

Composition schemes: q -enumerations and phase transitions in Gibbs models

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Abstract

Composition schemes are ubiquitous in combinatorics, statistical mechanics and probability theory. We give a unifying explanation to various phenomena observed in the combinatorial and statistical physics literature in the context of q -enumeration (this is a model where objects with a parameter of value k have a Gibbs measure/Boltzmann weight q^k). For structures enumerated by a composition scheme, we prove a phase transition for any parameter having such a Gibbs measure: for a critical value $q = q_c$, the limit law of the parameter is a two-parameter Mittag-Leffler distribution, while it is Gaussian in the supercritical regime ($q > q_c$), and it is a Boltzmann distribution in the subcritical regime ($0 < q < q_c$). We apply our results to fundamental statistics of lattice paths and quarter-plane walks. We also explain previously observed limit laws for pattern-restricted permutations, and a phenomenon uncovered by Krattenthaler for the wall contacts in watermelons.

2012 ACM Subject Classification Mathematics of computing → Generating functions; Mathematics of computing → Enumeration; Mathematics of computing → Distribution functions

Keywords and phrases Composition schemes, q -enumeration, generating functions, Gibbs distribution, phase transitions, limit laws, Mittag-Leffler distribution, chi distribution, Boltzmann distribution

Funding *Cyril Banderier*: supported by the French-Austrian PHC Amadeus project “Asymptotic behaviour of combinatorial structures”.

Stephan Wagner: supported by the Swedish research council (VR), grant 2022-04030.

Michael Wallner: supported by the Austrian Science Fund (FWF): P 34142 and OeAD WTZ project FR 01/2023.

1 Introduction

1.1 q -enumeration and Gibbs distributions

Let \mathcal{T} be a family of combinatorial objects, let $|\cdot|$ denote the size of objects, and let $\mathcal{X}: \mathcal{T} \rightarrow \mathbb{N}$ be a statistic defined on \mathcal{T} . The statistic \mathcal{X} on objects of \mathcal{T} of size n can be encoded by the sum

$$f_n(q) = \sum_{T \in \mathcal{T}: |T|=n} q^{\mathcal{X}(T)}. \quad (1)$$

This sum reduces for $q = 1$ to the total number $f_n(1) = f_n$ of objects of size n . In combinatorics, for any given $q \in \mathbb{R}$, it is called the q -enumeration of \mathcal{T} of size n with respect to \mathcal{X} (see, e.g., [1, 31]). In the language of statistical mechanics, $f_n(q)$ is a partition function with Boltzmann weight q . The bivariate generating function $F(z, q)$ is then defined as

$$F(z, q) = \sum_{T \in \mathcal{T}} z^{|T|} q^{\mathcal{X}(T)} = \sum_{n \geq 0} f_n(q) z^n = \sum_{n \geq 0} \sum_{k \geq 0} f_{n,k} z^n q^k. \quad (2)$$

Here $f_{n,k}$ denotes the number of objects of \mathcal{T} of size n for which \mathcal{X} equals k . It is usual to associate with the statistic \mathcal{X} the random variables X_n , $n \geq 1$, defined as

$$\mathbb{P}(X_n = k) = \frac{f_{n,k}}{f_n} = \frac{[q^k]f_n(q)}{f_n(1)} = \frac{[z^n q^k]F(z, q)}{[z^n]F(z, 1)}, \quad (3)$$

such that each object from \mathcal{T} of size n is equally likely. The associated probability generating function is given by $\mathbb{E}(q^{X_n}) = \frac{[z^n]F(z, q)}{[z^n]F(z, 1)}$. In Equation (3), the reader is probably used to consider q as a formal variable, but in this work, like in statistical mechanics, we shall consider q as an adjustable parameter (weight $\in \mathbb{R}_+$) of the underlying combinatorial and physical structures. This is also the spirit of the Boltzmann sampling method [18], where q is tuned to minimize the number of rejection steps in the sampling algorithm.

More precisely, in this article, we put a Gibbs measure on the statistic \mathcal{X} ; that is, one has the following probabilistic model.

► **Definition 1** (Gibbs distribution). *Let a family \mathcal{T} of combinatorial objects and a statistic $\mathcal{X}: \mathcal{T} \rightarrow \mathbb{N}$ be given. For real $q > 0$, the Gibbs distribution of this statistic is the law of the random variable $X_n(q)$ with probability mass function*

$$\mathbb{P}(X_n(q) = k) = \frac{f_{n,k}q^k}{f_n(q)}, \quad k \geq 0.$$

In terms of the probability generating function $p(v) = \mathbb{E}(v^{X_n(1)})$, we have $\mathbb{E}(v^{X_n(q)}) = \frac{p(vq)}{p(q)}$.

A well-known example in probability theory is the Mallows distribution [35] on permutations with respect to the inversion statistic.

In many applications, one is interested in the limit distribution of $X_n(q)$, which depends on the value of $q > 0$; see, e.g., [9, 11, 12, 14, 33, 34, 36, 39]. Let us pinpoint the result of Krattenthaler [33], who uncovered a phase transition in the normalized *mean number* of wall contacts at $q = 2$ in watermelons. Using methods from analytic combinatorics, we will show that similar phase transitions naturally occur in a great many instances, not only with respect to the expectation but also for the *limit laws*. We will use the framework of composition schemes, which often provide a direct and unifying way to explain why phase transitions occur [5, 6, 25]. In Section 2, we establish in which way the phase transitions in the Gibbs model depend on the value of q . Particular instances of similar phenomena have been observed in [13, 41]. We give further examples in Sections 3 and 4.

