w}_1 = \begin{pmatrix} 1-f_F \\ -g_F \\ W_F \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} -f_G \\ 1-g_G \\ W_G \end{pmatrix}, \quad \vec{w}_3 = \begin{pmatrix} -f_H \\ -g_H \\ W_H \end{pmatrix},$$
$$\vec{r} = \begin{pmatrix} f_z \\ g_z \\ W_z \end{pmatrix} \cdot \rho'(1) + \begin{pmatrix} f_v \\ g_v \\ W_v \end{pmatrix},$$
$$\tau_1'(1) = -\frac{\det(\vec{r}, \vec{w}_2, \vec{w}_3)}{\det(\vec{w}_1, \vec{w}_2, \vec{w}_3)}, \quad \tau_2'(1) = -\frac{\det(\vec{w}_1, \vec{r}, \vec{w}_3)}{\det(\vec{w}_1, \vec{w}_2, \vec{w}_3)}, \quad \tau_3'(1) = -\frac{\det(\vec{w}_1, \vec{w}_2, \vec{r})}{\det(\vec{w}_1, \vec{w}_2, \vec{w}_3)},$$
$$\begin{aligned}
\vec{s} = {}& \vec{x}_{zz}(\rho'(1))^2 + \vec{x}_{FF}(\tau_1'(1))^2 + \vec{x}_{GG}(\tau_2'(1))^2 + \vec{x}_{HH}(\tau_3'(1))^2 + \vec{x}_{vv} + 2\vec{x}_{zF}\rho'(1)\tau_1'(1)\\
&+ 2\vec{x}_{zG}\rho'(1)\tau_2'(1) + 2\vec{x}_{zH}\rho'(1)\tau_3'(1) + 2\vec{x}_{zv}\rho'(1) + 2\vec{x}_{FG}\tau_1'(1)\tau_2'(1) + 2\vec{x}_{FH}\tau_1'(1)\tau_3'(1)\\
&+ 2\vec{x}_{Fv}\tau_1'(1) + 2\vec{x}_{GH}\tau_2'(1)\tau_3'(1) + 2\vec{x}_{Gv}\tau_2'(1) + 2\vec{x}_{Hv}\tau_3'(1),
\end{aligned}$$

and where all functions and vectors of functions appearing, let us denote them by $\psi = \psi(z, F, G, H, v)$ *and* $\vec{\psi} = (\psi_1(z, F, G, H, v), \psi_2(z, F, G, H, v), \psi_3(z, F, G, H, v))$*, are evaluated at* $z = \tilde{\rho}$*,* $F = \tau_1$*,* $G = \tau_2$*,* $H = \tau_3$ *and* $v = 1$*.*

The results of Theorem 2 concerning the size of the kernel K_n follow again by establishing the computations of Lemma 6 for $\varphi(t) = (1+t)^2$: binary trees, $\varphi(t) = \frac{1}{1-t}$: planted plane trees and $\varphi(t) = e^t$: rooted labelled trees.

7. Proof of the existence of type I and type II trees

7.1. **Node-independence number.** We want to show that for every simply generated tree family $\mathcal{T}$ satisfying Assumption 1 there exists a constant n_0, such that the total weight of type $\mathrm{I}^{[N]}$ trees and the total weight of type $\mathrm{II}^{[N]}$ trees are positive, $F_n > 0$ and $G_n > 0$, for all $n > n_0$. We do this by constructing for all n large enough a tree $T \in \mathcal{T}$ of size n, which has a positive weight $w(T)$.

Due to condition (iii) of Assumption 1 there exist positive integers $0 < a < b$, with $\gcd(a, b) = 1$, such that the multiplicative weights of nodes of out-degree a and b: φ_a and φ_b, given by the degree-weight sequence $(\varphi_k)_{k\in\mathbb{N}}$, satisfy $\varphi_a > 0$ and $\varphi_b > 0$. By definition it also holds that $\varphi_0 > 0$. We are considering from now on only trees T, whose nodes $v \in T$ have out-degree $d(v) \in \{0, a, b\}$. When counting the edges of such a tree T we obtain

$$|T| = n = 1 + \sum_{v\in T} d(v) = 1 + ka + \ell b,$$

where k gives the number of nodes of out-degree a and ℓ the number of nodes of out-degree b in T. Of course, the remaining $n - k - \ell$ nodes have out-degree 0 and are thus the leaves of T. The weight $w(T)$ is then given by $w(T) = \varphi_0^{n-k-\ell}\varphi_a^k\varphi_b^\ell > 0$.

We consider now the equation

$$n - 1 = ka + \ell b.$$

Since $\gcd(a, b) = 1$ it follows that a is a generating element of the group $\langle \mathbb{Z}_b, +\rangle$, which implies that for all integers n there exists a non-negative integer k, with $0 \le k \le b - 1$, such that

$$n - 1 - ka \equiv 0 \pmod b.$$

If n is large enough, $n - 1 \ge ab$, then it follows further that there exists a positive integer $\ell \ge 1$ such that

$$n - 1 = ka + \ell b.$$

This leads now to the well-known fact that for $n > n_0 := ab$ the total weight T_n of trees is positive, since we can construct a tree T with $k \ge 0$ nodes of out-degree a, $\ell \ge 1$ nodes of out-degree b and $n - k - \ell$ nodes of out-degree 0, which has a positive weight.

