Equiprojective polytopes in high dimension

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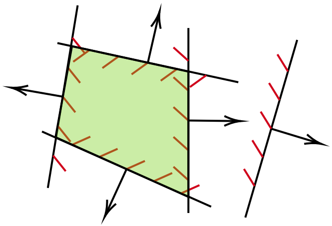
A bit of vocabulary on polyhedral geometry

- A polyhedron is the intersection of finitely many half-spaces of form

$$\{x, \langle x, \eta \rangle \leq \alpha\}$$

- A polytope is a bounded polyhedron, equivalently it is the convex hull of finitely many points.
- $\{x, \langle x, \eta_i \rangle = \alpha_i\} = \{x, \langle x, \eta_i \rangle \leq \alpha_i\} \cap \{x, \langle x, -\eta_i \rangle \leq -\alpha_i\} \rightarrow$ a new smaller polyhedron: a face
- A face of dimension k is called a k-face. 0-faces = vertices; 1-faces = edges; $(d-1)$-faces are facets



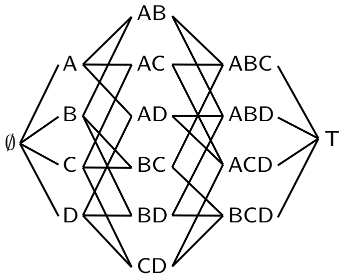
Definition (equiprojectivity)

On an example.

- Any *k*-gon-based prism is $(k+2)$-equiprojective
- The face lattice of a polyhedron is the set of all its faces ordered by inclusion.
- Two polyhedra have the same combinatorial type if they have "the same" face lattice.
- Let A and B be two sets, their Minkowski sum $A+B$ is defined as

$$\{a+b, a \in A, b \in B\}$$

- Zonotopes are Minkowski sums of finitely many segments. They are k-equiprojective.



Equiprojectivity in any dimension

→ If "degeneracy" in higher dimension, it is still because of (at least) one 2-dimensional face!

- (i) : "a 2-face is degenerating on the boundary"
- (ii) : "a 2-face is degenerating **anywhere** in the projection"

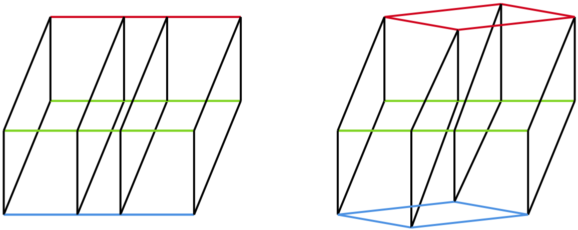


Figure: The two definitions are not equivalent

Good news is, zonotopes are still equiprojective!

Lemma (Hasan–Lubiw, 2008)

It is possible to modify continuously the direction of projection so that we can go from any admissible projection to any other while making sure that no two non-parallel 2-face are simultaneously degenerating.

The Grassmannian $\mathscr{G}(k,d)$ is the set of all k-dimensional spaces of $\mathbb{R}^d$

Definition (connectedness in the Grassmannian)

Let $V_0, V_1 \subset \mathbb{R}^d$ two vector spaces of dimension k. We say that $\gamma : t \in [0,1] \mapsto \gamma(t) \in (\mathbb{R}^d)^k$ is a *continuous arc (or path) between* V_0 *and* V_1 if:

1. for all $t \in [0,1]$, $\gamma(t)$ is a free family
2. $\mathrm{Vect}(\gamma(0)) = V_0$ and $\mathrm{Vect}(\gamma(1)) = V_1$
3. γ is continuous except in a finite number of points $t_1, \ldots, t_n$ such that $0 < t_1 < \ldots < t_n < 1$ and admits limits above and below at these points that satisfy $\mathrm{Vect}(\gamma(t_i^-)) = \mathrm{Vect}(\gamma(t_i^+)) = \mathrm{Vect}(\gamma(t_i))$

We will say that *we can continuously go from* V_0 *to* V_1 and we will denote $V_t = \mathrm{Vect}(\gamma(t))$.

