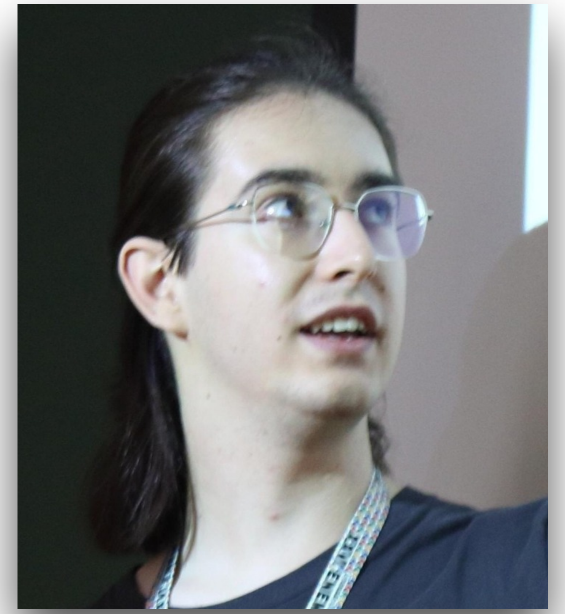
On records of trees and forests

Mercedes Rosas
Universidad de Sevilla

Enrica Ducci

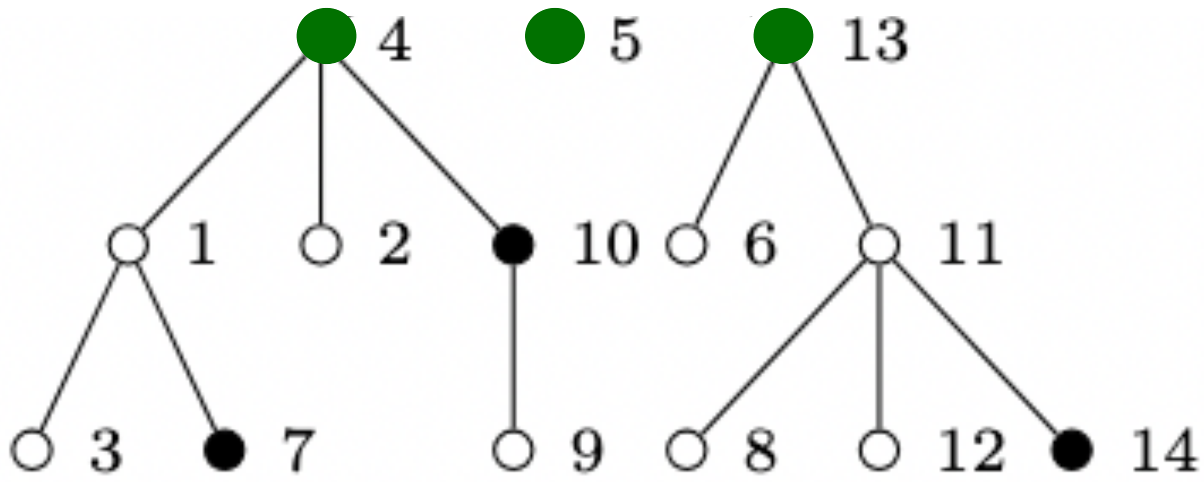
Adrián Lillo



Pablo Puerto

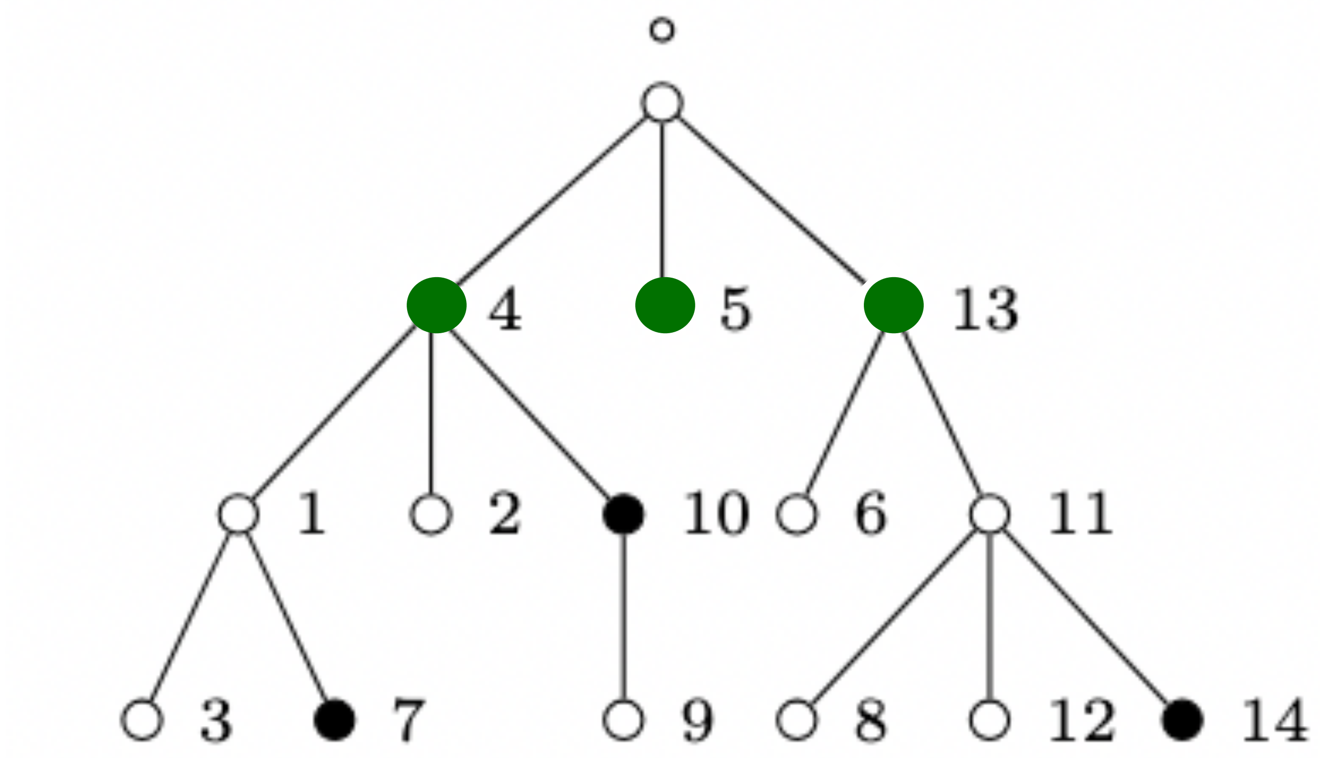
Stefan Trandafir

Records of trees and forests



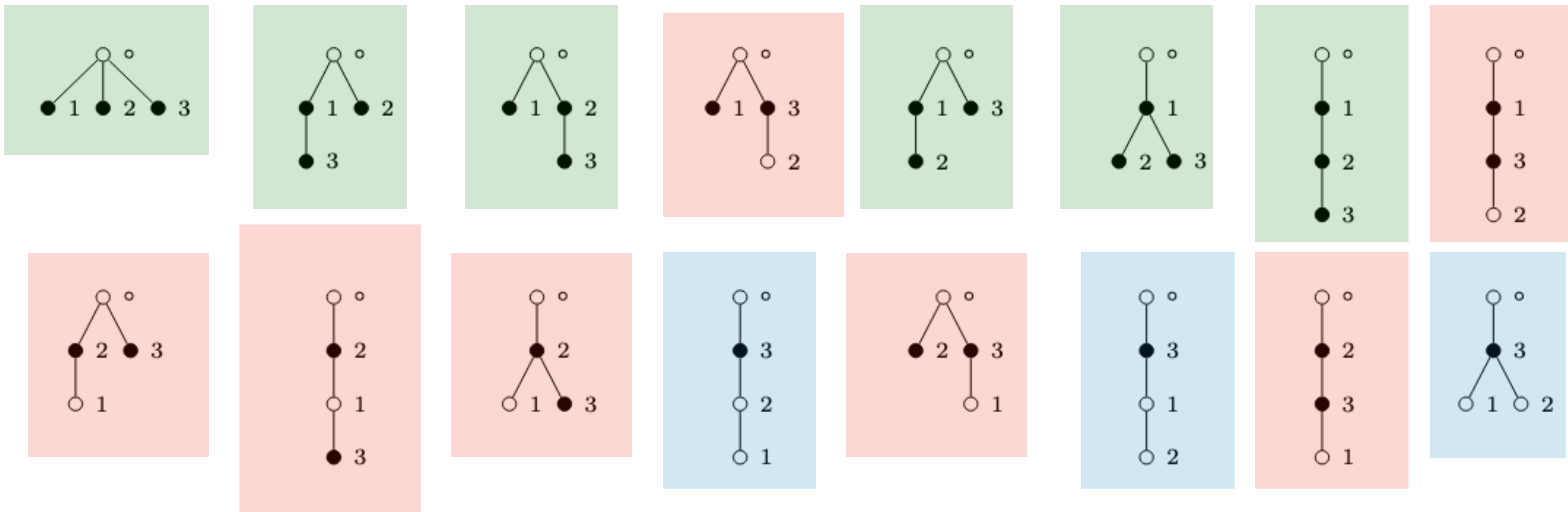
A node v of a rooted tree/forest is a record if its label is the largest along the unique path from v to a root.

Records of trees and forests



A node v of a rooted tree/forest is a record if its label is the largest along the unique path from v to a root.

The forest record numbers

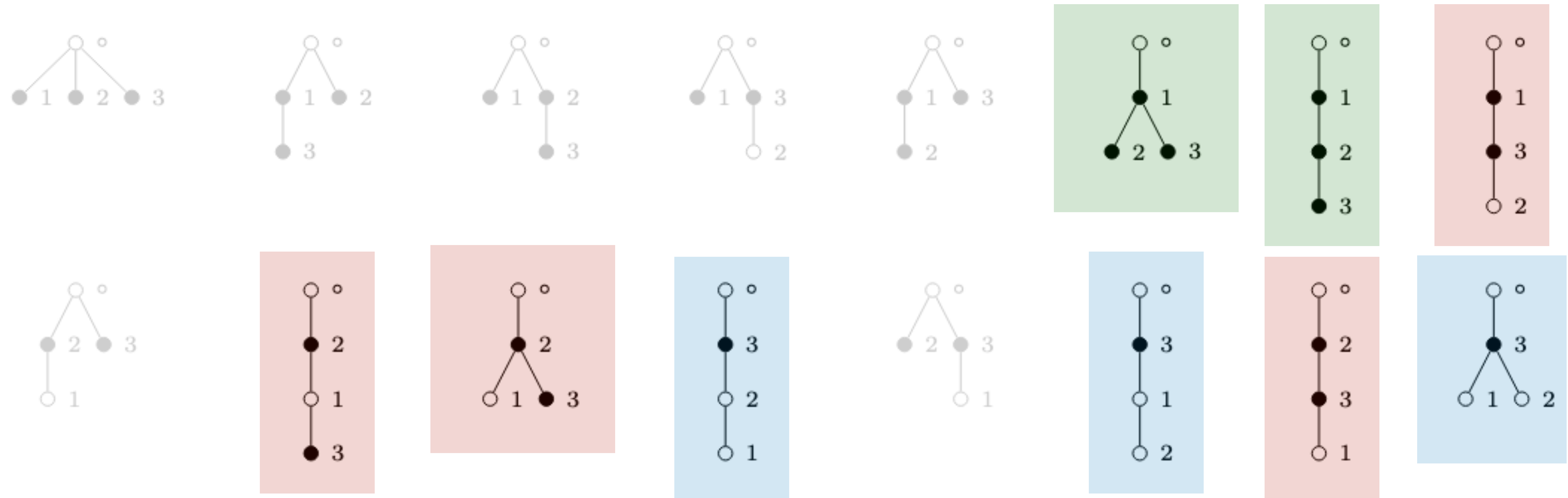


$$R(3,1) = 3,\ R(3,2) = 7,\ and\ R(3,3) = 6.$$

R(n,n) = # increasing trees

R(n,1) # unrooted trees

The tree record numbers



$$R_\bullet(3,1) = 3,\ R_\bullet(3,2) = 4,\ \textit{and}\ R_\bullet(3,3) = 2.$$

Generating functions:

$$\mathcal{T}(z) = \sum_{n\geq 1} n^{n-1} \frac{z^n}{n!},$$ *Cayley tree function.*

$$\mathcal{R}_\bullet(z,t) = \sum_{n,k\geq 0} R_\bullet(n,k)\frac{z^n}{n!}t^k,$$ *tree record function*

$$\mathcal{R}(z,t) = \sum_{n,k\geq 0} R(n,k)\frac{z^n}{n!}t^k,$$ *forest record function*

