Scaling limits of random spanning trees

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Random spanning trees

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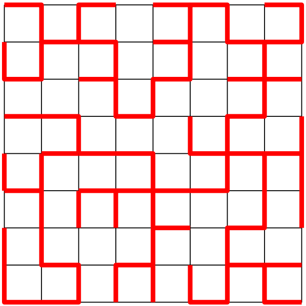
A **spanning tree of G** is a connected subset of edges of G touching all vertices of G and containing no cycles.



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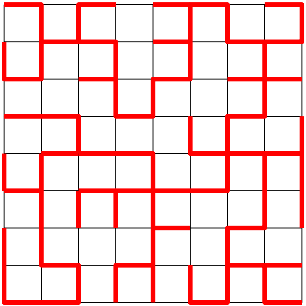

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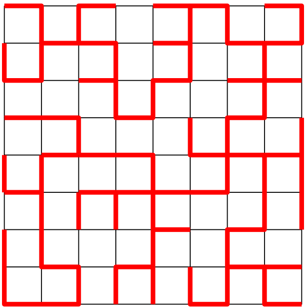

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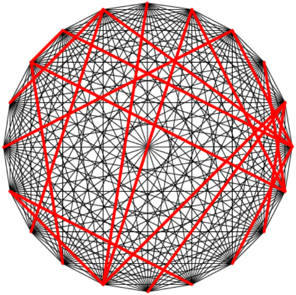
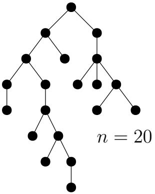
Popular model in statistical physics:
nice sampling algorithms/connections to electrical networks/enumeration techniques.

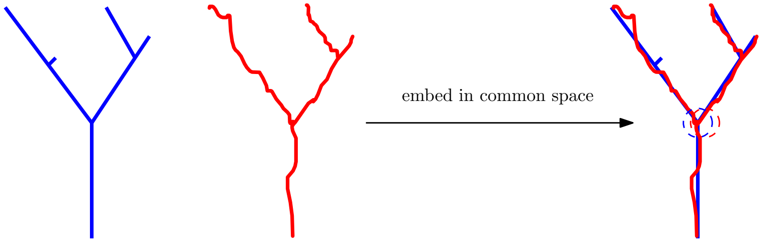
Classical example: UST of complete graph

1. UST(K_n) is a uniformly chosen labelled tree on n vertices.
2. Therefore $\left(\mathrm{UST}(K_n), \frac{1}{\sqrt{n}} d_n, \mu_n\right) \xrightarrow{(d)} \mathrm{CRT}$ as $n \to \infty$ (Aldous, Le Gall), wrt GHP topology.





GHP topology



Gromov-Hausdorff-Prohorov topology

1. Topology for *metric-measure spaces.*
2. The *Hausdorff distance* d_H between two sets $A, A' \subset E$ is defined as

$$d_H^E(A, A') = \max \left\{ \sup_{a \in A} d(a, A'), \sup_{a' \in A'} d(a', A) \right\}.$$

3. We define the *Prohorov distance* $d_P^E(\mu, \nu)$ between μ and ν by

$$\inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \; \forall \text{ closed } A \subset E\}.$$

4. For two metric-measure spaces (E, d, μ) and (E', d', μ') we define the *Gromov-Hausdorff-Prohorov* distance between them as

$$d_{GHP}(E, E') = \inf\{d_H^F(\varphi(E), \varphi'(E')) \vee d_P^F(\mu \circ \varphi^{-1}, \mu' \circ \varphi'^{-1})\},$$

where the infimum is taken over all isometric embeddings $\varphi : E \to F$, $\varphi' : E' \to F$ into some metric space (F, δ).

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- Rescaled diameters converge: $\frac{\mathrm{Diam}(\mathcal{T}_n)}{n^{1/2}} \xrightarrow{(d)} \mathrm{Diam}(\mathrm{CRT})$.