Elements of proof

Lemma (Isometry Lemma)

Up to a isometry of $\mathbb{R}^d$, we can suppose that $\mathrm{Vect}(e_1, e_d)$ is admissible.

The proof then relies on a concatenation of perturbations and paths of the likes of:

$$t \in [0,1] \mapsto (u_1 - tv, u_2, \ldots, u_{d-2})$$

$$t \in [0,1] \mapsto \begin{pmatrix} 0 & \ldots & 0 \\ 1 & & (0) \\ & \ddots & \\ (0) & & 1 \\ (1-t)x_2 & \ldots & (1-t)x_{d-1} \end{pmatrix}$$

Theorem (Walking in the Grassmannian)

After adapting the vocabulary, the previous lemma can be generalised to higher dimensions.

$\rightarrow$ To study equiprojectivity, it is enough to focus on what is happening when "one" 2-face is degenerating.

A first characterisation

Definition (edge-facet, compensation)

Construction in dimension 3 on an example.

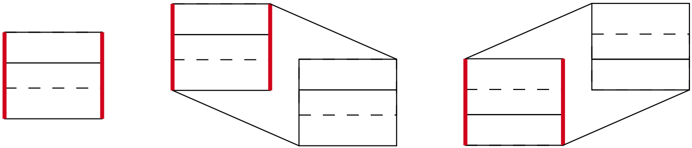


Figure: On the left: degenerated projection of a cube ; middle and on the right : degenerated projections of an hypercube

$\rightarrow$ Notion of *edge-2-faces*.

Theorem

A polytope is equiprojective if and only if its edges-2-faces can be partitioned into compensating pairs.

First implication ($\Longleftarrow$) : “easy” with the walk in the Grassmannian.
Second implication ($\Longrightarrow$) : by contraposition, building two projections of different size.

A second characterisation

- Normal cone, definition on an example.
- The normal fan of a polyhedron is the given of all of its normal cones.

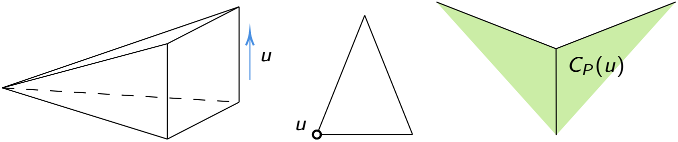


Figure: On the left, an example of an edge direction; middle, the projection along u; on the right, the aggregated cone at u.

Theorem (Buffière–Pournin, 2024)

A 3-polytope P is equiprojective if and only if for all edge direction u:

1. either the aggregated cone $C_P(u)$ is equal to $u^\perp$ or
2. the relative interior of $-C_P(u)$ is equal to $u^\perp \backslash C_P(u)$.

- The proof relies on the Hasan–Lubiw characterisation

Not (yet) established in higher dimensions!

Theorem (Buffière-Pournin, 2024)

If P is an equiprojective polytope, let us denote $\kappa(P)$ its constant of equiprojectivity. If P, Q and $P+Q$ are equiprojective polytopes, then

$$\kappa(P+Q) = \kappa(P) + \kappa(Q) - \lambda(P,Q)$$

Where $\lambda(P,Q)$ depends on the number of common edge directions of P and Q.

Corollary

If P and Q are equiprojective and do not share any edge directions, then $P+Q$ is also equiprojective and

$$\kappa(P+Q) = \kappa(P) + \kappa(Q)$$

$\rightarrow$ Application: building many k-equiprojective polytope when k is odd.

What's next?

- Projecting on higher dimensional spaces
  - What is the quantity we want to preserve?
  - Is it really interesting?

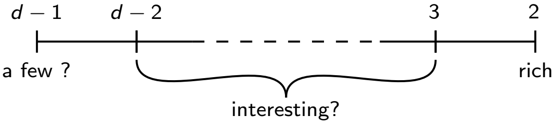


Figure: Projecting a polytope onto different spaces (speculations)

- How about unbounded polyhedra?

Thank you for your attention!