Generating functions:

$$R_\bullet(3,1) = 3,\ R_\bullet(3,2) = 4,\ and\ R_\bullet(3,3) = 2.$$

$$\mathcal{R}_\bullet(z,t) = tz + \left(t + t^2\right)\frac{z^2}{2!} + \left(3t + 4t^2 + 2t^3\right)\frac{z^3}{3!} + \left(16t + 24t^2 + 18t^3 + 6t^4\right)\frac{z^4}{4!} + \left(125t + 200t^2 + 180t^3 + 96t^4 + 24t^5\right)\frac{z^5}{5!} + \cdots$$

$$R(3,1) = 3,\ R(3,2) = 7,\ and\ R(3,3) = 6.$$

$$\mathcal{R}(z,t) = 1 + tz + \left(t + 2t^2\right)\frac{z^2}{2!} + \left(3t + 7t^2 + 6t^3\right)\frac{z^3}{3!} + \left(16t + 39t^2 + 46t^3 + 24t^4\right)\frac{z^4}{4!} + \left(125t + 310t^2 + 415t^3 + 326t^4 + 120t^5\right)\frac{z^5}{5!} + \cdots$$

exponential formula

$$\mathcal{T}(z) = z \exp\left(\mathcal{T}(z)\right).$$

$$\mathcal{R}(z,t) = \exp(\mathcal{R}_{\bullet}(z,t)).$$

arXiv > math > arXiv:2510.12536

Mathematics > Combinatorics

[Submitted on 14 Oct 2025]

On the enumeration of records of rooted trees and rooted forests

Adrián Lillo, Mercedes Rosas, Stefan Trandafir

we find expressions for the

Record generating functions

in terms of the

Cayley tree function.

arXiv > math > arXiv:2510.12536

Mathematics > Combinatorics

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On the enumeration of records of rooted trees and rooted forests

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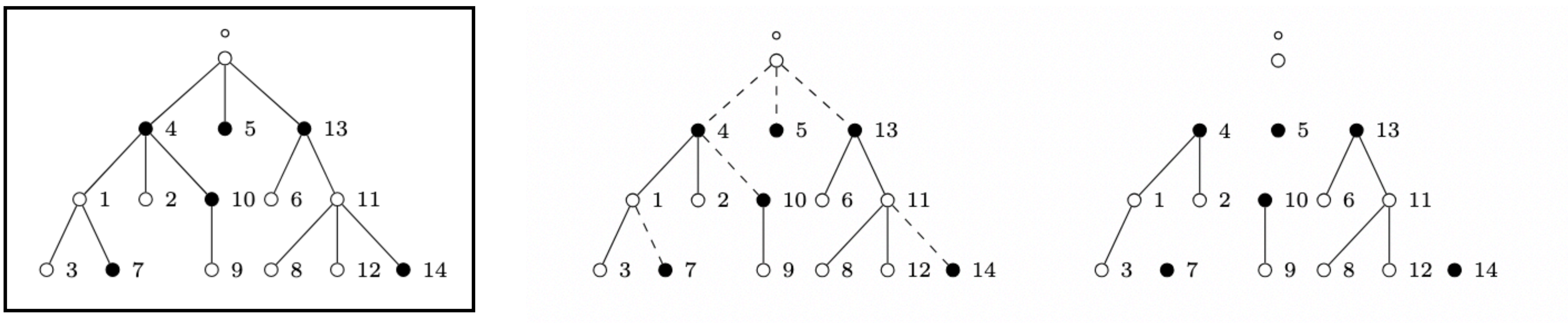
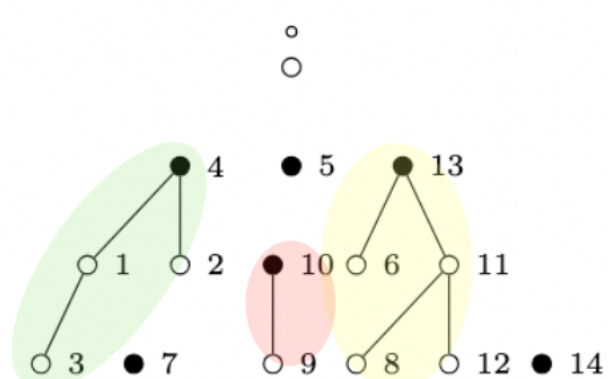
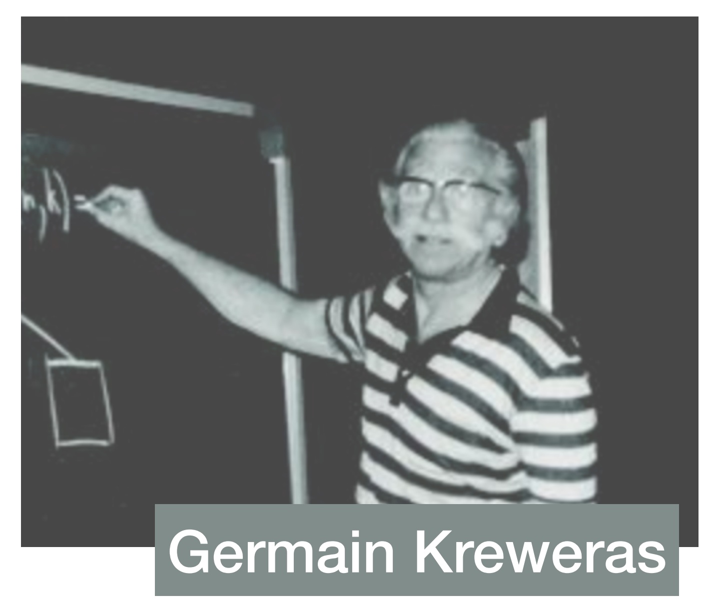