Moreover, if $n - 1 \ge 3ab$ and thus $n - 1 - 2ab \ge ab$ it follows that there exist $k_1 \ge 0$ and $\ell_1 \ge 1$ such that $n - 1 - 2ab = k_1a + \ell_1b$, which implies that

$$n - 1 = (k_1 + b)a + (\ell_1 + a)b,$$

and thus that for $n > n_0 := 3ab$ we can construct a tree T with $k \ge b$ nodes of out-degree a, $\ell \ge a+1$ nodes of out-degree b and $n - k - \ell$ nodes of out-degree 0.

In order to prove that F_n and G_n are positive for n large enough we show how to construct a type $\mathrm{I}^{[N]}$ tree and a type $\mathrm{II}^{[N]}$ tree of size n with k nodes of out-degree a, ℓ nodes of out-degree b and $n - k - \ell$ nodes of out-degree 0. Since the given construction of type $\mathrm{II}^{[N]}$ trees requires $\ell \ge 1$ we can guarantee the existence of such trees for $n > n_0 = ab$, whereas the construction of type $\mathrm{I}^{[N]}$ trees requires $\ell \ge 2$ and $k \ge b - 1$ and the existence of such trees is guaranteed for $n > n_0 = 3ab$.

- A tree T of type $\mathrm{II}^{[N]}$ can be built by a root of out-degree b, which has $b-1$ branches $B_2, \dots, B_b$ that are leaves, and one branch B_1, which is a chain of $\ell - 1$ nodes of out-degree b attached to a chain of k nodes of out-degree a, which are filled up with leaves.

- A tree T of type $\mathrm{I}^{[N]}$ can be built by a root of out-degree b, which has $b-1$ branches $B_2, \dots, B_b$ that consist of a root of out-degree a with a children that are leaves, and one branch B_1, which is a type $\mathrm{II}^{[N]}$ tree constructed as described previously with $\ell - 1 \geq 1$ nodes of out-degree b and $k - b + 1 \geq 0$ nodes of out-degree a, filled up with leaves.

These constructions are visualized in Figure 4-5.

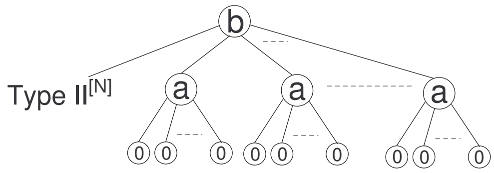

FIGURE 4. A tree T of type $\mathrm{I}^{[N]}$.

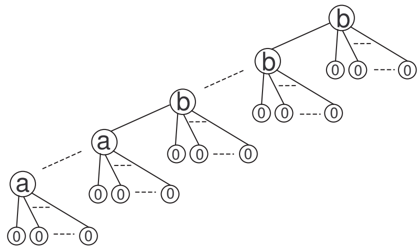

FIGURE 5. A tree T of type $\mathrm{II}^{[N]}$.

7.2. **Path node-covering number.** We show now that for every simply generated tree family $\mathcal{T}$ satisfying Assumption 1 there exists a constant n_0, such that the total weight of type $\mathrm{I}^{[P]}$ trees and the total weight of type $\mathrm{II}^{[P]}$ trees are positive, $F_n > 0$ and $G_n > 0$, for all $n > n_0$ by constructing for all n large enough a tree $T \in \mathcal{T}$ of size n, which has a positive weight $w(T)$.

To do this we follow the proof in Subsection 7.1 and choose $0 < a < b$ with $\gcd(a, b) = 1$, such that $\varphi_a > 0$ and $\varphi_b > 0$. First we treat the special case $b = 2$, which implies $a = 1$. Here, for $n \geq 1$, we can construct a type $\mathrm{II}^{[P]}$ tree T with $k = n - 1$ nodes of out-degree 1, 0 nodes of out-degree 2 and exactly one node of out-degree 0 by building a chain of the k nodes of out-degree 1 and attaching the leaf. Also, for $n \geq 3$, we can construct a type $\mathrm{I}^{[P]}$ tree T with $k = n - 3$ nodes of out-degree 1, one node of out-degree 2 and two nodes of out-degree 0 in the following way: the root of T consists of the node with out-degree 2, one branch is given by a leaf and the other branch is a chain of k nodes of out-degree 1 followed by a leaf.