1.2 Composition schemes and Gibbs distributions

Functional composition schemes such as $F(z) = G(H(z))$ are of great importance in combinatorics [5, 6, 25] and probability theory [38]. The main focus is to analyse probabilistic properties of compositions like

$$F(z, u) = G(uH(z)) \quad (4)$$

as a multitude of parameters \mathcal{X} can be modelled in this way. Here u marks the so-called size of the core, i.e., the involved G -component; see [5, 6, 25]. The distribution of the corresponding random variable X_n is then readily defined by

$$\mathbb{P}(X_n = k) = \frac{[z^n u^k]F(z, u)}{[z^n]F(z, 1)}.$$

Structurally, such schemes are at the heart of many fascinating phase transition phenomena (analytically corresponding, e.g., to coalescing saddle points or to confluence of singularities), related to the Gibbs measure in statistical physics and probability theory [38].

First, we relate composition schemes to q -enumeration.

► **Lemma 2** (Composition schemes and Gibbs distributions). *Let a combinatorial statistic \mathcal{X} with bivariate generating function $F(z, u)$ be given. Then, for real $q > 0$ the Gibbs distribution of \mathcal{X} has a probability mass function given in terms of F by*

$$\mathbb{P}(X_n(q) = k) = \frac{[z^n u^k]F(z, qu)}{[z^n]F(z, q)}.$$

In many (combinatorial) applications of composition schemes, the following assumptions hold [6]: all involved generating functions $G(z)$, $H(z)$, and $F(z)$ have nonnegative coefficients and aperiodic support, are analytic in a Δ -domain with a finite radius of convergence ρ_F and possess singular expansions of the form

$$F(z) = P\left(1 - \frac{z}{\rho_F}\right) + c_F \cdot \left(1 - \frac{z}{\rho_F}\right)^{\lambda_F} (1 + o(1)), \quad (5)$$

where $\lambda_F \in \mathbb{R} \setminus \{0, 1, 2, \dots\}$ is called the *singular exponent* (at $z = \rho_F$), and where $P(x) \in \mathbb{C}[x]$ is a polynomial (of degree ≥ 1 for $\lambda_F > 1$, of degree 0 for $0 < \lambda_F < 1$, and $P = 0$ for $\lambda_F < 0$). Then, singularity analysis [25] can often be used to compute the asymptotics of the coefficients.

For the reader's convenience, we recall the classical terminology for composition schemes in the following definition.

► **Definition 3** (Classification of composition schemes). *Let $\tau_H = H(\rho_H)$. A composition scheme $F(z) = G(H(z))$ is called subcritical if it satisfies $\tau_H < \rho_G$, critical if it satisfies $\tau_H = \rho_G$, and supercritical if it satisfies $\tau_H > \rho_G$.*

We note that each individual class of critical schemes leads to very diverse combinatorial and probabilistic phenomena [3, 5, 6, 25, 38].

One typical example of a combinatorial construction of shape $F(z, u) = G(uH(z))$ is given by the sequence construction.

► **Example 4** (Sequence of objects). Given a combinatorial structure \mathcal{H} , let $\mathcal{F} = \text{SEQ}(\mathcal{H})$. Using the variable u to encode the core size (i.e., the number the \mathcal{H} components), one has $F(z, u) = 1/(1 - uH(z))$. This is a composition scheme with $G(z) = 1/(1 - z)$, with $\rho_G = 1$ and $\lambda_G = -1$.

We observe that the parameter q can thus directly influence the nature of the underlying singular structure when the total mass (obtained by $u = 1$) changes from $[z^n]F(z, 1)$ to $[z^n]F(z, q)$. Let us make this more precise for the following class of functions that includes the sequence construction.

► **Lemma 5** (Nature and asymptotics of q -enumerated composition schemes). *Let a composition scheme $F(z, u) = G(uH(z))$ with singular exponents $\lambda_G < 0$ and $0 < \lambda_H < 1$ be given. Let $q_c := \frac{\rho_G}{\tau_H} = \frac{\rho_G}{H(\rho_H)} > 0$. The nature of the scheme then splits into three different regimes:*

- for $0 < q < q_c$, the scheme is subcritical;
- for $q = q_c$, the scheme is critical;
- for $q > q_c$, the scheme is supercritical.

Accordingly, if one imposes a Gibbs measure on the number of \mathcal{H} -components, this impacts the asymptotics of their q -enumeration $f_n(q)$ as follows:

$$f_n(q) \sim \begin{cases} \frac{c_H q G'(q\tau_H)}{\Gamma(-\lambda_H)} \rho_H^{-n} n^{-\lambda_H-1}, & \text{for } 0 < q < q_c, \\ c_G \left(-\frac{c_H}{\tau_H}\right)^{\lambda_G} \frac{1}{\Gamma(-\lambda_H \lambda_G)} \rho_H^{-n} n^{-\lambda_H \lambda_G-1}, & \text{for } q = q_c, \\ c_G \left(\frac{q\rho H'(\rho)}{\rho_G}\right)^{\lambda_G} \frac{1}{\Gamma(-\lambda_G)} \rho^{-n} n^{-\lambda_G-1}, & \text{for } q > q_c. \end{cases}$$

Proof of Lemma 5. First, we turn to the nature of the scheme $F_q(z) = G(qH(z))$. Since $F_q(z)$ has nonnegative coefficients, Pringsheim's Theorem [25] implies that there is a singularity on the real axis. Further, $H(z)$ is monotonically increasing on the real axis from 0 to ρ_H , where it attains the value $\tau_H = H(\rho_H) < \infty$. Thus, the nature of $F_q(z)$ depends on the relation between the singularity ρ_G of $G(z)$ and $q\tau_H$ as claimed. Next, we look at the singular expansions. We start with those of $G(z)$ and $H(z)$. By the assumptions on λ_G and λ_H we have

$$G(z) \sim c_G \left(1 - \frac{z}{\rho_G}\right)^{\lambda_G} \quad \text{and} \quad H(z) \sim \tau_H + c_H \left(1 - \frac{z}{\rho_H}\right)^{\lambda_H}. \quad (6)$$

In the subcritical regime $0 < q < \frac{\rho_G}{\tau_H}$, the outer function $G(z)$ is analytic at $q\tau_H$ and we combine its expansion with the singular expansion of $H(z)$ around $z = \rho_H$ to obtain

$$\begin{aligned} F_q(z) &= G(q\tau_H) + G'(q\tau_H)(qH(z) - q\tau_H)(1 + o(1)) \\ &\sim G(q\tau_H) + c_H q G'(q\tau_H) \left(1 - \frac{z}{\rho_H}\right)^{\lambda_H}. \end{aligned}$$

Basic singularity analysis [25] provides the stated expansion.