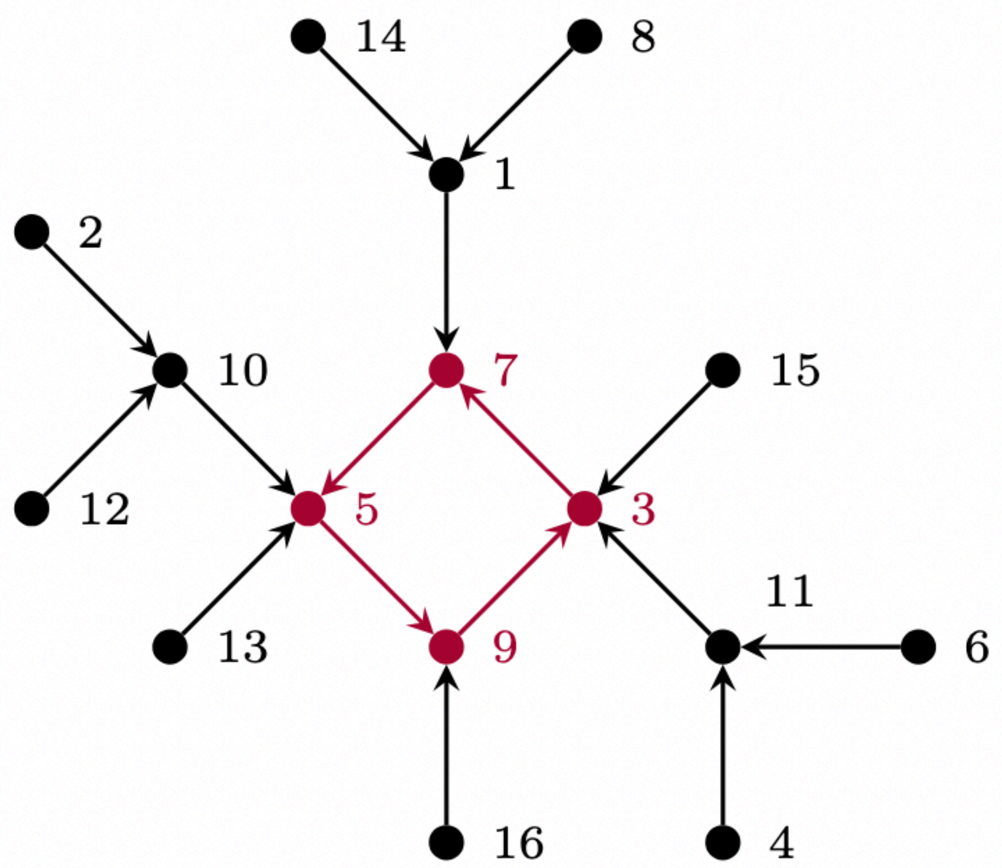




I

Records of trees and connected endofunctions

Girth of connected endofunctions



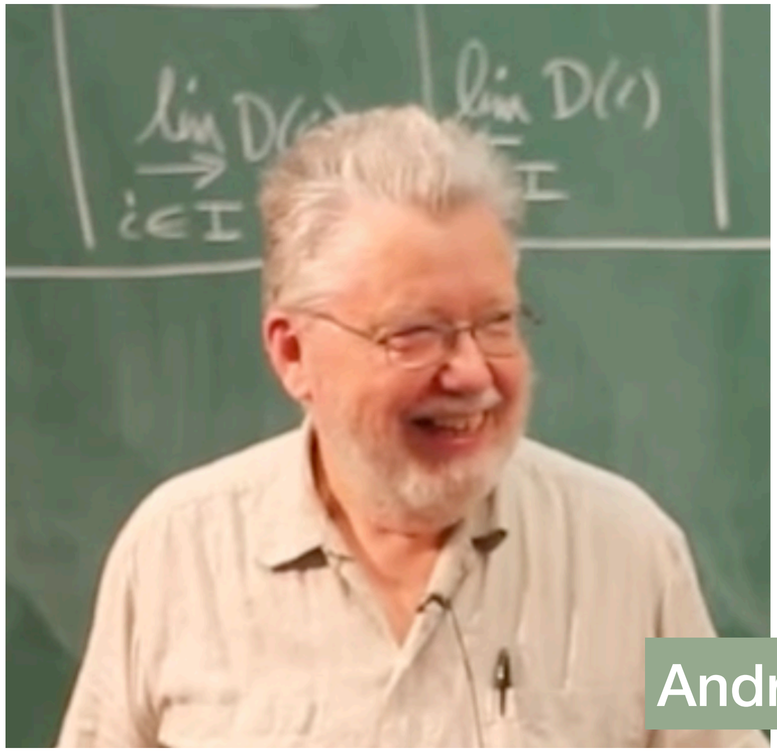
Girth 4

Cycles of trees

Bijection 1

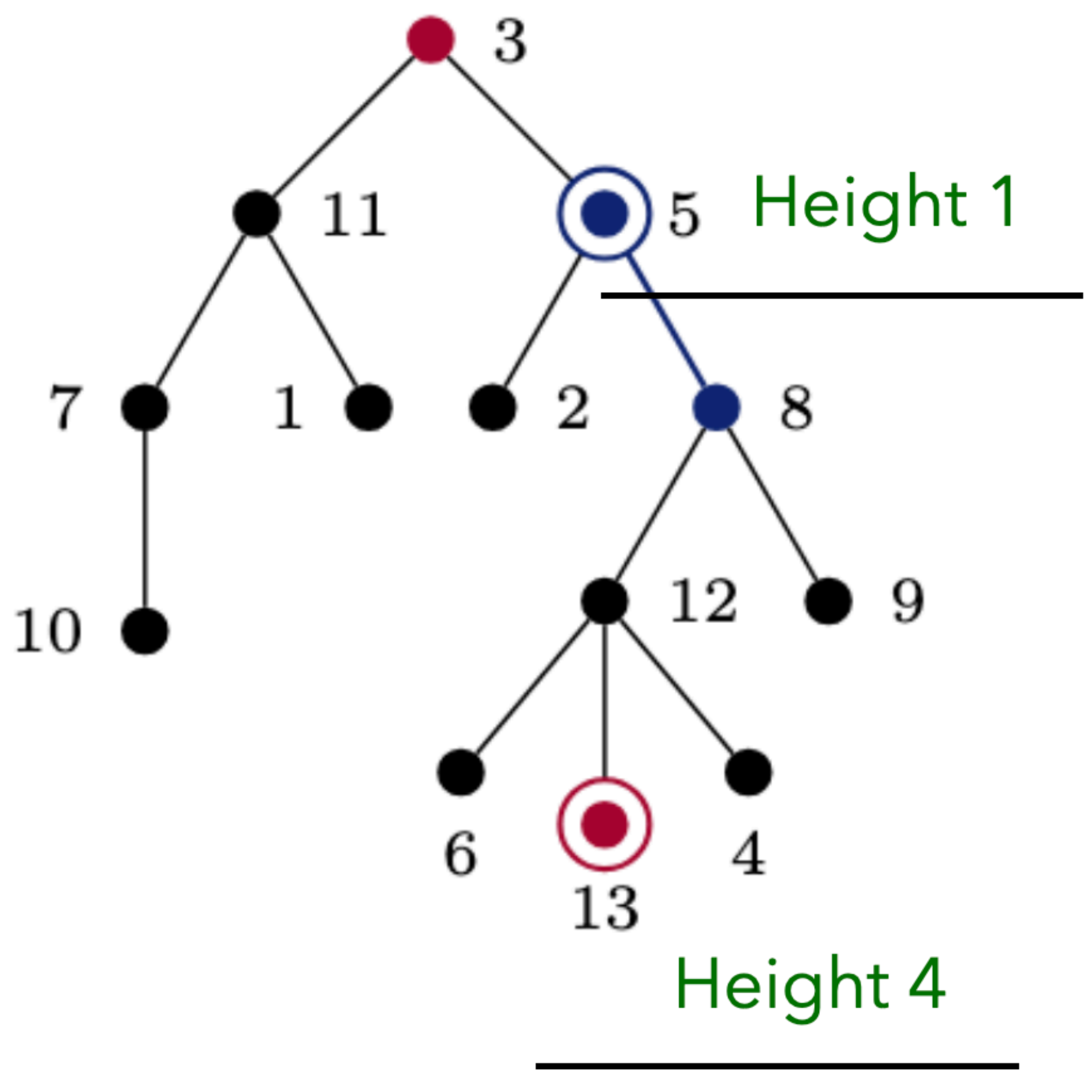
Rooted trees of *order n* with a distinguished record at height k-1 ⟷ Connected endofunction *on [n]* of girth k

Inspired on

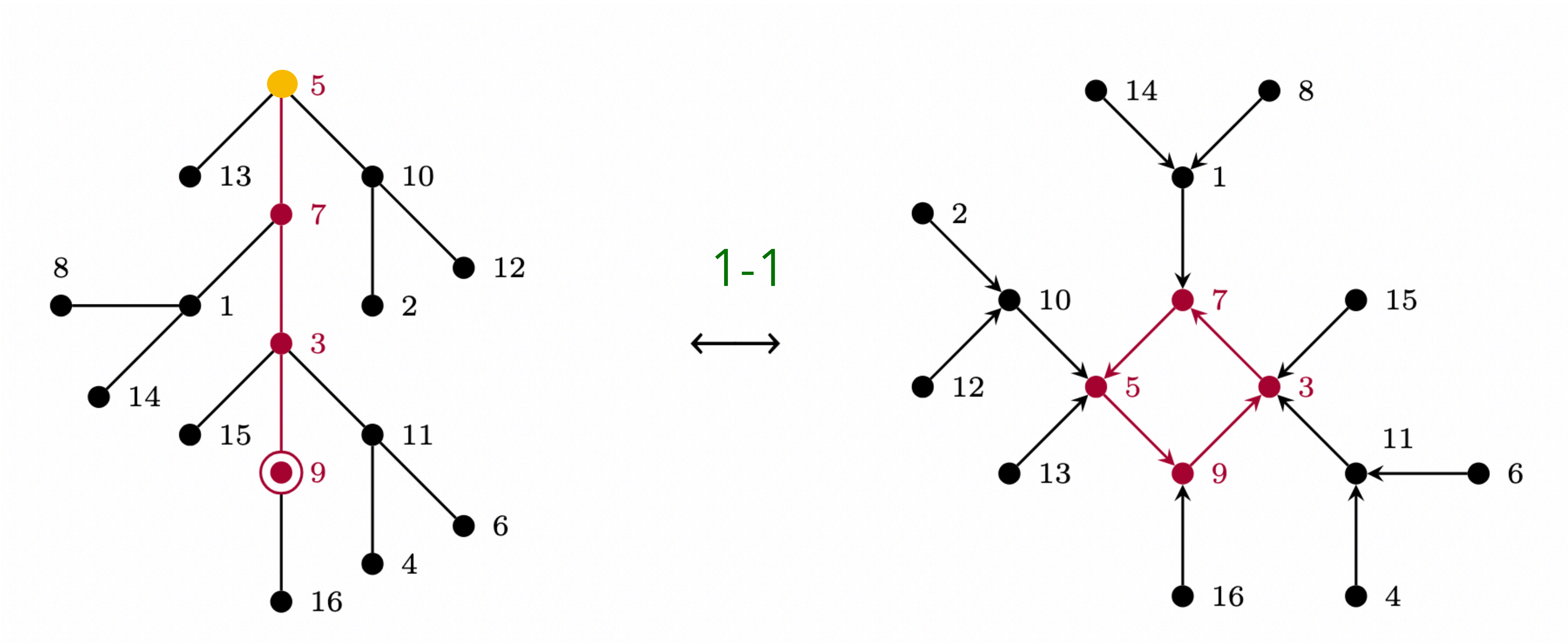


André Joyal

The height of a record



Rooted trees of order n
with a distinguished record at height k-1



Connected endofunction on [n] of girth k

The number of rooted trees of order n with a distinguished record at height $k-1$ coincides with the number of connected endofunctions on $[n]$ of girth k.

https://oeis.org/A001865

$$C(z,t) = \sum_{(T,r)} \frac{z^{|T|} t^{ht(r)-1}}{|T|!}$$

$$C(z,t) = \log\left(\frac{1}{1 - t\,\mathcal{T}(z)}\right)$$

the total number of records in all rooted trees with n nodes is equal to the number of connected endofunctions on $[n]$

https://oeis.org/A001865

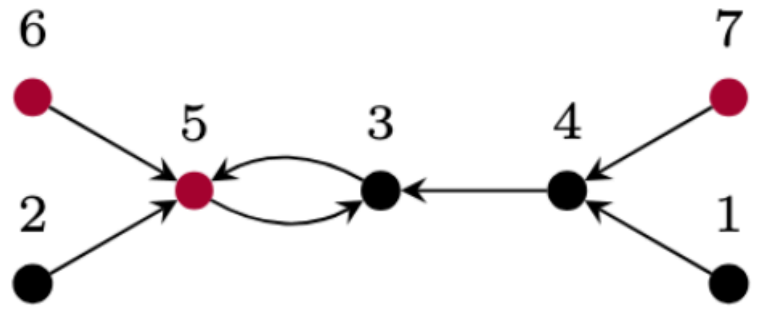
II

Girths and records of endofunctions

Record of a connected endofunction

$$f : [n] \to [n]$$

$$i \geq f^k(i) \text{ for all } k$$



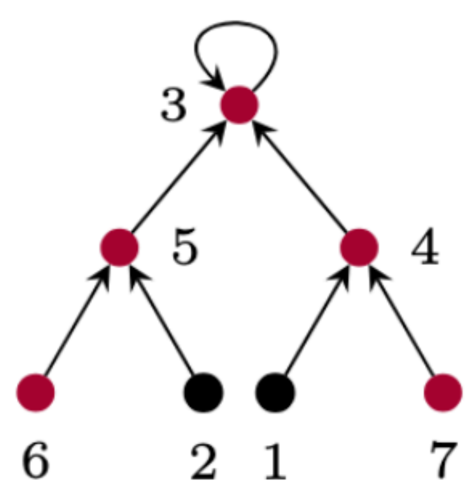
Bijection 2

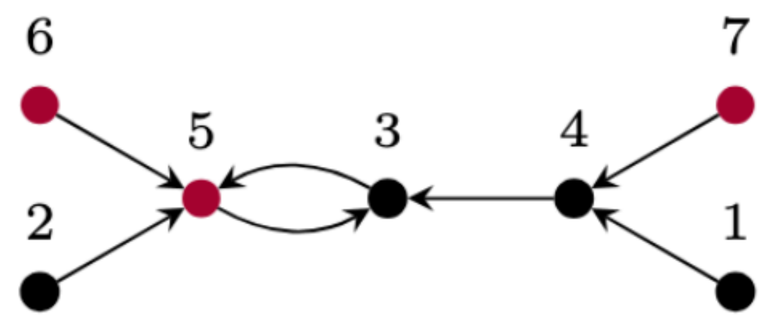
connected endofunctions on $[n]$ of girth m and at least $k + 1$ records

$\longleftrightarrow$

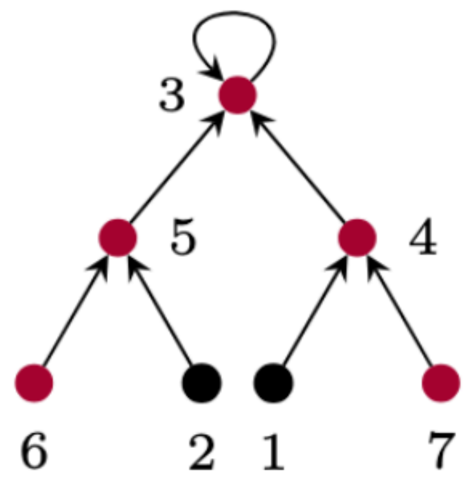
connected endofunctions on $[n]$ of girth $m + 1$ and at least k records.

Select any record other that the smallest one





An endofunction with at least 3+1 records



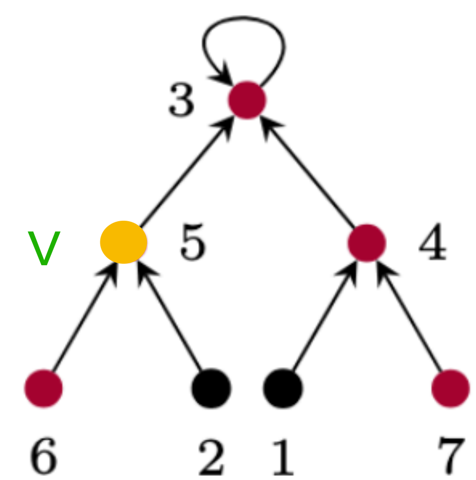
An endofunction with at least 3+1 records

k=3

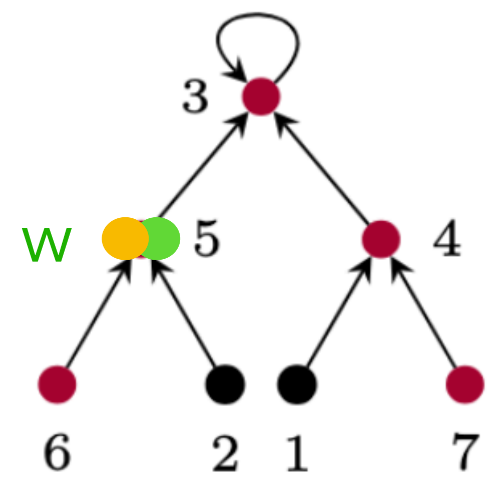
Let v be the kth (3rd) greatest record

The smallest record belongs to the cycle thus ● is outside

Select a node other than the root

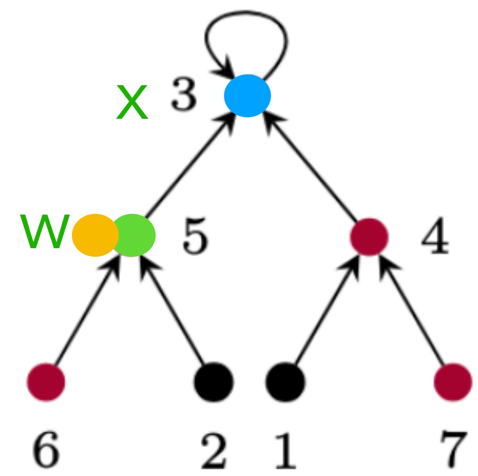


An endofunction with at least 3+1 records



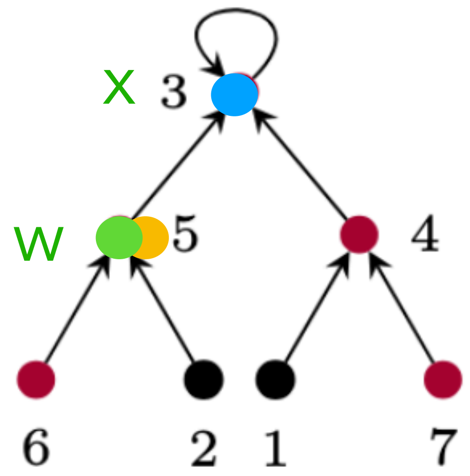
Let w be the first element in the orbit of v before reaching the cycle σ

