SL_n-TILINGS OF LATTICES

IN COLLABORATION WITH

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FOR FIXED VECTORS ρ, v, w IN $\mathbb{R}^2$ CONSIDER THE AFFINE LATTICE

$$\mathcal{L}_\rho := \{\rho + iv + jw \mid i,j \in \mathbb{Z}\}$$

WE WRITE $[i,j]$

FOR $\rho + iv + jw$

$$\mathcal{L} = \mathcal{L}_0 := \{iv + jw \mid i,j \in \mathbb{Z}\}.$$

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"A $\mathcal{L}$-INDEXED MATRIX"

FOR A FIELD $\mathbb{K}$
CONSIDER

$$A : \mathcal{L} \longrightarrow \mathbb{K}$$

$$[i,j] \longmapsto A[i,j]$$

FOR ANY $I, J \subseteq \mathbb{Z}$
WE HAVE THE SUBMATRIX

$$A(I \times J) := \Big(A[i,j] \Big)_{\substack{i \in I \\ j \in J}}$$

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AS USUAL $[m] := \{0, 1, \ldots, m-1\}$.

LET $x \in \mathcal{L}_0$, THEN WE MAY WRITE:

$$A_x([2]\times[2]) = \begin{pmatrix} A(x+[0,0]) & A(x+[0,1]) \\ A(x+[1,0]) & A(x+[1,1]) \end{pmatrix}$$

$\mathcal{A}$ IS A $\mathbb{K}$ VALUED SL_n-TILING OF $\mathcal{L}$

IFF

$$\det \mathcal{A}_x([n]\times[n]) = 1$$

FOR ALL $x \in \mathcal{L}_0$

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$$A : \mathcal{L} \longrightarrow \mathbb{K}$$

$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$\cdots$	887	567	247	174	101	28	11	5	4	3	2	1	1	$\cdots$
$\cdots$	158	101	44	31	18	5	2	1	1	1	1	1	2	$\cdots$
$\cdots$	61	39	17	12	7	2	1	1	2	3	4	5	11	$\cdots$
$\cdots$	25	16	7	5	3	1	1	2	5	8	11	14	31	$\cdots$
	14	9	4	3	2	1	2	5	13	21	29	37	82	
$\cdots$	3	2	1	1	1	1	3	8	21	34	47	60	133	$\cdots$
$\cdots$	1	1	1	2	3	4	13	35	92	149	206	263	583	$\cdots$
$\cdots$	1	2	3	7	11	15	49	132	347	562	777	992	2199	$\cdots$
$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

SL_2-TILING

$$\det \boxed{\begin{matrix} a & b \\ c & d \end{matrix}} = 1$$

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A SL_n-TILING $\mathcal{A}$ IS SAID TO BE <u>TAME</u>

IFF $\mathrm{RANK}(\mathcal{A}) = n$

A SL_n-tiling $\mathcal{A}$ is said to be **tame** iff $\mathrm{rank}(\mathcal{A}) = n$

i.e.: it is of minimal rank

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A is <u>WILD</u> if RANK(A) > n

$$\begin{array}{cccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 1 & x_{11} & -1 & x_{12} & 1 & x_{13} & -1 & x_{14} & 1 & x_{15} & \cdots \\
\cdots & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots \\
\cdots & -1 & x_{21} & 1 & x_{22} & -1 & x_{23} & 1 & x_{24} & -1 & x_{25} & \cdots \\
\cdots & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \\
\cdots & 1 & x_{31} & -1 & x_{32} & 1 & x_{33} & -1 & x_{34} & 1 & x_{35} & \cdots \\
\cdots & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots \\
\cdots & -1 & x_{41} & 1 & x_{42} & -1 & x_{43} & 1 & x_{44} & -1 & x_{45} & \cdots \\
\cdots & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \\
\cdots & 1 & x_{51} & -1 & x_{52} & 1 & x_{53} & -1 & x_{54} & 1 & x_{55} & \cdots \\
\cdots & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

0-FREE SL_n-TILING

$$\det A_x([n-1]\times[n-1]) \neq 0$$

FOR ALL $x \in \mathcal{L}_0$

PROP ANY 0-FREE TILING IS TAME

THM IF A IS A TAME SL_m-TILING, THEN

$$\left(\det A(I \times J)\right)_{I,J \in \binom{\mathbb{Z}}{m}}$$

IS OF RANK 1.