For the critical regime $q = \frac{\rho_G}{\tau_H}$ we obtain

$$F_q(z) \sim c_G \left(1 - q \frac{\tau_H}{\rho_G} - q \frac{c_H}{\rho_G} \left(1 - \frac{z}{\rho_H}\right)^{\lambda_H}\right)^{\lambda_G} = c_G \left(-\frac{c_H}{\tau_H}\right)^{\lambda_G} \left(1 - \frac{z}{\rho_H}\right)^{\lambda_H \lambda_G},$$

which leads to the desired result.

Finally, in the supercritical regime $q > \frac{\rho_G}{\tau_H}$, there exists a unique $0 < \rho < \rho_H$ such that $qH(\rho) = \rho_G$. As $\rho < \rho_G$, we may expand $H(z)$ around ρ . This leads to

$$qH(z) = qH(\rho) + qH'(\rho)(z - \rho) + \mathcal{O}((z - \rho)^2).$$

Note that by the positivity of the coefficients of H one has $H'(\rho) \neq 0$. Thus, we have

$$F_q(z) \sim c_G \left(\frac{q\rho H'(\rho)}{\rho_G}\right)^{\lambda_G} \left(1 - \frac{z}{\rho}\right)^{\lambda_G},$$

and the asymptotic formula for $f_n(q)$ follows by singularity analysis. \blacktriangleleft

2 Main theorem: Gibbs models and phase transitions with respect to q

The following result is our main theorem. It describes the dependency of the limit law on the parameter q . In this extended abstract, we present only the case of sequence-like schemes.

► **Theorem 6** (Gibbs distribution and phase transitions of sequence-like schemes). *Let a composition scheme $F(z, u) = G(uH(z))$, with singular exponents $\lambda_G < 0$ and $0 < \lambda_H < 1$, be given. Then, the Gibbs distribution of $X_n = X_n(q)$ associated with $F(z, qv)$ has the following limit laws and phase transition diagram that depend on $q_c = \frac{\rho_G}{\tau_H}$:*

- *In the subcritical regime $0 < q < q_c$, the random variable $X_n - 1$ converges to a discrete distribution, a Boltzmann distribution $\mathcal{B}_{G'}(q\tau_H)$ with explicit probability generating function given by:*

$$\mathbb{P}(X_n - 1 = k) \rightarrow [v^k] \frac{G'(vq\tau_H)}{G'(q\tau_H)}.$$

In particular, if $G(z) = \frac{1}{(1-z)^m}$, the limit law of $X_n - 1$ is a negative binomial distribution $\text{NegBin}(m+1, 1 - q\tau_H)$, where $X \sim \text{NegBin}(r, p)$ is defined by $\mathbb{P}(X = k) = \binom{k+r-1}{k} p^r (1-p)^k$ for $k \geq 0$.

Parameter q	$0 < q < q_c$	$q = q_c$	$q > q_c$
Regime	subcritical	critical	supercritical
Singularity	ρ_H	ρ_H	$\rho < \rho_H$
Singular exponent	Z^{λ_H}	$Z^{\lambda_G \lambda_H}$	Z^{λ_G}
Limit law	discrete (Boltzmann)	continuous (Mittag-Leffler)	continuous (Gaussian)

- In the critical regime $q = q_c$, the random variable X_n/n^{λ_H} converges in distribution:

$$\frac{-c_H X_n}{\tau_H n^{\lambda_H}} \xrightarrow{\mathcal{L}} X,$$

where the random variable X follows the two-parameter Mittag-Leffler distribution $\text{ML}(\alpha, \beta)$ (with $\alpha := \lambda_H$ and $\beta := -\lambda_G \lambda_H$) of density $f_X(x) = \frac{\Gamma(\beta+1)}{\alpha \Gamma(\frac{\beta}{\alpha}+1)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \Gamma(-n\alpha)} x^{n+\beta/\alpha-1}$ that is determined by its moments $\mathbb{E}(X^r) = \frac{\Gamma(\beta)\Gamma(r+\beta/\alpha)}{\Gamma(\beta/\alpha)\Gamma(\alpha r+\beta)}$.

In particular, for $\lambda_G = -1$ and $\lambda_H = \frac{1}{2}$, X follows the Rayleigh distribution $\mathcal{R}(\sqrt{2})$, where $X \sim \mathcal{R}(\sigma)$ is defined by the density $\frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}$ for $x \geq 0$.

- In the supercritical regime $q > q_c$, the centred and normalized random variable $(X_n - \mu_n)/\sigma_n$ converges in distribution to a standard normal distribution $\mathcal{N}(0, 1)$, where mean μ_n and variance σ_n^2 are both of order n : we have, with $\rho \equiv \rho(q)$ given by $qH(\rho) = \rho_G$,

$$\mu_n \sim \frac{\rho_G}{q\rho H'(\rho)} \cdot n, \quad \sigma_n^2 \sim \left(\frac{\rho_G^2}{q^2 \rho^2 H'(\rho)^2} - \frac{\rho_G}{q\rho H'(\rho)} + \frac{\rho_G^2 H''(\rho)}{q^2 \rho H'(\rho)^3} \right) \cdot n.$$