An endofunction with at least 3+1 records

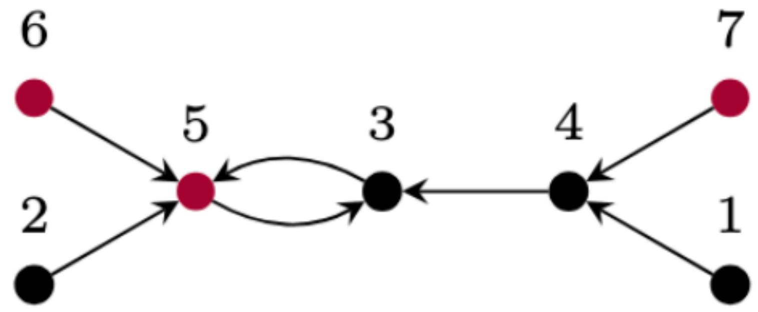


Let $x = \sigma^{-1}(f(w))$ The preimage of x inside of the cycle

At least 3+1 records
girth 1

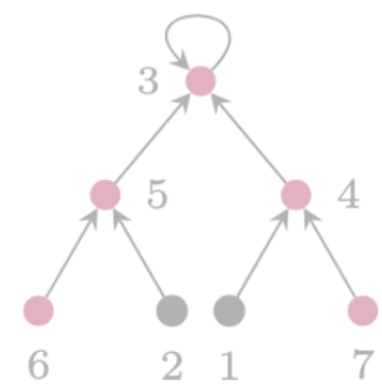


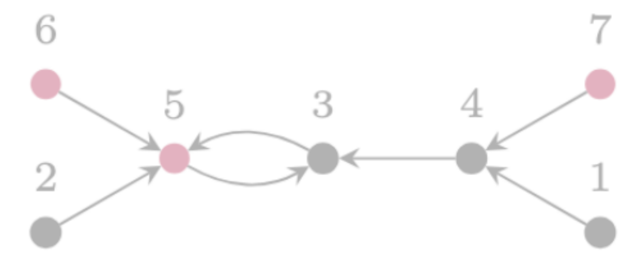
At least 3 records,
and girth 1+1



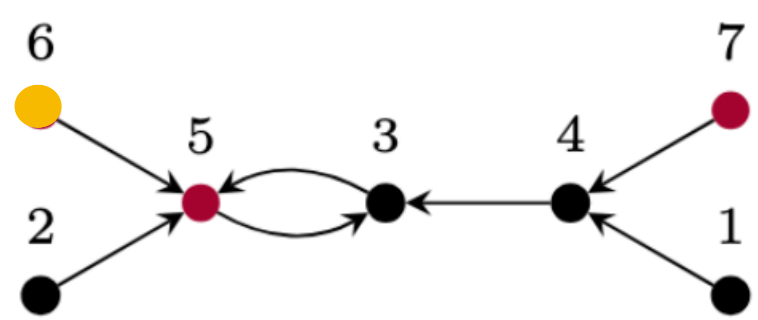
Define g :

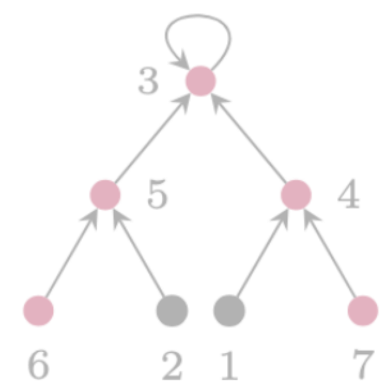
$g(z) = f(z)$ if $z \neq x$, $g(x) = w$.