$$M^{(k)}_{ij} := \det A_{ij}([k] \times [k])$$

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DODGSON "CONDENSATION LAW OF DETERMINANTS"

$$M_{ij}^{(r+1)}\, M_{i+1,j+1}^{(r-1)} = \det\begin{pmatrix} M_{ij}^{(r)} & M_{i,j+1}^{(r)} \\ M_{i+1,j}^{(r)} & M_{i+1,j+1}^{(r)} \end{pmatrix}$$

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Example

$$\det\begin{pmatrix} a_{ij} & a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j} & a_{i+1,j+1} & a_{i+1,j+2} \\ a_{i+2,j} & a_{i+2,j+1} & a_{i+2,j+2} \end{pmatrix} \det\left(a_{i+1,j+1}\right) =$$

$$\det\begin{pmatrix} \begin{vmatrix} a_{ij} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} & \begin{vmatrix} a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j+1} & a_{i+1,j+2} \end{vmatrix} \\ \\ \begin{vmatrix} a_{i+1,j} & a_{i+1,j+1} \\ a_{i+2,j} & a_{i+2,j+1} \end{vmatrix} & \begin{vmatrix} a_{i+1,j+1} & a_{i+1,j+2} \\ a_{i+2,j+1} & a_{i+2,j+2} \end{vmatrix} \end{pmatrix}$$

DUAL OF A TAME SL_m-TILING

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$$\partial_m A : \mathcal{L}_{(m-1)\rho} \longrightarrow \mathbb{K}$$

$$(\partial_m A)[i,j] := \det A_{ij}([m]\times[m])$$

$$\rho := \frac{1}{2}(v+w)$$

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$$\mathcal{A}^* := \partial_{n-1}\mathcal{A} \qquad \text{DUAL}$$

THM IF $\mathcal{A}$ IS A TAME SL_n-TILING, THEN $\mathcal{A}^*$ IS A TAME SL_n-TILING AND $(\mathcal{A}^*)^* = \mathcal{A}$

$$\mathcal{L}_{n\rho} = \begin{cases} \mathcal{L} & \text{if } n \in 2\mathbb{Z} \\ \mathcal{L}_\rho & \text{if } n \in 2\mathbb{Z}+1 \end{cases}$$

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A SL_3-Tiling and its Dual

⋱	⋮		⋮		⋮		⋮		⋮		⋮		⋮		⋮	⋰
⋯	8997		1782		353		70		14		3		1		1	⋯
		417		131		42		14		5		2		1		
⋯	1782		353		70		14		3		1		1		2	⋯
		131		42		14		5		2		1		1		
⋯	353		70		14		3		1		1		2		5	⋯
		42		14		5		2		1		1		3		
⋯	70		14		3		1		1		2		5		14	⋯
		14		5		2		1		1		3		14		
⋯	14		3		1		1		2		5		14		42	⋯
		5		2		1		1		3		14		70		
⋯	3		1		1		2		5		14		42		131	⋯
		2		1		1		3		14		70		353		
⋯	1		1		2		5		14		42		131		417	⋯
⋰	⋮		⋮		⋮		⋮		⋮		⋮		⋮		⋮	⋱

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SL_m-TILING ASSOCIATED TO A PATH