For the general case $b \geq 3$ we use the following construction, which leads to a type $\mathrm{I}^{[P]}$ tree and a type $\mathrm{II}^{[P]}$ tree of size n with k nodes of out-degree a, ℓ nodes of out-degree b and $n - k - \ell$ nodes of out-degree 0. Since the given construction of type $\mathrm{I}^{[P]}$ trees requires $\ell \geq 1$ we can guarantee the existence of such trees for $n > n_0 = ab$, whereas the construction of type $\mathrm{II}^{[P]}$ trees requires $\ell \geq a$ and $k \geq 1$ and the existence of such trees is guaranteed for $n > n_0 = 3ab$.

- A tree T of type $\mathrm{I}^{[P]}$ can be built by a root of out-degree b, which has $b-1$ branches $B_2, \dots, B_b$ that are leaves, and one branch B_1, which is a chain of $\ell - 1$ nodes of out-degree b attached to a chain of k nodes of out-degree a, which are filled up with leaves.
- A tree T of type $\mathrm{II}^{[P]}$ can be built by a root of out-degree a, which has $a - 1$ branches $B_2, \dots, B_a$ that consist of a root of out-degree b with b children that are leaves, and one branch B_1, which is a type $\mathrm{I}^{[P]}$ tree constructed as described previously with $\ell - a + 1 \geq 1$ nodes of out-degree b and $k - 1 \geq 0$ nodes of out-degree a, filled up with leaves.

These constructions are visualized in Figure 6-7.

7.3. **Size of the kernel.** We show here that for every directed simply generated tree family $\tilde{\mathcal{T}}$ satisfying Assumption 1 there exists a constant n_0, such that the total weight of type $\mathrm{I}^{[K]}$ trees and the total weight of type $\mathrm{II}^{[K]}$ trees are positive, $F_n > 0$ and $G_n > 0$, for all $n > n_0$ by constructing for all n large enough a tree $\tilde{T} \in \tilde{\mathcal{T}}$ of size n, which has a positive weight $w(\tilde{T})$.

We follow the proof in Subsection 7.1 and choose $0 < a < b$ with $\gcd(a, b) = 1$, such that $\varphi_a > 0$ and $\varphi_b > 0$. The following construction generates a type $\mathrm{I}^{[K]}$ tree and a type $\mathrm{II}^{[K]}$ tree of size n with k nodes of out-degree a, ℓ nodes of out-degree b and $n - k - \ell$ nodes of out-degree 0. Since the construction requires $\ell \geq 1$ we can guarantee the existence of such trees for $n > n_0 = ab$.

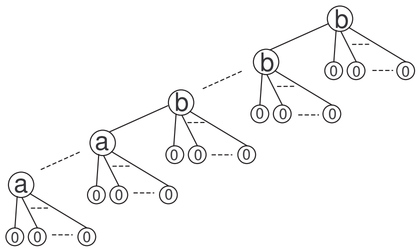

FIGURE 6. A tree T of type $\mathrm{I}^{[P]}$.

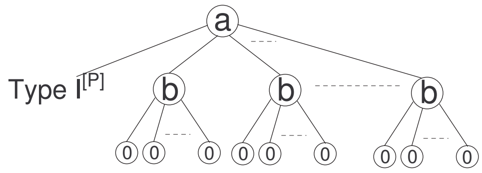

FIGURE 7. A tree T of type $\mathrm{II}^{[P]}$.

- A directed tree $\tilde{T}$ of type $\mathrm{II}^{[K]}$ can be built in the following way. First we construct an undirected tree T by a root r of out-degree b, which has $b-1$ branches $B_2, \ldots, B_b$ that are leaves, and one branch B_1, which is a chain of $\ell - 1$ nodes of out-degree b attached to a chain of k nodes of out-degree a, which are filled up with leaves. Then $\tilde{T}$ is constructed from T by orientating all edges, such that every edge leads away from the root r.
- A directed tree $\tilde{T}$ of type $\mathrm{I}^{[K]}$ can be built in the following way. First we construct an undirected tree T by a root r of out-degree b, which has $b-1$ branches $B_2, \ldots, B_b$ that are leaves, and one branch B_1, which is a chain of $\ell - 1$ nodes of out-degree b attached to a chain of k nodes of out-degree a, which are filled up with leaves. Then $\tilde{T}$ is constructed from T by orientating all edges, such that every edge is directed to the root r.

These constructions are visualized in Figure 8-9.

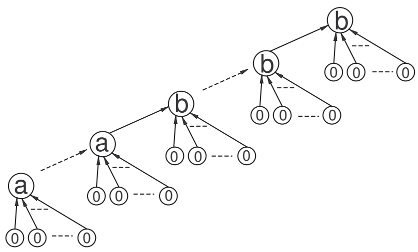

FIGURE 8. A tree T of type $\mathrm{I}^{[K]}$.

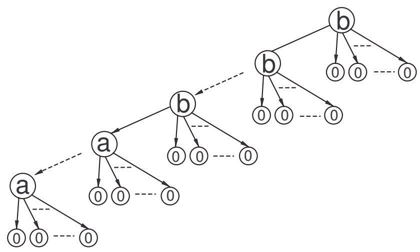

FIGURE 9. A tree T of type $\mathrm{II}^{[K]}$.

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