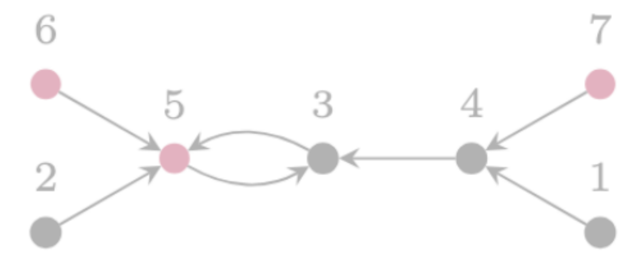




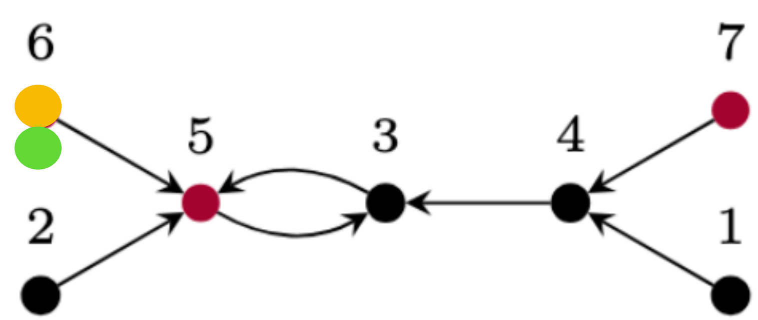
At least 2+1 records,
and girth 2

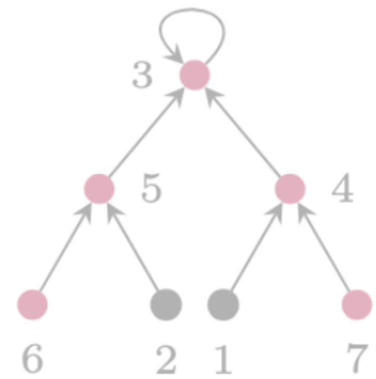


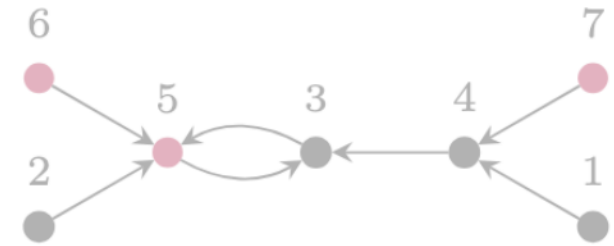




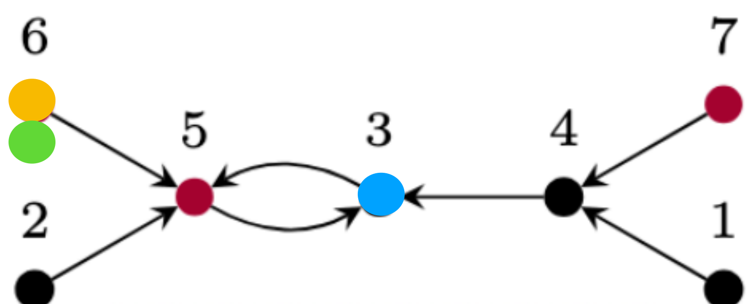
At least 2+1 records,
and girth 2



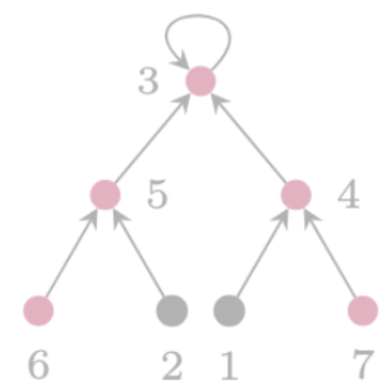


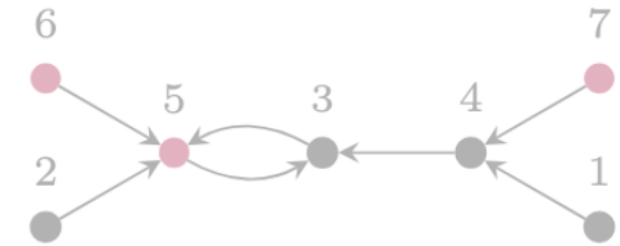


At least 2+1 records, and girth 2

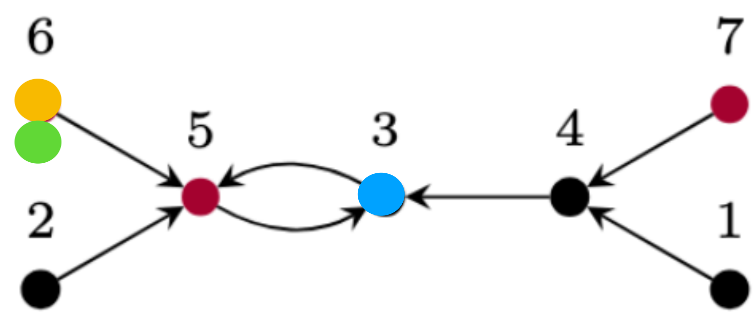


$$x = \sigma^{-1}(f(w))$$

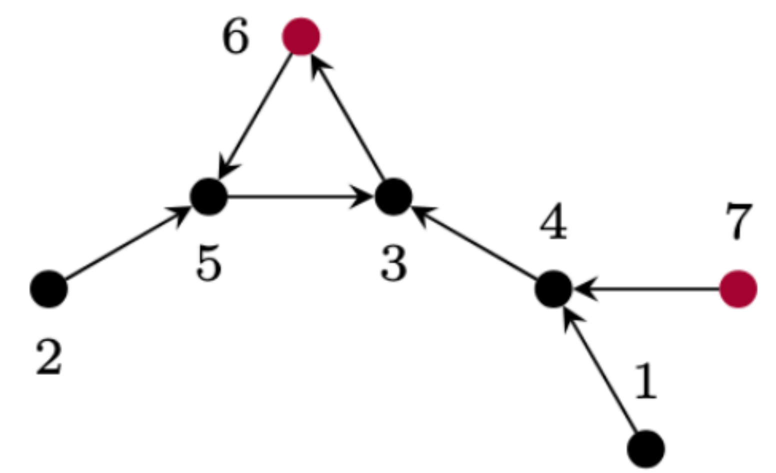




At least 2+1 records,
and girth 2

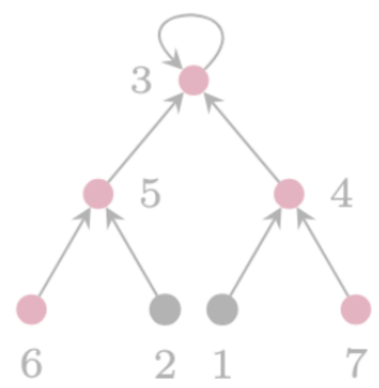


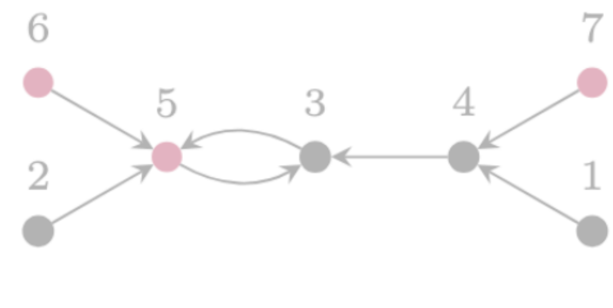
At least 2 records,
and girth 2+1

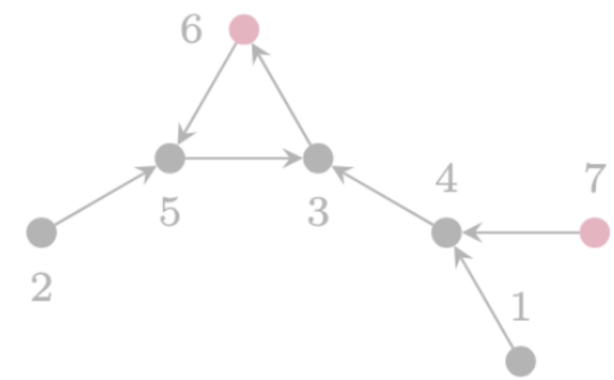


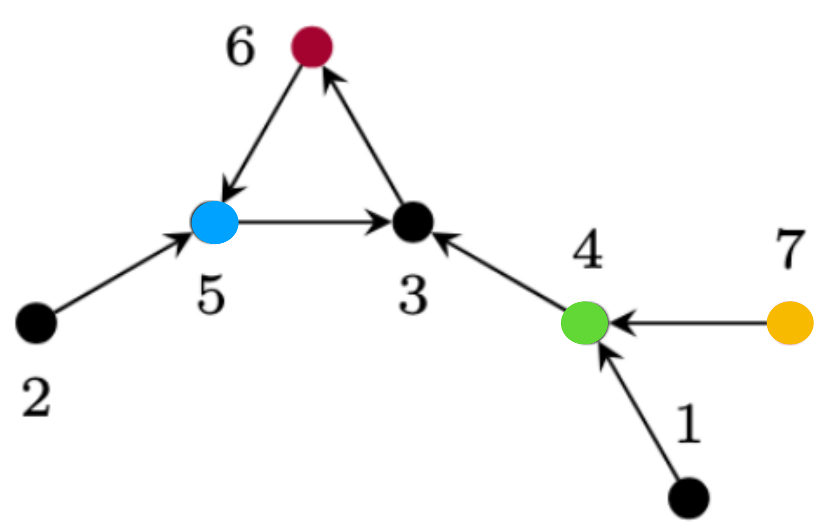
Define g :

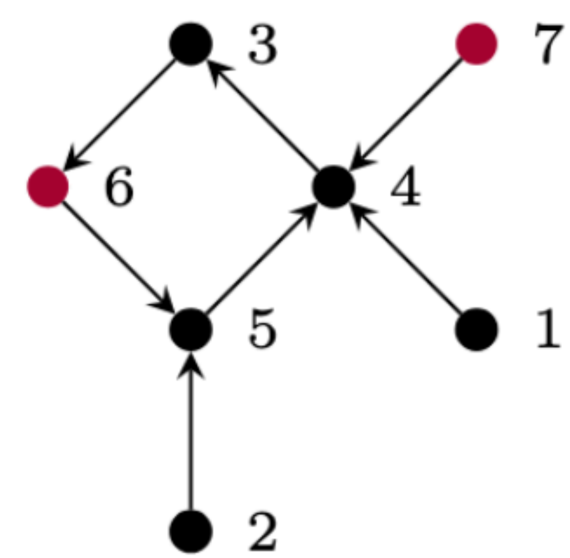
$g(z) = f(z)$ if $z \neq x$, $g(x) = w$.











At least 1+1 records,
and girth 3

Define g :

$g(z) = f(z)$ if $z \neq x$, $g(x) = w$.

connected endofunctions on $[n]$ of
girth m and at least $k + 1$ records

are in bijection with

connected endofunctions on $[n]$ of
girth $m + 1$ and at least k records.

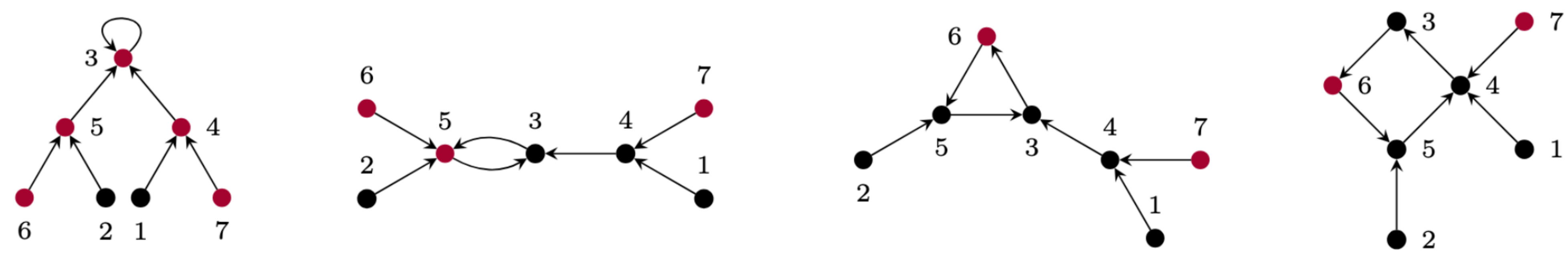
connected endofunctions on $[n]$ of
girth m and at least $k + 1$

are in bijection with

connected endofunctions on $[n]$ of
girth $m + 1$ and at least k records.

The number of connected endofunctions on [n]
of girth m and at least k records
only depends on $m + k$

endofunctions of girth 1 are rooted trees



The number of rooted trees with n nodes and
at least k records coincides with
the number of connected endofunctions on $[n]$ of girth k

The number of rooted trees with n nodes and

at least k records coincides with

the number of connected endofunctions on $[n]$ of girth k

cycles of k rooted trees of order n.

GF for rooted trees with at least k forests

$$\mathcal{R}_{\geq k}(z) = \sum_{l \geq k} \mathcal{R}_l(z) \qquad \mathcal{R}_{\geq k}(z) = \frac{\mathcal{T}^k(z)}{k}$$

GF for rooted trees with at least k forests

$$\mathcal{R}_{\geq k}(z) = \sum_{l \geq k} \mathcal{R}_l(z)$$

$$\mathcal{R}_{\geq k}(z) = \frac{\mathcal{T}^k(z)}{k}$$

cycles of trees of order n of k rooted trees.

GF for rooted trees with k forests

$$\mathcal{R}_k(z) = \sum_{n \geq 0} R(n,k)\ z^n$$

$$\mathcal{R}_k = \frac{\mathcal{T}^k(z)}{k} - \frac{\mathcal{T}^{k+1}(z)}{k+1}$$

The number of rooted trees with n nodes and
at least k records coincides with
the number of connected endofunctions on $[n]$ of girth k

cycles of trees of order n of k rooted trees.

$$R_\bullet(n,k) = k\,(n-1)\cdots(n-k+1)\,n^{n-k-1}$$

At least k records

Choose the set of roots

$$kn^{n-k-1}$$

labeled rooted forests with a fixed k-set of roots

$$n(n-1)\ldots(n-k+1)/k$$

Order the roots cyclically

Plus some minor simplification (subtract the count for k+1 from the count for k).

$$\mathcal{R}_{\geq k}(z) = \frac{\mathcal{T}^k(z)}{k}$$

$$\mathcal{R}_k = \frac{\mathcal{T}^k(z)}{k} - \frac{\mathcal{T}^{k+1}(z)}{k+1}$$

$$R_\bullet(n, k) = k\,(n-1)\cdots(n-k+1)\,n^{n-k-1}$$

III

The first entry of the OEIS

The genesis sequence of the OEIS

The Number Collector (with Neil Sloane) - Numberphile Podcast

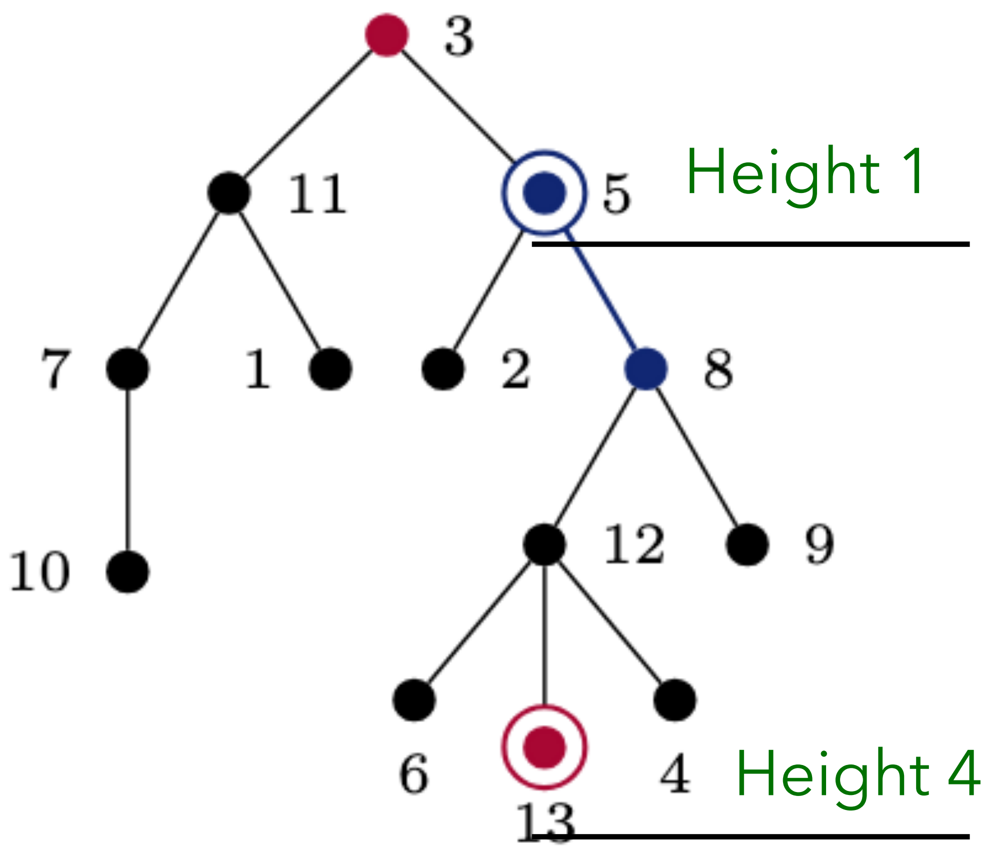
A000435 Normalized total height of all nodes in all rooted trees with n labeled nodes. 21
(Formerly M4558 N1940)

0, 1, 8, 78, 944, 13800, 237432, 4708144, 105822432, 2660215680, 73983185000, 2255828154624, 74841555118992, 2684366717713408, 103512489775594200, 4270718991667353600, 187728592242564421568, 8759085548690928992256, 432357188322752488126152, 22510748754252398927872000

(list; graph; refs; listen; history; text; internal format)

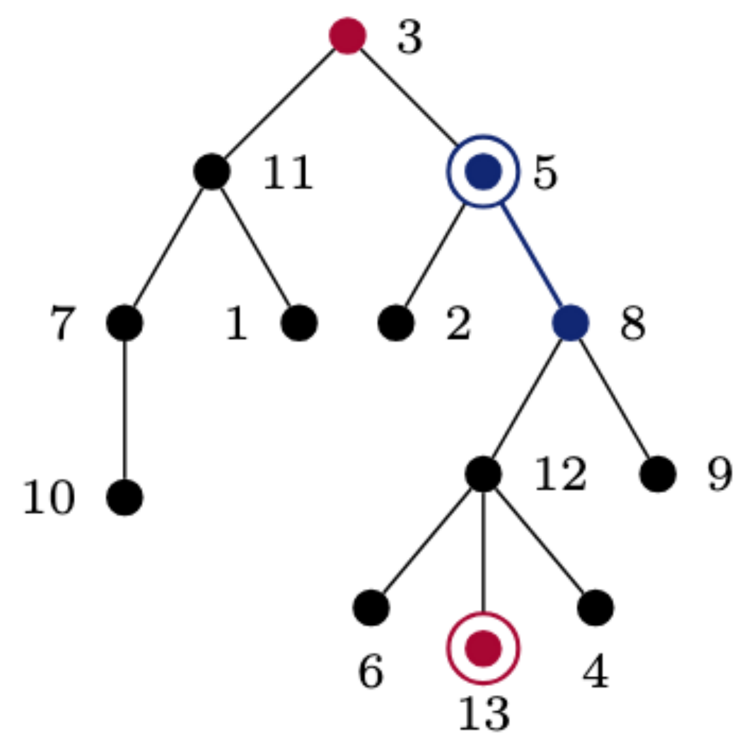
OFFSET 1,3

COMMENTS This is the sequence that started it all: the first sequence in the database!
The height h(V) of a node V in a rooted tree is its distance from the root. a(n) = Sum_{all nodes V in all n^(n-1) rooted trees on n nodes} h(V)/n.