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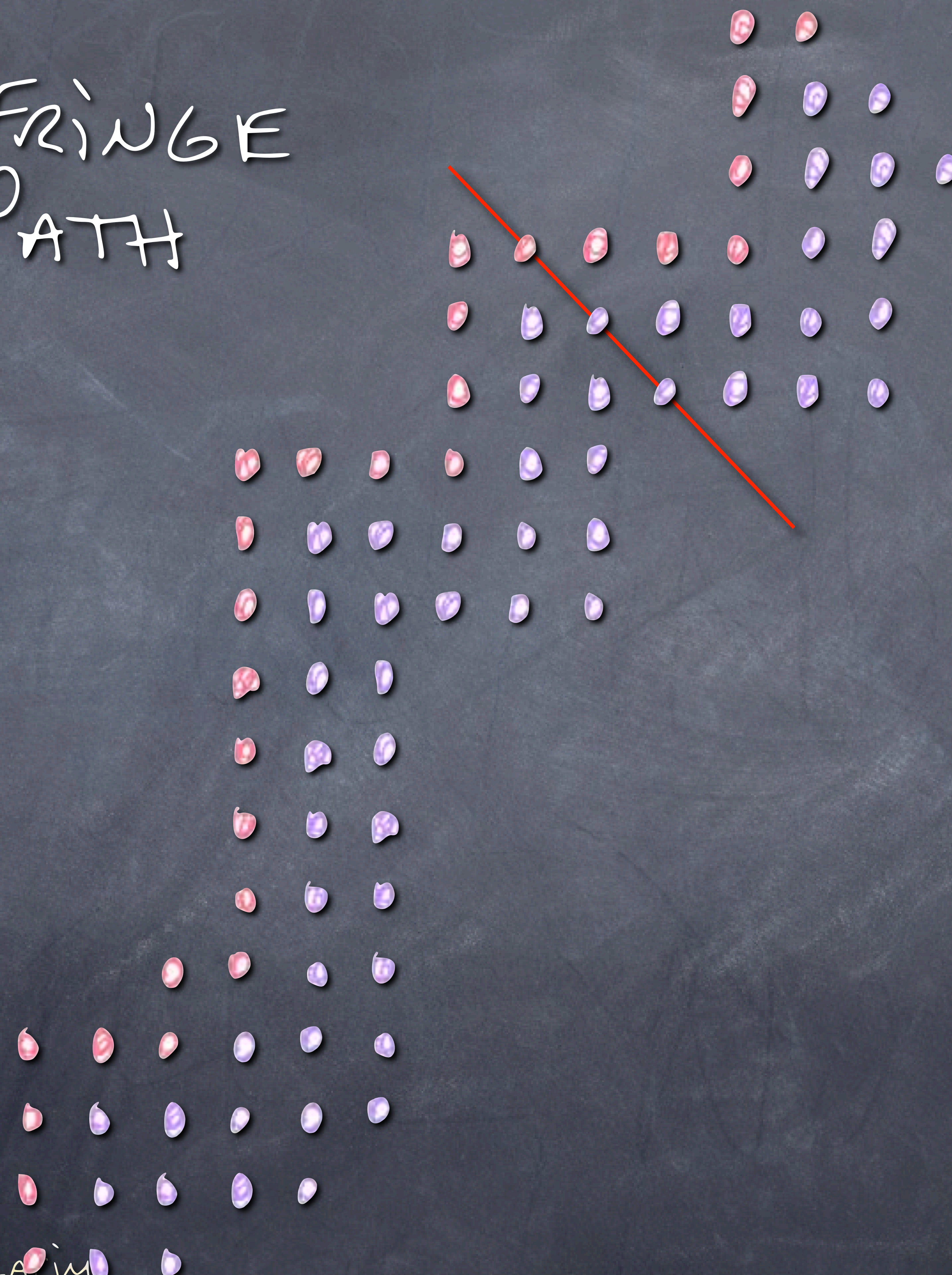
THE 3-FRINGE OF A PATH