In particular, the expected value of X_n is for $n \rightarrow \infty$ asymptotically equivalent to

$$\mathbb{E}(X_n) \sim \begin{cases} 1 + \frac{q\tau_H G''(q\tau_H)}{G'(q\tau_H)}, & \text{for } 0 < q < q_c, \\ \frac{\lambda_G \tau_H \Gamma(-\lambda_G \lambda_H)}{c_H \Gamma((1-\lambda_G)\lambda_H)} \cdot n^{\lambda_H}, & \text{for } q = q_c, \\ \frac{\rho_G}{q\rho H'(\rho)} \cdot n, & \text{for } q > q_c. \end{cases}$$

Proof of Theorem 6 (Sketch). For $q < q_c$ we are in the subcritical regime and follow the proof of Lemma 5. We build on the results of [6, 25]. We expand $F(z, qv)$ for $0 < v < 1$ to obtain

$$F(z, qv) \sim G(qv\tau_H) + c_H qv G'(qv\tau_H) \left(1 - \frac{z}{\rho_H}\right)^{\lambda_H}.$$

This implies that the probability generating function satisfies

$$\lim_{n \rightarrow \infty} \mathbb{E}(v^{X_n(q)}) = \lim_{n \rightarrow \infty} \frac{[z^n]F(z, qv)}{f_n(q)} = \frac{vG'(qv\tau_H)}{G'(q\tau_H)},$$

leading to a Boltzmann distribution $\mathcal{B}_{G'}(q\tau_H)$. In particular, for $G(z) = 1/(1-z)$ we have $G'(z) = 1/(1-z)^2$, leading to a negative binomial distribution.

For $q = q_c$ we are at the critical value and we thus apply [6, Theorem 4.1] with $\lambda_G < 0$ and $0 < \lambda_H < 1$. This yields the stated limit law, as discussed in [6, Remark 4.2].

In the supercritical regime for $q > q_c$, our claim results from the approach of Bender [25, Propositions IX.6 and IX.7]. The singularity $\rho = \rho(qu)$ becomes an analytic function of u while the nature of the singularity remains unchanged for u in a sufficiently small neighbourhood of 1. In particular, the expected value, the variance, and the normal limit law follow by an application of Hwang's quasi-power theorem [25, 32]. \blacktriangleleft

► Remark 7. The constant in the asymptotics of σ_n^2 in the supercritical case is always strictly positive, thus degenerate limit laws are not possible. We refer to the full version for a proof.

3 Applications

We start our list of applications with the case of fixed-point-biased permutations avoiding a pattern of length three, whose asymptotic behaviour was recently considered in [13]. Then we consider several instructive examples from the theory of lattice paths, namely the returns to zero in Dyck bridges and Dyck excursions, as well as in Motzkin bridges and Motzkin excursions. Thereby, we add to the existing classical results (that is, for the uniform distribution) the phase transitions stemming from q -enumeration (that is, for the Gibbs distribution on the parameter). We also consider the number of boundary interactions in some quarter-plane walk models, and the number of contacts between 2 paths in some watermelon models.

3.1 Fixed point biased permutations avoiding a pattern of length three

Let $\mathcal{S}_n(p)$ be the set of permutations of $1, 2, \dots, n$ that avoid any given pattern p of length 3. Elizalde [19] obtained the generating function counting the statistic $\text{fp}(\sigma)$ (number of fixed points) of such permutations:

$$F(z, u) = 1 + \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{S}_n(p)} u^{\text{fp}(\sigma)} z^n = \frac{2}{1 + 2(1-u)z + \sqrt{1-4z}}.$$

Later, Chelikavada and Panzo [13] used it to establish a phase transition which we rederive now.

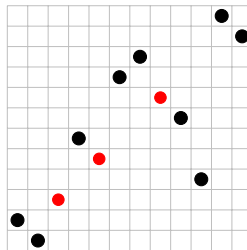
► **Theorem 8** (Phase transition for fixed-point-biased permutations). *The limit Gibbs distribution of the fixed-point statistic in permutations avoiding any given pattern of length 3 has a phase transition with critical value $q_c = 3$:*

Parameter q	$0 < q < 3$	$q = 3$	$q > 3$
Limit law of $X_n(q)$	Negative binomial $\text{NegBin}(2, 1 - q/3)$	Rayleigh $\mathcal{R}(\sqrt{2})$	Gaussian $\mathcal{N}(0, 1)$

Proof. In order to deduce the phase transitions from Theorem 6, we write $F(z, u)$ as a sequence of components $H(z)$ marked with u

$$F(z, u) = \frac{H(z)}{z} \cdot \frac{1}{1 - uH(z)} = \frac{1}{uz} \cdot \frac{1}{1 - uH(z)} - \frac{1}{uz}, \quad \text{where} \quad H(z) = \frac{2z}{1 + 2z + \sqrt{1 - 4z}}.$$

Thus, one has $\rho_G = 1$, $\rho_H = \frac{1}{4}$ and $\tau_H = H(\rho_H) = \frac{1}{3}$. Theorem 6 then implies the phase transition at $q_c = \rho_G/\tau_H = 3$ with NegBin, Rayleigh (since $\lambda_H = \frac{1}{2}$), and Gaussian distributions. ◀



■ **Figure 1** An example of a 312-avoiding permutation of length 12 with 3 fixed points marked by red dots. (A permutation $\pi = (\pi_1, \dots, \pi_n)$ avoids the pattern $p = 312$ if there is no triplet $1 \leq i < j < k \leq n$ such that $\pi_k < \pi_i < \pi_j$.)

3.2 Returns to zero in Dyck and Motzkin paths

We consider two classical directed models: Dyck and Motzkin paths. Dyck paths consist of the steps up $(1, 1)$ and down $(1, -1)$, while Motzkin paths allow additionally some horizontal steps $(1, 0)$. They are called *bridges* if they start at $(0, 0)$ and end at $(2n, 0)$, and *excursions* if it is additionally required that they never cross the x -axis.

The random variable X_n , counting the number of returns to the x -axis in a random excursion and bridge of size $2n$, is a very well studied object [4, 25], leading for excursions to a negative binomial limit law for X_n and leading for bridges to a Rayleigh limit law for X_n/\sqrt{n} . Note that the root degree in plane trees behaves the same due to a classical bijection between plane trees and Dyck excursions.