The sum of the heights of
all nodes of all trees
of order k, divided by k.

A catalyst for T is a pair (u, v) of distinct vertices of T such that v is an ancestor of u



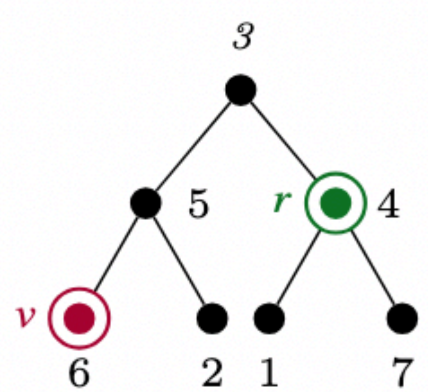
Let $C(T)$ the set of catalysts for T

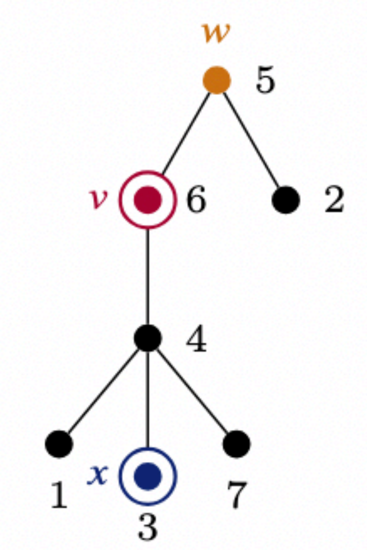
Bijection 3

R(n) = set of pairs (T, r):
T is a rooted tree labeled with $[n]$ and
r is a non-root record of T

$$[n] \times R(n)$$

$$C(n) = \bigsqcup_{T} C(T).$$



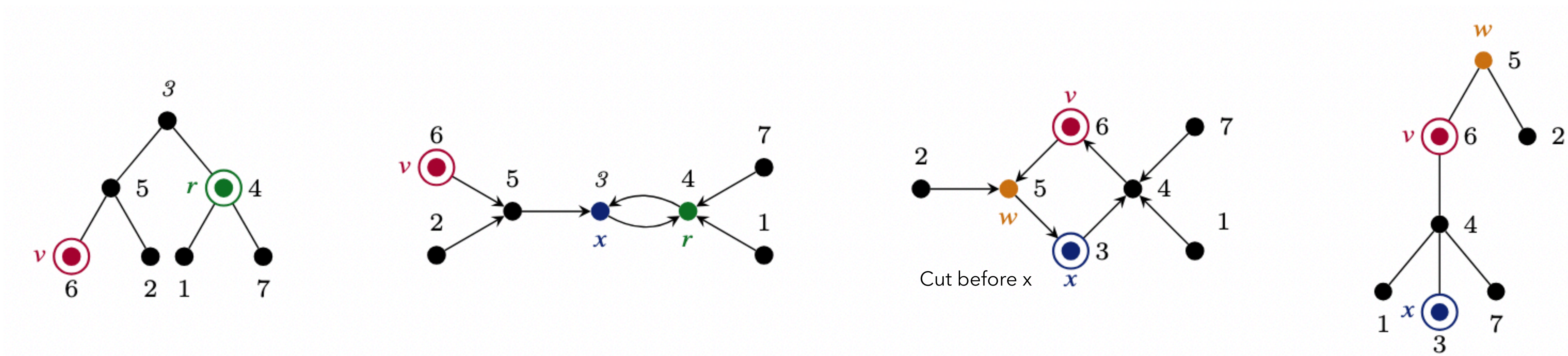


Bijection 3

The number of catalysts for T is equal to the sum of the heights of all vertices in T

$$[n] \times R(n)$$

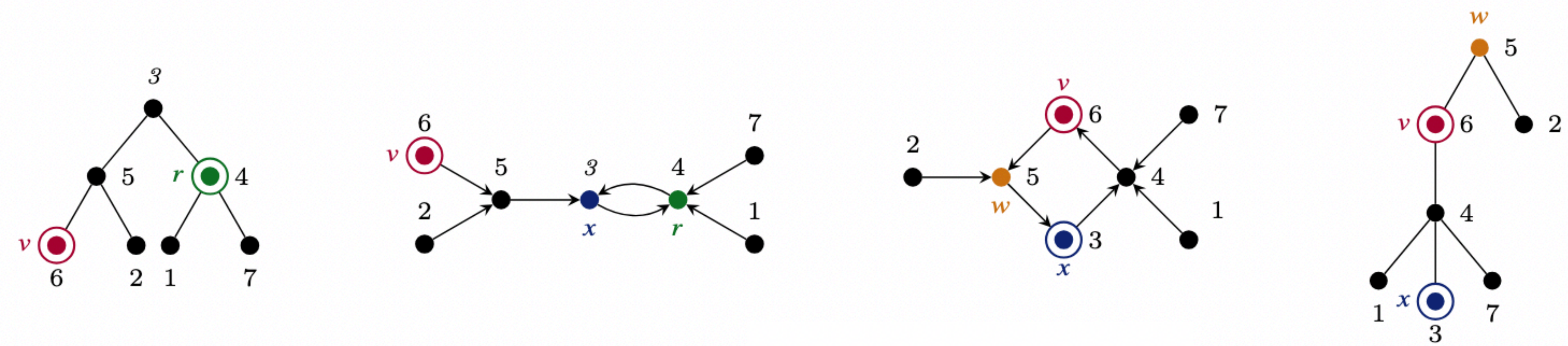
$$C(n) = \bigsqcup_{T} C(T).$$



R(n) = set of pairs (T, r):
T is a rooted tree labeled with $[n]$ and
r is a non-root record of T

The result is a Catalyst

The number of catalysts for T
is equal to the sum of the heights of all
vertices in T



$$[n] \times R(n) \longrightarrow C(n) = \bigsqcup_{T} C(T).$$

The first sequence and the record numbers

$$\sum_{k=1}^{n} (k-1) R_{\bullet}(n,k)$$

Sloan + Riordan

$$(n-1)! \sum_{k=0}^{n-1} \frac{n^k}{k!}$$

There exists a bijection between

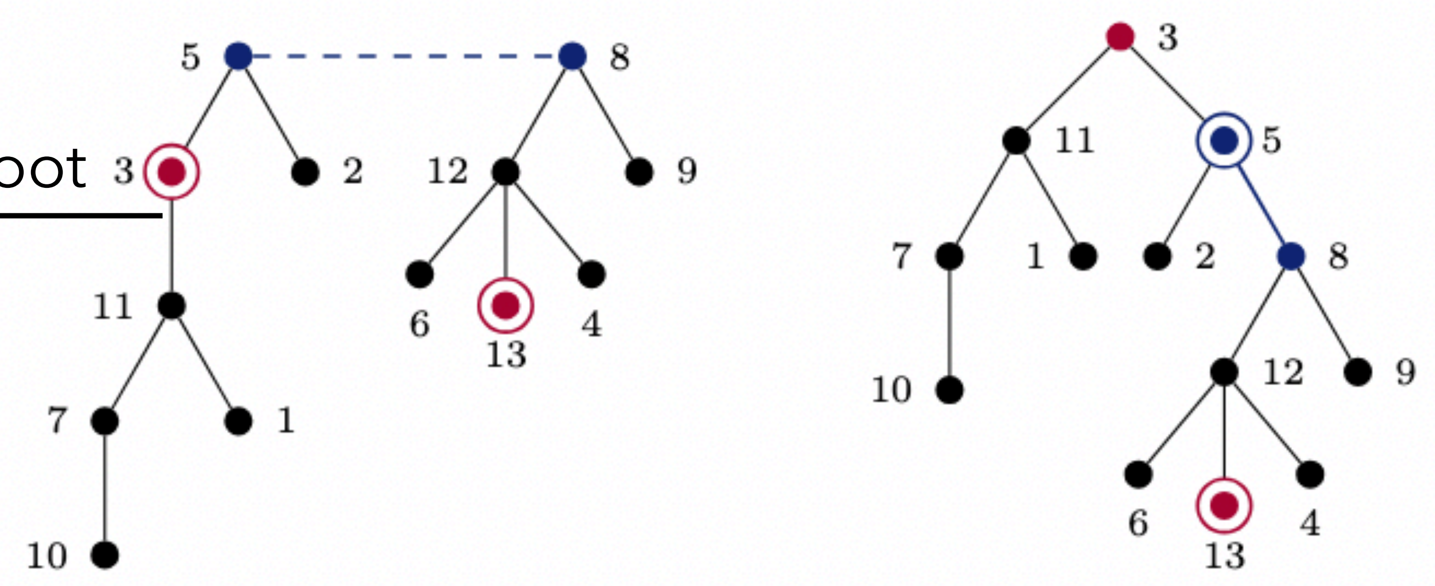
the set of rooted trees with a catalyst on each,

and

the set of pairs of rooted tree
with a selected node on each.

$$(n-1)!\sum_{k=0}^{n-1}\frac{n^k}{k!}$$

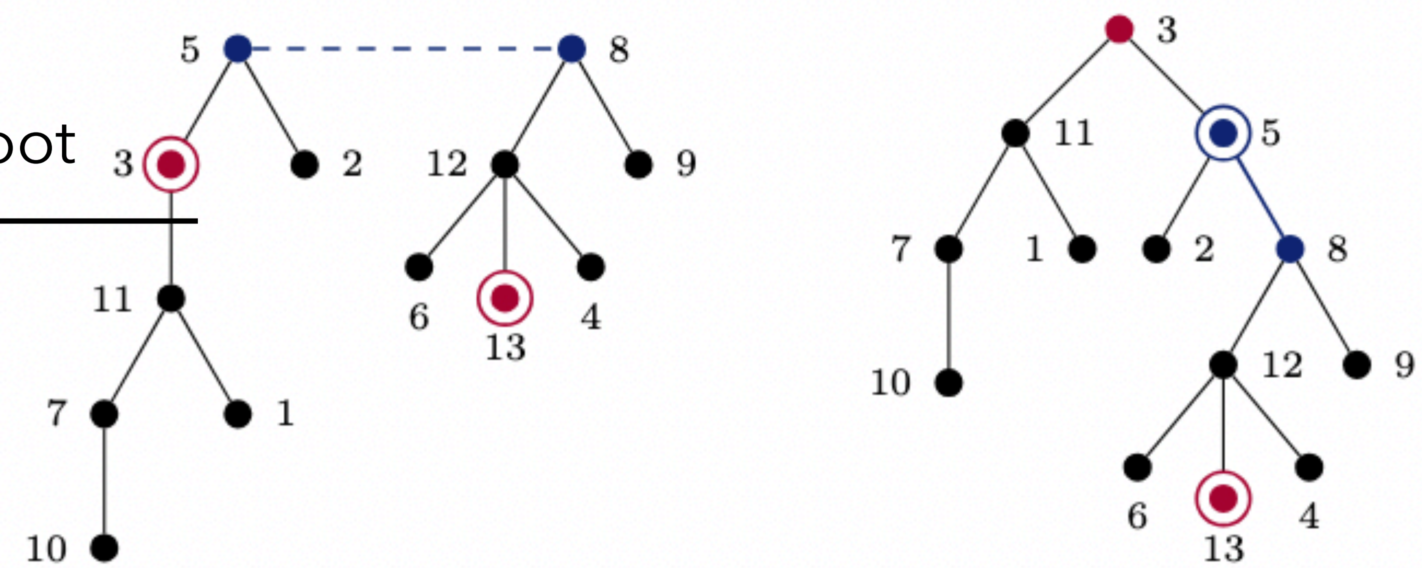
Becomes the root



pairs of rooted tree
with a selected node on each.