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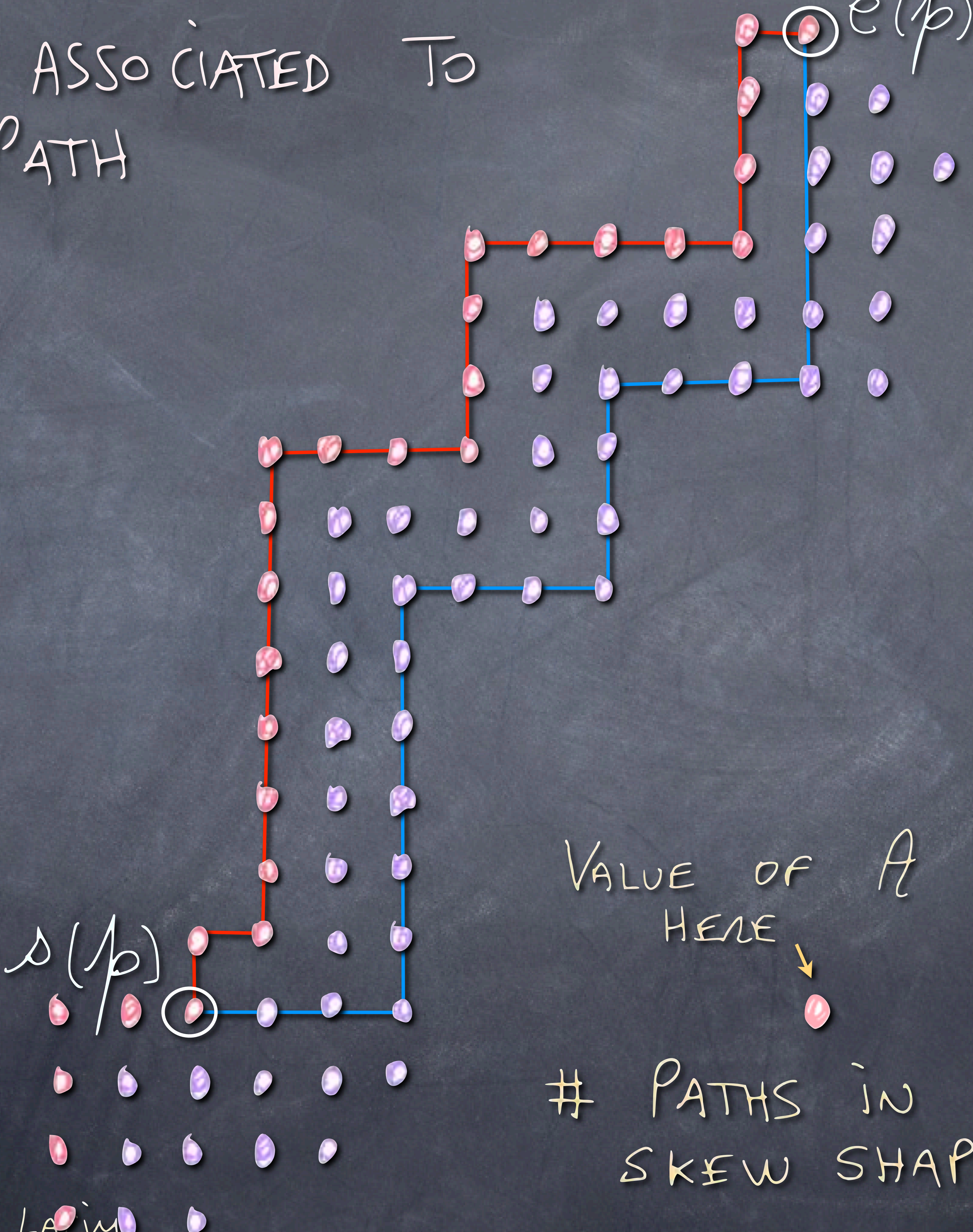
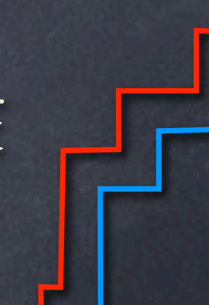
SL_n-Tiling associated to a path



Value of A here

\# Paths in the skew shape



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CHEMINS DE DYCK DE HAUTEUR ≤ 3

1157954	229347	45425	8997	1782	353	70	14	3	1	1
229347	45425	8997	1782	353	70	14	3	1	1	2
45425	8997	1782	353	70	14	3	1	1	2	5
8997	1782	353	70	14	3	1	1	2	5	14
1782	353	70	14	3	1	1	2	5	14	42
353	70	14	3	1	1	2	5	14	42	131
70	14	3	1	1	2	5	14	42	131	417
14	3	1	1	2	5	14	42	131	417	1341
3	1	1	2	5	14	42	131	417	1341	4334
1	1	2	5	14	42	131	417	1341	4334	14041
1	2	5	14	42	131	417	1341	4334	14041	45542

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THM

$$\underbrace{\text{FOR ANY ADMISSIBLE PATH}}_{}$$

GOES FROM $(-\infty,-\infty)$ TO (∞,∞)

FOR ANY ADMISSIBLE PATH, THERE IS A UNIQUE TAME SL_m-TILING, WITH VALUES GIVEN BY PATH ENUMERATION FOR POINTS LYING BELOW THE PATH.