► **Theorem 9** (*q*-enumerations: limit laws for returns to zero). *The Gibbs distribution of the number $X_n = X_n(q)$ of returns to zero in Dyck excursions and bridges of length $2n$ as well as Motzkin excursions and bridges of length n , has a phase transition at q_c and follows, after suitable rescaling for $n \rightarrow +\infty$, either a negative binomial, Rayleigh, or Gaussian distribution.*

Parameter q	Limit law	$q_c = \begin{cases} 2 & \text{for Dyck excursions,} \\ 1 & \text{for Dyck bridges,} \\ \frac{3}{2} & \text{for Motzkin excursions,} \\ 1 & \text{for Motzkin bridges.} \end{cases}$
$0 < q < q_c$	$X_n - 1 \xrightarrow{\mathcal{L}} \text{NegBin}(2, 1 - q\tau_H)$	
$q = q_c$	$\frac{-c_H}{\tau_H} \frac{X_n}{\sqrt{n}} \xrightarrow{\mathcal{L}} \text{Rayleigh}(\sqrt{2})$	
$q > q_c$	$\frac{X_n - \mu \cdot n}{\sigma \cdot \sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$	

Here, τ_H is $\frac{1}{2}, 1, \frac{2}{3}, 1$, and c_H is $-\frac{1}{2}, -1, -\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}$ for Dyck excursions, Dyck bridges, Motzkin excursions and Motzkin bridges, respectively.

Proof. Cutting each time the path returns to the x -axis [25, p. 636], one directly sees that the generating functions $D(z)$ and $B_D(z)$ of Dyck excursions and bridges, respectively, satisfy the relations $D(z) = \frac{1}{1-z^2D(z)}$ and $B_D(z) = \frac{1}{1-2z^2D(z)}$. The generating functions $D(z, u)$ and $B_D(z, u)$ of Dyck excursions and bridges marking the number of returns are

$$D(z, u) = \frac{1}{1 - z^2uD(z)} \quad \text{and} \quad B_D(z, u) = \frac{1}{1 - 2z^2uD(z)}. \quad (7)$$

We can readily apply Theorem 6 to the q -enumerated returns to zero.

Motzkin excursions have the generating function

$$M(z) = \frac{1-z-\sqrt{(1+z)(1-3z)}}{2z^2},$$

with dominant singularity $\rho = \frac{1}{3}$ [25]. With the same ideas as for Dyck paths, we directly get the bivariate generating functions $M(z, u)$ and $B_M(z, u)$ of excursions and bridges, respectively, as

$$M(z, u) = \frac{1}{1 - zu(1 + zM(z))} \quad \text{and} \quad B_M(z, u) = \frac{1}{1 - zu(1 + 2zM(z))}. \quad (8)$$

This leads again to similar phase transitions. ◀

► **Remark 10** (Weighted paths). Let $p_{-1}, p_0, p_1 \geq 0$ be the weights of the steps $(1, -1), (1, 0), (1, 1)$, respectively. The weight of a path is the product of its weights. Weighted Dyck excursions and bridges behave exactly as unweighted ones, as each path of length $2n$ has a weight $(p_{-1}p_1)^n$. However, weighted Motzkin excursions and bridges behave differently. With the same techniques it is easy to show that for weighted Motzkin excursions, the phase transition occurs at




$$q_c = \frac{p_0 + 2\sqrt{p_{-1}p_1}}{p_0 + \sqrt{p_{-1}p_1}} = 1 + \frac{1}{1 + \frac{p_0}{\sqrt{p_{-1}p_1}}} \in (1, 2].$$




Finally, for weighted Motzkin bridges the phase transition again always occurs at $q_c = 1$, because $\tau_H = 1$ is independent of the weights.

3.3 Boundary contacts for quarter-plane walks

A direct byproduct of our results are phase transitions for Hadamard models of quarter-plane walks. Beaton, Owczarek, and Rechnitzer [7] initiated the study of quarter-plane walks with wall interactions. We are interested in walks restricted to the quarter plane, starting and ending at the origin, and their interaction with the walls (that is, their number of contacts with x - or y -axis). It turns out that in many models the generating functions are rather complicated [7], but for three families of walks called *Hadamard models*, the analysis of contacts can be done in a fairly simple way. Such models are enumerated by a *Hadamard product* of generating functions

$$A(z) \odot B(z) := \sum_{n \geq 0} a_n b_n z^n, \quad \text{where } A(z) = \sum_{n \geq 0} a_n z^n \quad \text{and} \quad B(z) = \sum_{n \geq 0} b_n z^n.$$

They correspond to the diagonal, diablo, and king walk models with stepsets , , , respectively (for the last two models, one has in fact a slight variant of the usual diablo or king models: here we allow additionally the step $(0, 0)$, as indicated by the centre dot in the stepset representation). Walks of length $2n$ in the quarter plane that start and end at $(0, 0)$ decompose into two independent directed excursions of length n . Therefore, these models are in bijection with pairs of Dyck and Motzkin excursions. This is summarized in the following table, where $D(z, u)$ and $M(z, u)$ are the generating functions from (7) and (8), respectively.

Model	Steps	Generating function $Q(z, u_1, u_2)$	Sequence Q_{2n}	OEIS
Diagonal		$D(z, u_1) \odot D(z, u_2)$	$C_n \cdot C_n$	A001246
Diablo		$D(z, u_1) \odot M(z, u_2)$	$C_n \cdot M_n$	A151362
King		$M(z, u_1) \odot M(z, u_2)$	$M_n \cdot M_n$	A133053

The limit laws for the number of wall interactions with the x -axis or y -axis depend on the particular values of the q -enumerations;

compare with the results in Theorem 9. More precisely, we get the following proposition.

By symmetry it also translates to y -axis contacts in the missing cases.