catalyst

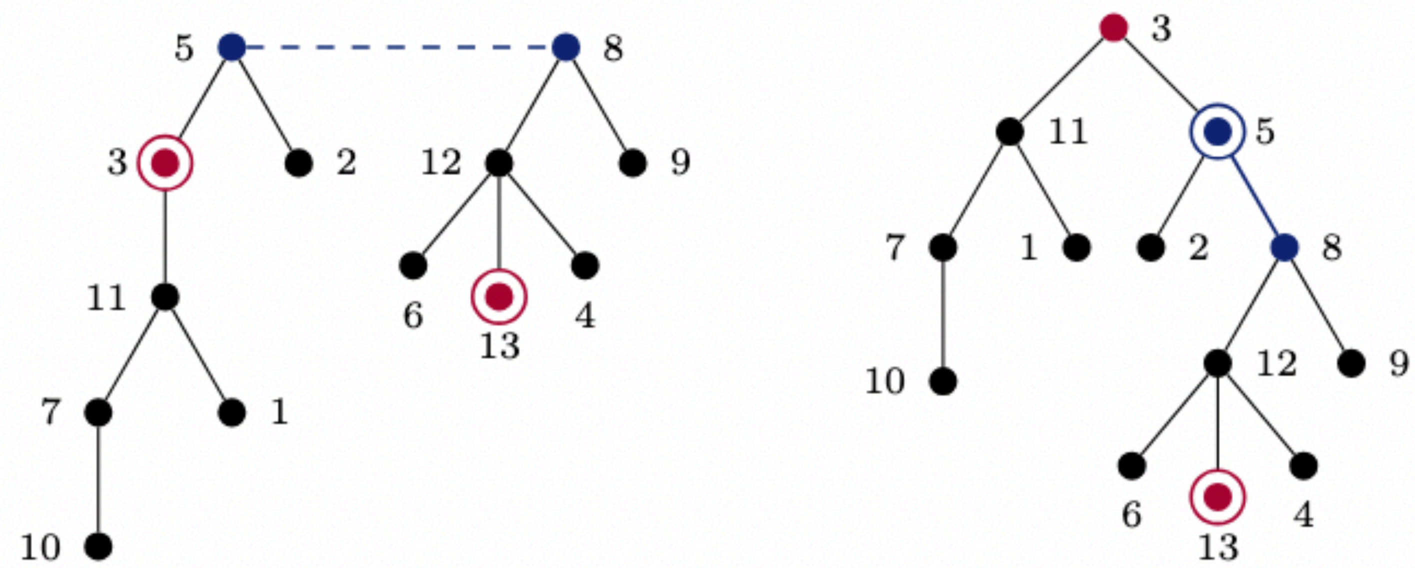
Becomes the root



The number of catalysts for T
is equal to the sum of the heights of all vertices in T

$$(z\mathcal{T}'(z))^2$$

Generating function
for the genesis sequence



The number of catalysts for T
is equal to the sum of the heights of all vertices in T

$$\left(z\mathcal{T}'(z)\right)^2$$

Generating function
for the genesis sequence

$$\sum_{k=1}^{n}(k-1)R_{\bullet}(n,k) = (n-1)!\sum_{k=0}^{n-1}\frac{n^k}{k!}$$

The genesis sequence, tree records, and endofunctions.

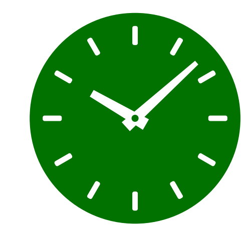
Enrica Duchi, Adrián Lillo, Pablo Puerto, Mercedes Rosas, Stefan Trandafir

December 14, 2025

IV

Dominique Foata

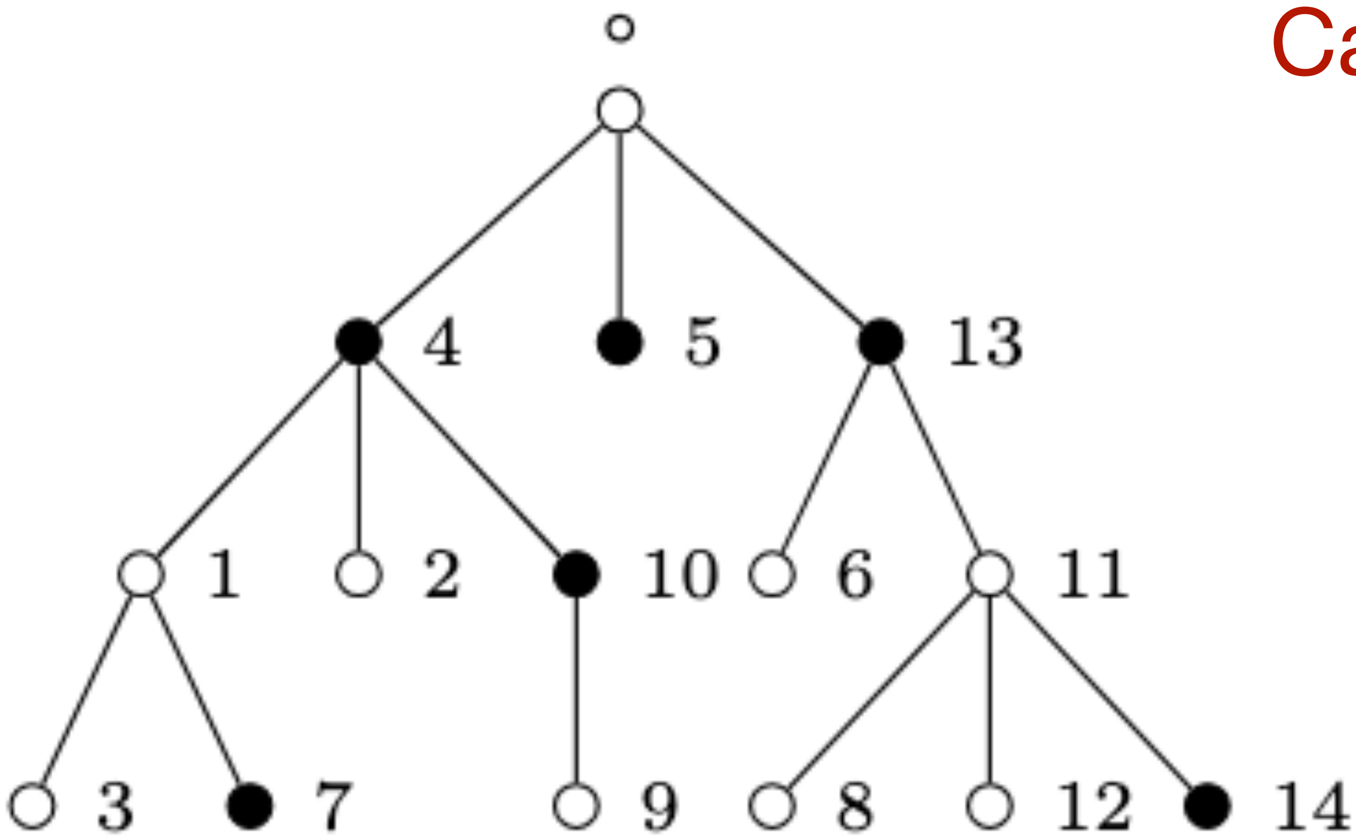
As they drive, clusters appear



Overtaking is not allowed !

Foata’s fundamental transformation.

Consider the possibility of junctions



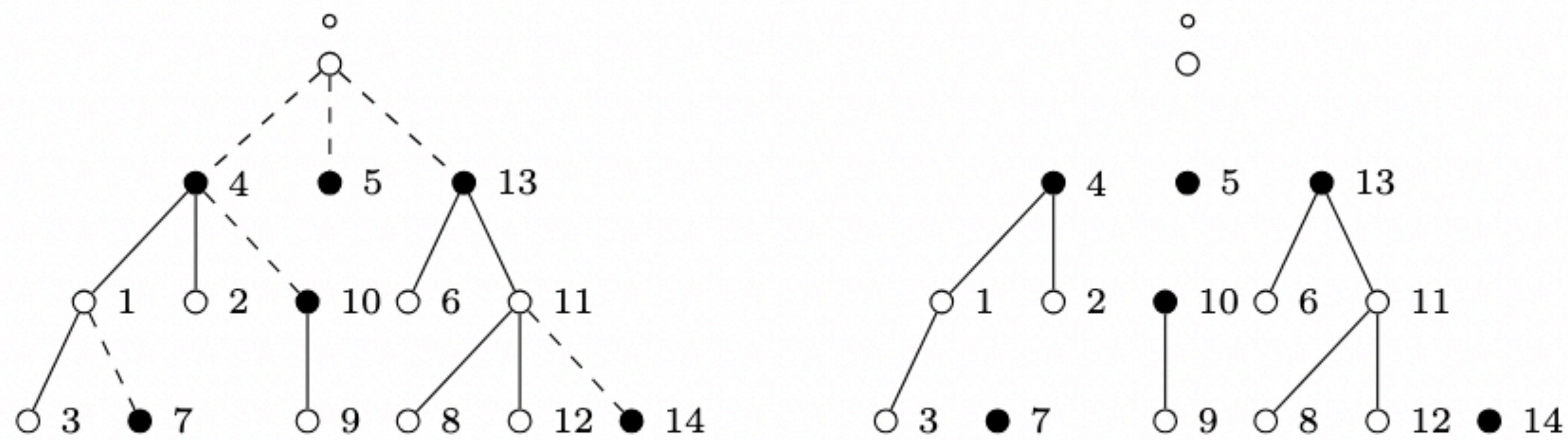
Cayley tree
rooted
labelled
tree

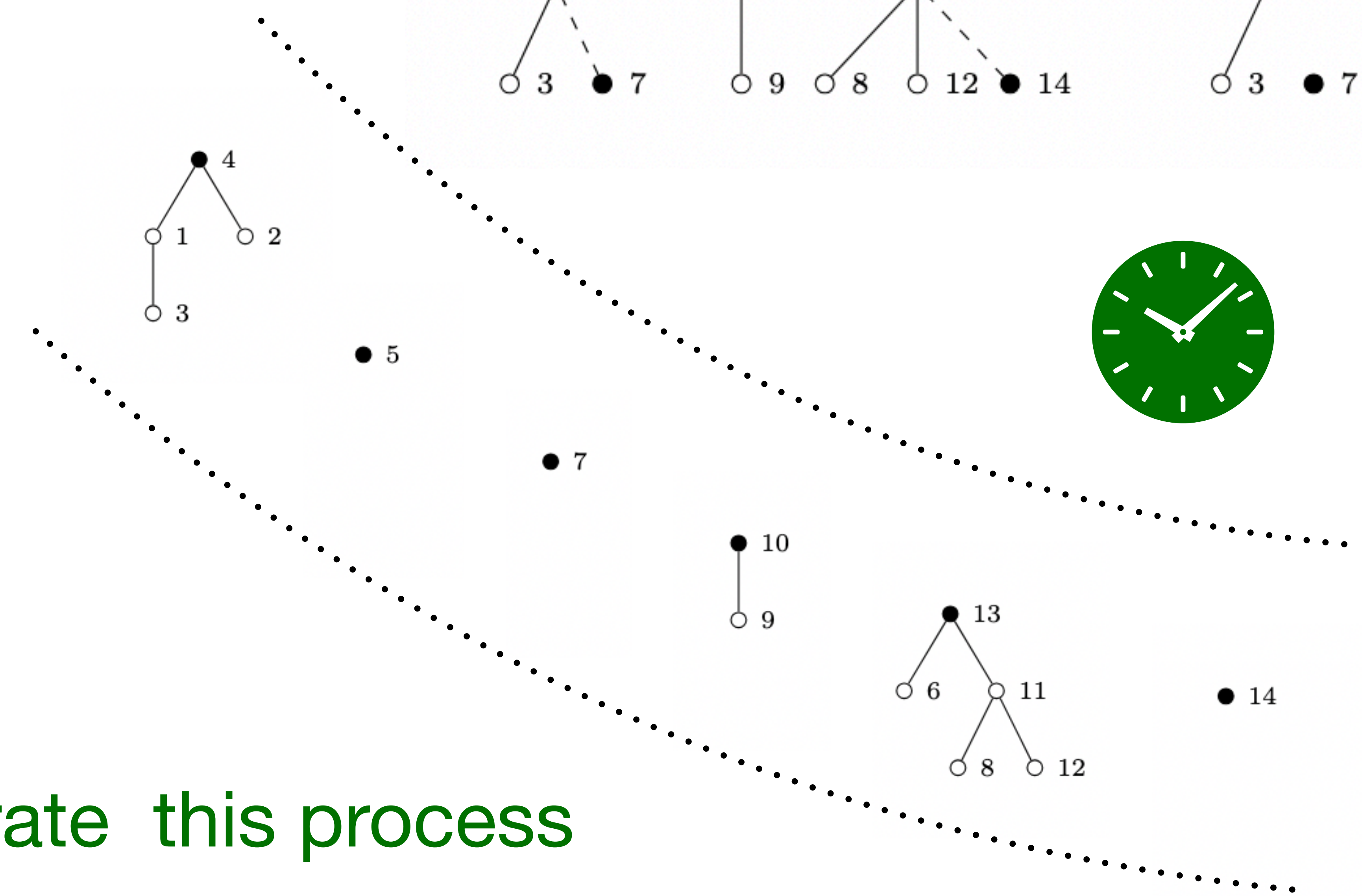
with time clusters will form.