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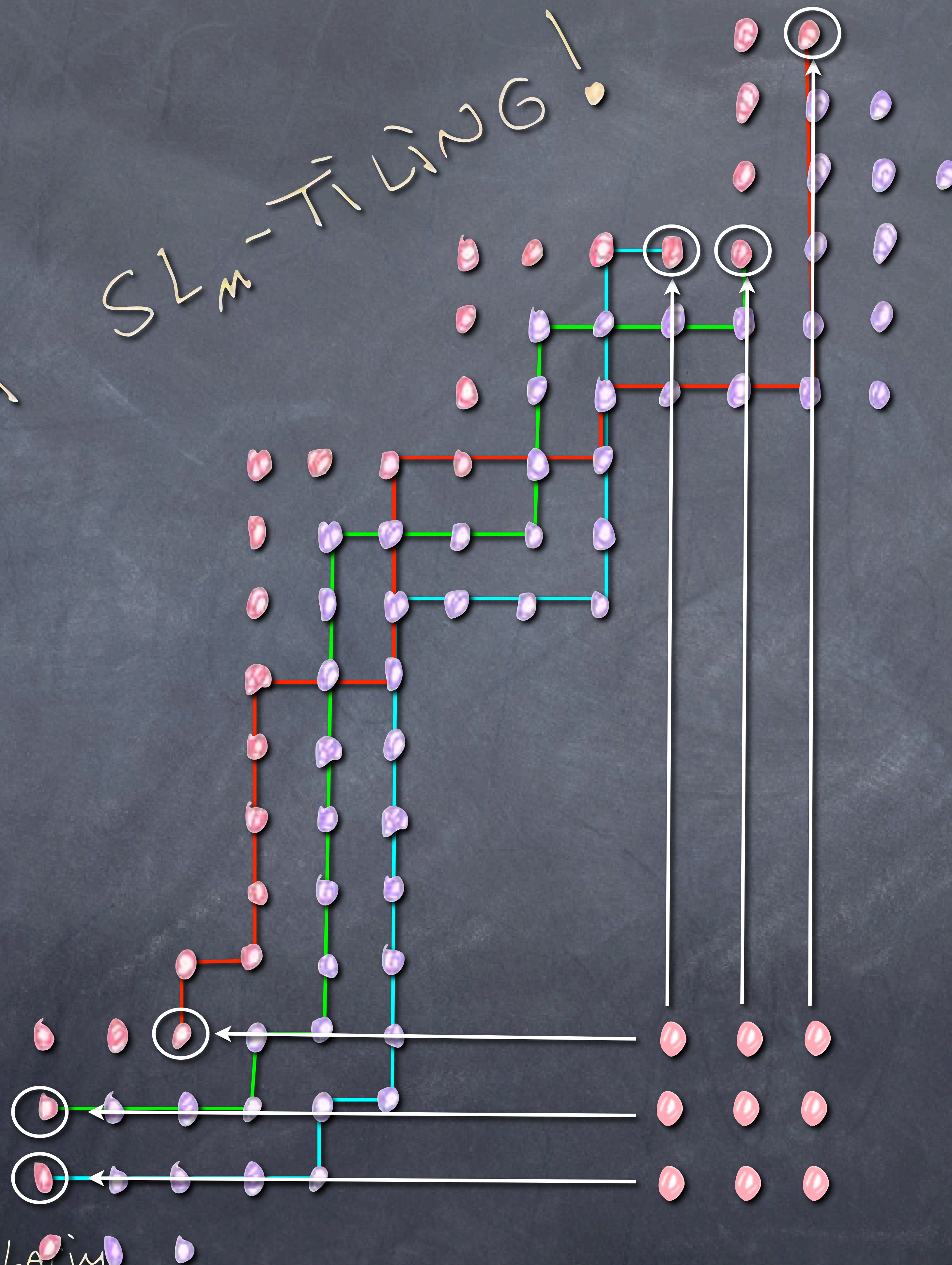
IT IS A SL_m-TILING!