► **Theorem 11** (Boundary interactions for some quarter-plane walks). *The number of x -axis contacts of diagonal walks and the number of y -axis contacts of diablo walks follows the law of the q -enumeration of Dyck returns to 0. The number of x -axis contacts of diagonal, diablo, and king walks follows the law of the q -enumeration of Motzkin returns to 0. Accordingly, the phase transitions are the same as in Theorem 9.*

Proof (sketch). In the diagonal model, the generating function of the x -axis contacts is equal to $D(z, qu) \odot D(z)$. Therefore, by Lemma 2 we have

$$\mathbb{P}(X_n(q) = k) = \frac{[z^n u^k] D(z, qu) \odot D(z)}{[z^n] D(z, qu) \odot D(z)} = \frac{[z^n u^k] D(z, qu)}{[z^n] D(z, q)}.$$

Thus, the result follows directly from Theorem 9. The other models follow in the same fashion. ◀

► **Remark 12** (x -axis plus y -axis contacts). The law of the number of x -axis plus y -axis contacts is more involved, as it requires the study of the analytic behaviour of Hadamard products of the shape $(1 - z)^a \odot (1 - z)^b$. These products were studied by Fill, Flajolet, and Kapur, who gave the corresponding Puiseux expansions; see [23, Proposition 8]. Note that, as pinpointed in [23], the case $a + b$ integer requires the use of additional hypergeometric identities.

3.4 Friendly two-watermelons without wall: contacts and returns

We consider a pair of directed walkers with Dyck steps $(1, -1)$ and $(1, 1)$. The walkers start and end at the same point, may meet and share edges but not cross. Such walker configurations \mathcal{W} are also called friendly two-watermelons (see the related work of Roitner [37] and Krattenthaler, Guttmann, and Viennot [34]). We are interested in walks of length n and the number of contacts C of the two walkers. A contact in a two-watermelon is a point (not counting the starting point) where both paths occupy the same lattice point. (See Figure 2.)

► **Theorem 13** (Phase transition for contacts in friendly two-watermelons). *With the renormalizations of Theorem 6, the limit Gibbs distribution of the number of contacts in friendly two-watermelons has a phase transition with critical value $q_c = 4/3$:*

Parameter q	$0 < q < \frac{4}{3}$	$q = \frac{4}{3}$	$q > \frac{4}{3}$
Limit law of $X_n(q)$	Negative binomial $\text{NegBin}(2, 1 - \frac{3}{4}q)$	Rayleigh $\mathcal{R}(\sqrt{2})$	Gaussian $\mathcal{N}(0, 1)$

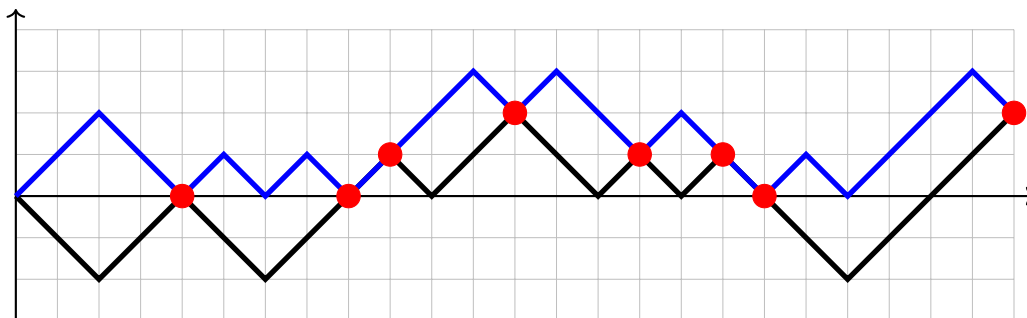
Proof. Let $F(z, u)$ denote the bivariate generating function of friendly two-watermelons with respect to the number of contacts:

$$F(z, u) = \sum_{w \in \mathcal{W}} z^{|w|} u^{C(w)}.$$

This generating function was determined by Roitner [37], using a reduction to weighted Motzkin paths. It is given by

$$F(z, u) = \frac{1}{1 - u(z^2 W(z) + 2z)}, \quad W(z) = \frac{1 - 2z - \sqrt{1 - 4z}}{2z^2}. \quad (9)$$

Under the uniform distribution model, Roitner also obtained a discrete limit law for the parameter (number of contacts). We note in passing that closely related problems in families of osculating walkers have been considered before by Bousquet-Mélou [10]. Under the Gibbs distribution model, we recognize a composition scheme with $G(z) = 1/(1 - z)$, $\rho_G = 1$, and $H(z) = z(zW(z) + 2)$ so that $\tau_H = \frac{3}{4}$. Therefore, Theorem 6 applies: we get the 3 phases, with the critical value $q_c = \frac{\rho_G}{\tau_H} = \frac{4}{3}$. ◀



■ **Figure 2** An example of a friendly two-watermelon without wall of length 24 with 8 contacts marked by red dots. Under the Gibbs model where such an object is given the weight q^8 , the distribution of the number of contacts then depends on the value of q , according to the phase transition given in Theorem 13.

4 Extensions to other constructions

It is appealing to extend Theorem 6 to even more general composition schemes such as

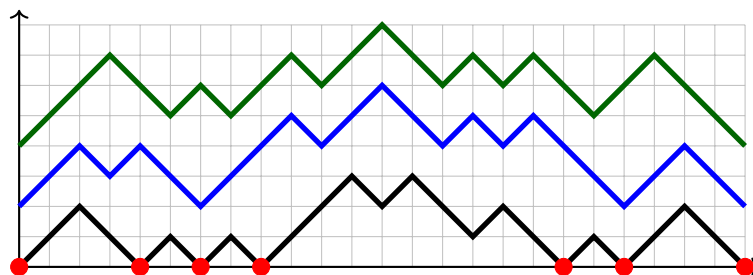
$$F(z, u) = M(z) \cdot G(uH(z)), \quad (10)$$

where an additional factor $M(z)$ appears. This extended scheme is of interest as it captures many classical combinatorial structures: some families of trees or lattice paths (meanders), Pólya urns, and other probabilistic processes like the Chinese restaurant model. In its critical phase, this scheme was recently analysed in [6] under the uniform distribution model for the associated combinatorial structures. This extends the work of pioneers like Bender, Flajolet, Soria, Drmota, Hwang, Gourdon [8, 16, 17, 26, 28, 32], later synthesized in [25].