Labels indicate each car's speed rank.

Black = record nodes

Overtaking is not allowed !

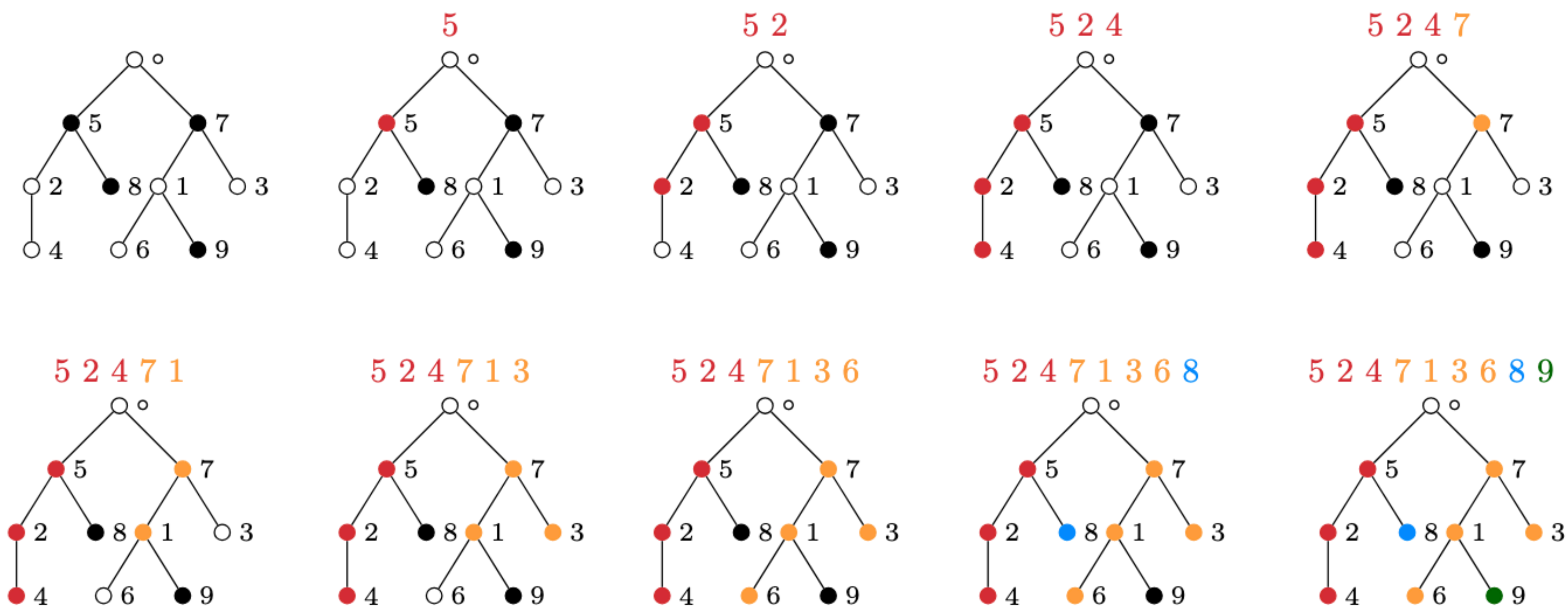




+ Iterate this process

Cars arrive and try to park on a long street.

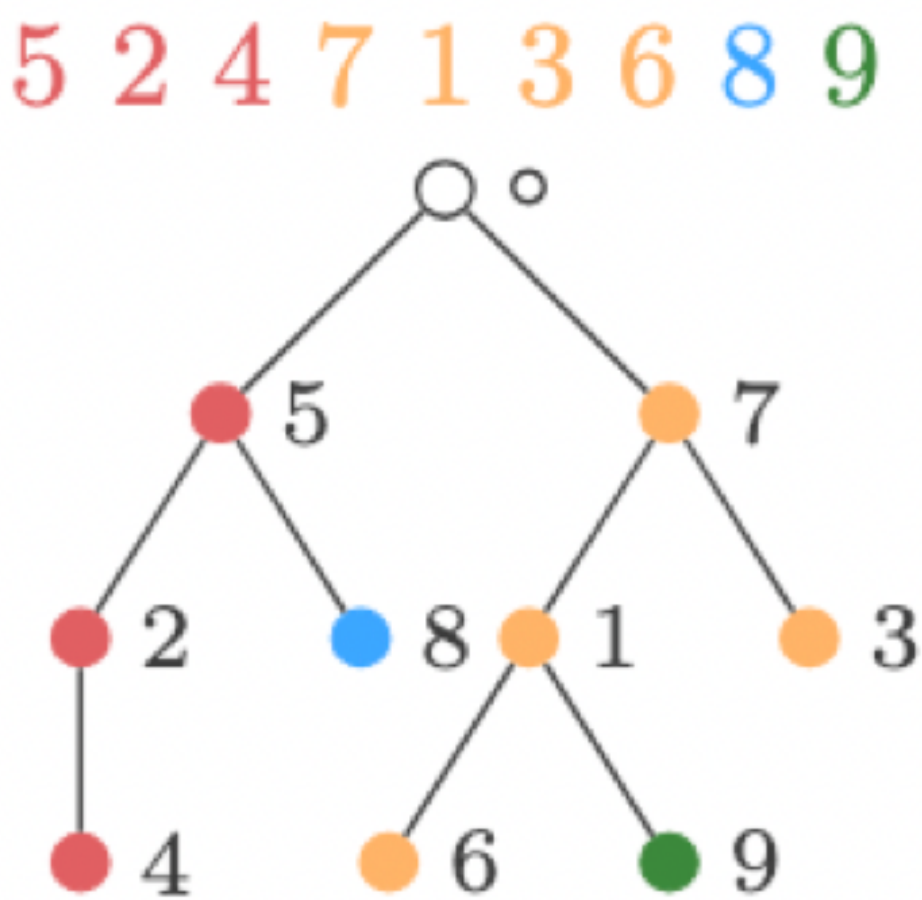
Weary permutation



Weary parking

Priority search

Since overtaking is impossible, the best drivers can hope for is to park immediately after the car in right in front.



Inverse
Bird's eye permutation

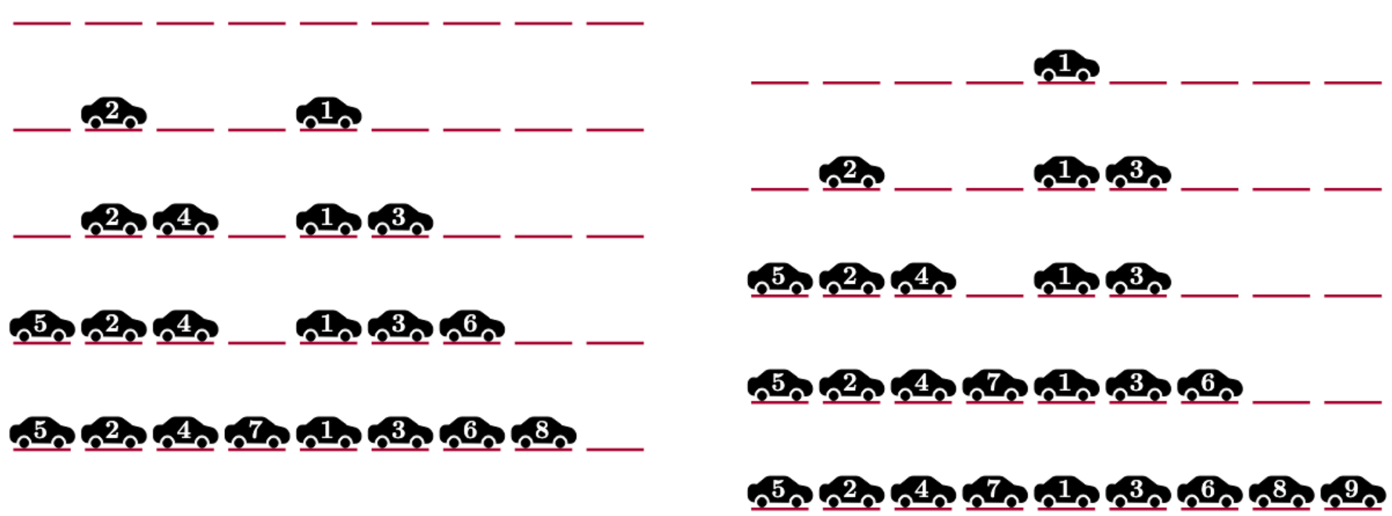
$$\pi_T(i) = \omega_T^{-1}(f_T(i)) + 1$$

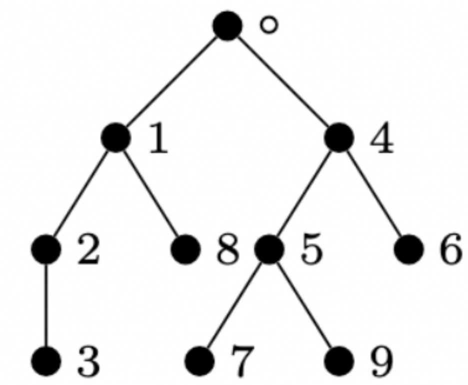
$$\pi_T = 5 \;\; 2 \;\; 5 \;\; 3 \;\; 1 \;\; 6 \;\; 1 \;\; 2 \;\; 6.$$

Where f_T is the parent function of the tree.

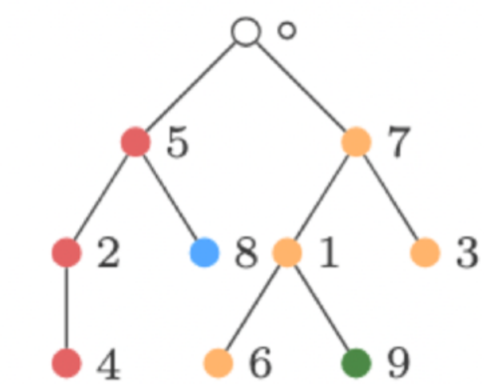
Tree records become permutation records

An equidistribution result:





$$\pi = 5\ 2\ 5\ 3\ 1\ 6\ 1\ 2\ 6$$



Parking Functions	Cayley Trees
parking function records	tree records
probes	waiting time
lucky cars	priority small ascents
1's	root degree
absent elements	leaves
multiplicity sequence	passport
length	order

arXiv > math > arXiv:2506.22145

On Weary Drivers, Records of Trees, and Parking Functions

Adrián Lillo, Mercedes Rosas, Stefan Trandafir

Merci beaucoup

Muchas gracias