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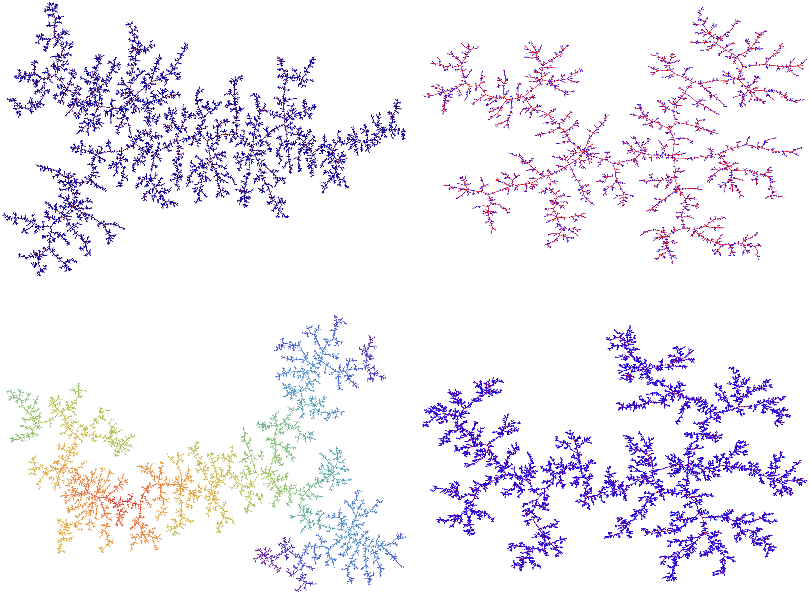
- Rescaled SRW converges wrt uniform topology on path space: There exists a probability space on which the GHP convergence is almost sure, upon which $P_n^{(O_n)}\left(\left(\frac{1}{\sqrt{n}}X_n(n^{\frac{3}{2}}t)\right)_{t\geq 0} \in \cdot\right) \to P^{(O)}\left((B_t)_{t\geq 0} \in \cdot\right)$ almost surely too.

The CRT



Pictures by Igor Kortchemski and Laurent Ménard.

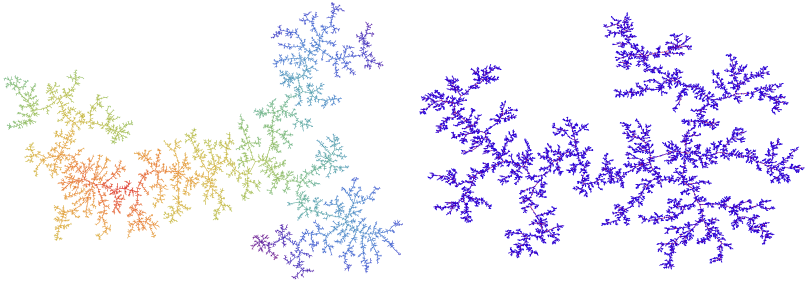
First aim today: universality of the CRT as the scaling limit

Let G_n be the torus of side length $\lfloor n^{1/d} \rfloor$ in dimension $d \geq 5$.

Theorem (A., Nachmias, Shalev 2024)

There exists $c_d \in (0, \infty)$ such that

$\left(UST(G_n), \frac{c_d}{\sqrt{n}} d_n, \mu_n \right) \xrightarrow{(d)} CRT$ *as $n \to \infty$, wrt GHP topology.*



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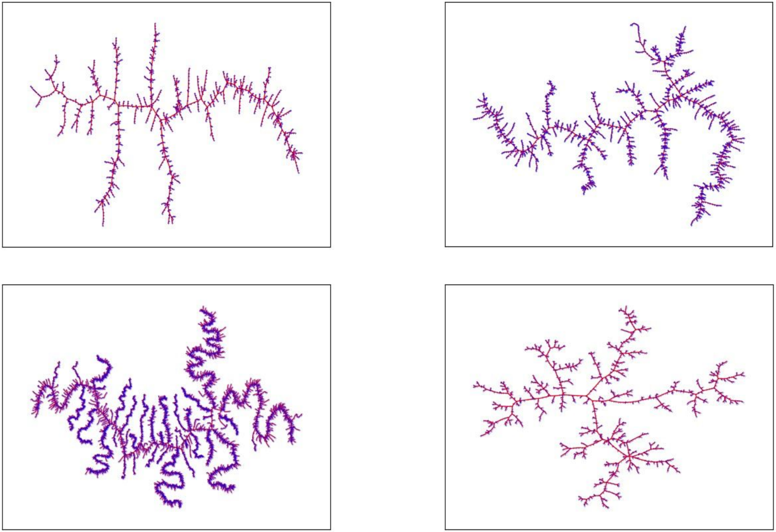
Second aim today: random choice spanning trees

$$\left(\mathrm{CST}_k(K_n), n^{-\frac{k}{k+1}} d_n, \mu_n\right) \xrightarrow[GHP]{(d)} (\mathcal{T}_k, d_{\mathcal{T}_k}, \mu_{\mathcal{T}_k}) \text{ as } n \to \infty.$$



Pictures by Matan Shalev.