In the long version of this article, we analyse this extended scheme under the Gibbs measure model, and we show that the phase transition for Gibbs distributions leads, in some cases, to the 3-parameter Mittag-Leffler distribution introduced in [6]. Further examples of combinatorial problems involving an extended scheme are the root degree in two-connected outerplanar graphs (see Drmota, Giménez, and Noy [15, Theorem 3.2]), the returns to zero in coloured walks, and the number of wall interactions in watermelons. We now analyse the phase transitions for these last two models.

4.1 Number of wall contacts in watermelons

An m -watermelon of length $2n$ consists of m walkers moving from $(0, 2i - 2)$ to $(2n, 2i - 2)$, $1 \leq i \leq m$, where every walker may either take an up step $(1, 1)$ or a down step $(1, -1)$, but walkers are not allowed to occupy the same position (they are thus called *vicious*). One says that the watermelon has a *wall* if the x -axis serves as a barrier which the lowest walker may touch but not cross. (See Figure 3.) Watermelons were introduced by Fisher [24] for modelling wetting and melting. We refer the reader to the work of Krattenthaler, Guttmann, and Viennot [33, 34] or Feierl [20–22] for more results on watermelons and related problems.



■ **Figure 3** An example of a 3-watermelon (with a wall) of length 24 with 7 x -axis contacts marked by red dots. Under the Gibbs model where such an object is given the weight q^7 , the distribution of the number of contacts then depends on the value of q , according to the phase transition given in Theorem 14.

We are interested in the number of contacts of the lowest walker with the x -axis. For $m = 1$ this reduces to Dyck excursions, discussed in Theorem 9. For $m \geq 1$, we get the following theorem.

► **Theorem 14** (Phase transition for wall contacts). *The limit Gibbs distribution of the number of x -axis contacts in m -watermelons with a wall has a phase transition with critical value $q_c = 2$:*

Parameter q	$0 < q < 2$	$q = 2$	$q > 2$
Limit law of $X_n(q)$	Negative binomial $\text{NegBin}(2m, 1 - \frac{q}{2})$	Chi distribution $\frac{X_n}{\sqrt{n}} \xrightarrow{\mathcal{L}} \chi(2m)$	Gaussian $\mathcal{N}(0, 1)$

Proof (sketch). The q -enumeration of the number of contacts in m -watermelons of length $2n$ ($n > 0$) with a wall was given by Krattenthaler [33, Theorem 4]:

$$f_n(q) = \frac{(n-1)! \prod_{i=0}^{m-1} (2i+1)! \prod_{i=0}^{m-2} (2n+2i)!}{\prod_{i=0}^{2m-2} (n+i)!} \sum_{\ell=2}^{n+1} \binom{2n-\ell}{n-1} \binom{\ell+2m-3}{\ell-2} q^\ell. \quad (11)$$

In the following, we denote by X_n the random variable counting the number of wall contacts, where we drop the dependence on m . Its probability generating function $\mathbb{E}(q^{X_n})$ satisfies

$$\mathbb{E}(q^{X_n}) = \frac{f_n(q)}{f_n(1)}. \quad (12)$$

Let $F(z, q)$ denote the generating function of the numerator of the reduced fraction (12), i.e.

$$F(z, q) = \sum_{n>0} z^n \sum_{\ell=2}^{n+1} \binom{2n-\ell}{n-1} \binom{\ell+2m-3}{\ell-2} q^\ell. \quad (13)$$

We change the order of summation and shift the index to get

$$F(z, q) = \sum_{\ell \geq 2} \binom{\ell+2m-3}{\ell-2} q^\ell \sum_{n \geq \ell-1} \binom{2n-\ell}{n-1} z^n = q^2 z \sum_{\ell \geq 0} \binom{\ell+2m-1}{\ell} q^\ell z^\ell \sum_{n \geq 0} \binom{2n+\ell}{n} z^n.$$

Introducing the Catalan generating function $C(z) = \frac{1-\sqrt{1-4z}}{2z}$, one has $\sum_{n \geq 0} \binom{2n+\ell}{n} z^n = \frac{C(z)^\ell}{\sqrt{1-4z}}$. While such a formula can be proven by convolution identities [29, Eq. (5.72)], it is pleasant to give a bijective proof; we invite the reader to pause here and find it before reading the next line. The bijection consists in taking a walk of length $2n + \ell$ ending at altitude ℓ , cutting it at the initial longest bridge, and after this, at the last passage at each altitude. This gives $\sum_{n \geq 0} \binom{2n+\ell}{n} z^{2n+\ell} = \frac{1}{\sqrt{1-4z}} (zC(z^2))^\ell$.

Going back to the quest for simplifying $F(z, q)$, we thus obtain

$$F(z, q) = \frac{q^2 z}{\sqrt{1-4z}} \cdot \frac{1}{(1-qzC(z))^{2m}}, \quad (14)$$

where in the last step we used the generating function identity $G(z) = \frac{1}{(1-z)^{2m}} = \sum_{\ell \geq 0} \binom{\ell+2m-1}{\ell} z^\ell$. Note that Eq. (14) suggests that there may be a bijective proof of Formula (11) using links with bridges and arches (instead of Krattenthaler's tour de force relying on determinants and jeu de taquin).

Now, in order to get the limit laws, our key observation is that Eq. (14) is a composition scheme of shape $F(z, q) = M(z)G(qH(z))$, where $H(z) = zC(z)$, and the probability generating function (under the Gibbs measure) of the number of contacts is given by

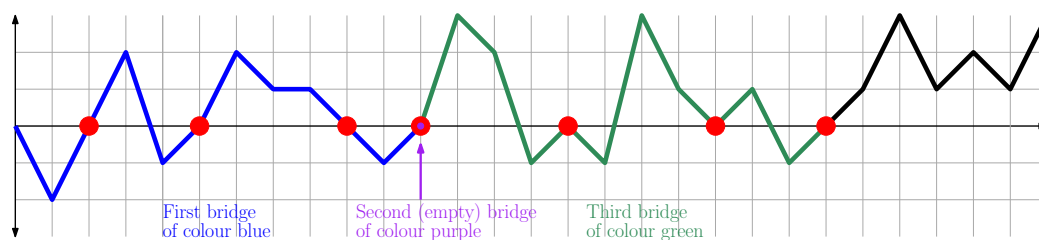
$$\mathbb{E}(v^{X_n(q)}) = \frac{[z^{n-1}v^k]F(z, qv)}{[z^{n-1}]F(z, q)}.$$

Next, we just apply singularity analysis with values $\lambda_G = -2m$, $\rho_G = 1$, $\lambda_H = \frac{1}{2}$. The scheme is critical for $q_c = 2$, where one gets the Mittag-Leffler distribution $\frac{1}{\sqrt{2}} \text{ML}(\frac{1}{2}, 2m - \frac{1}{2})$, which can be seen (from its moments) to be the same as the chi distribution $\chi(2m)$. This gives the theorem. ◀

4.2 Returns to zero in coloured walks

Let $m > 0$ be an integer. An m -coloured bridge is an m -tuple (B_1, \dots, B_m) of (possibly empty) bridges B_i . As a visual representation, we think of them appended one after the other, B_i is coloured in colour i . Note that not all colours need to be present. Let an m -coloured walk be an m -coloured bridge to which a final walk is appended that never returns to the x -axis. See Andrews [2] and [27, 30] for some combinatorial properties of these walks, and links with multicompositions.

We now prove that their number of returns to zero follows a $\chi(m)$ distribution. In particular, this gives the half-normal distribution for $m = 1$ (which extends Theorem 9 to unconstrained walks, see [40]), the Rayleigh distribution for $m = 2$, and the Maxwell distribution for $m = 3$.



■ **Figure 4** A 3-coloured walk (only 2 of the available 3 colours are present) with 7 returns to zero (red dots).

► **Theorem 15** (Phase transitions for returns to zero). *The Gibbs distribution of the number $X_n(q)$ of returns to zero in m -coloured walks of length n , has a phase transition at $q_c = 1$ and follows, after suitable rescaling for $n \rightarrow +\infty$, either a negative binomial, chi, or Gaussian distribution.*

Parameter q	$0 < q < 1$	$q = 1$	$q > 1$
Limit law of $X_n(q)$	Negative binomial $\text{NegBin}(m, 1 - q)$	Chi distribution $\chi(m)$	Gaussian $\mathcal{N}(0, 1)$

Proof (Sketch). We start by showing that returns to zero in m -coloured walks satisfy the composition scheme $F(z, q) = M(z)G(qA(z))$. First, let $B(z)$ be the generating function of bridges. Each bridge can be decomposed into a sequence of minimal bridges (or generalized arches), which have no return to zero between their extremities (yet jumps might cross the x -axis). Hence, $A(z) = 1 - \frac{1}{B(z)}$. Second, the m -tuple corresponding to an m -coloured bridge has the generating function $G(A(z))$, with $G(z) = \frac{1}{(1-z)^m}$. Third, the final part of the walk (that never returns to the x -axis) corresponds to $M(z) = W(z)/B(z)$, since each walk can be factored into an initial bridge and this final part.

Then, as before, the probability generating function (under the Gibbs measure) is given by $\mathbb{E}(v^{X_n(q)}) = \frac{[z^n v^k] F(z, qv)}{[z^n] F(z, q)}$. We get the values $\lambda_G = -m$, $\rho_G = 1$, and $\lambda_M = -1/2$. Furthermore, for any directed walk model [4], $B(z)$ has a critical exponent $\lambda_B = -1/2$, and therefore $\lambda_A = 1/2$. This also means that $B(z)$ diverges at the singularity, and thus $A(z)$ is equal to 1. Hence, the scheme is universally critical at $q_c = 1$, where we get $\frac{1}{\sqrt{2}} \text{ML}(\frac{1}{2}, \frac{m-1}{2})$, which can be seen (from its moments) to be the same as the chi distribution $\chi(m)$. Other cases follow by singularity analysis. ◀

5 Conclusion

We unified the recent results of [6] with the classical results of [8, 25] to obtain phase transitions for Gibbs models under the umbrella of composition schemes. This allows us to obtain a variety of limit laws, leading to new results, as well as summarizing and generalizing classical results in analytic combinatorics. It also explains the universality hidden behind some phase transitions up to now sporadically observed in the literature. In the full version of this article, we give several extensions of Theorem 6, thus treating many other examples of more general combinatorial constructions.

It is interesting to have an informal physicist look at our results: in statistical mechanics, the Gibbs measure can be seen as $q^k = \frac{\exp(-k/T)}{Z(1/T)}$, where T is the temperature of the model. Accordingly, $T \rightarrow 0$ (i.e. q is very small) gives a frozen “solid” phase (typically leading to a discrete distribution), while $T \rightarrow +\infty$ gives a “gaseous” phase (typically leading to a Gaussian distribution), and, around (or at) a critical temperature T_c , this gives a “liquid” phase (where the wild things are: one often observes at this location an unexpected fancy distribution). Our article is one more illustration of this informal paradigm.

Acknowledgments. The second author warmly thanks Thomas Feierl for many discussions about watermelons and Paul Schreivogl for discussions on partition functions.

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First version: November 28, 2023 (revised March 20, 2024)