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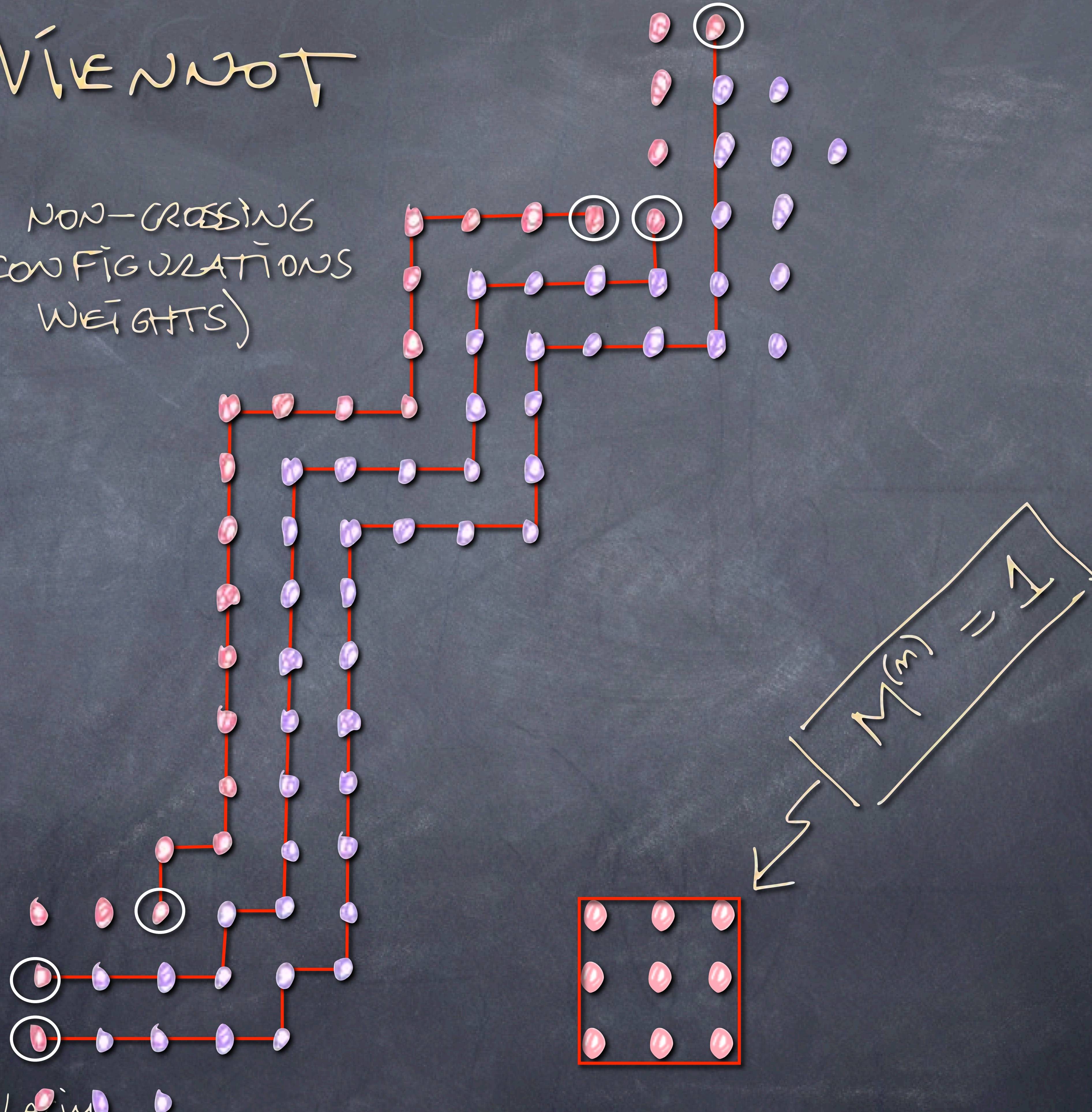
GESSEL-VIENNOT

det = # OF NON-CROSSING PATH CONFIGURATIONS (WITH WEIGHTS)



TAMENESS = NO $(m+1)$

NON CROSSING CONFIGURATION

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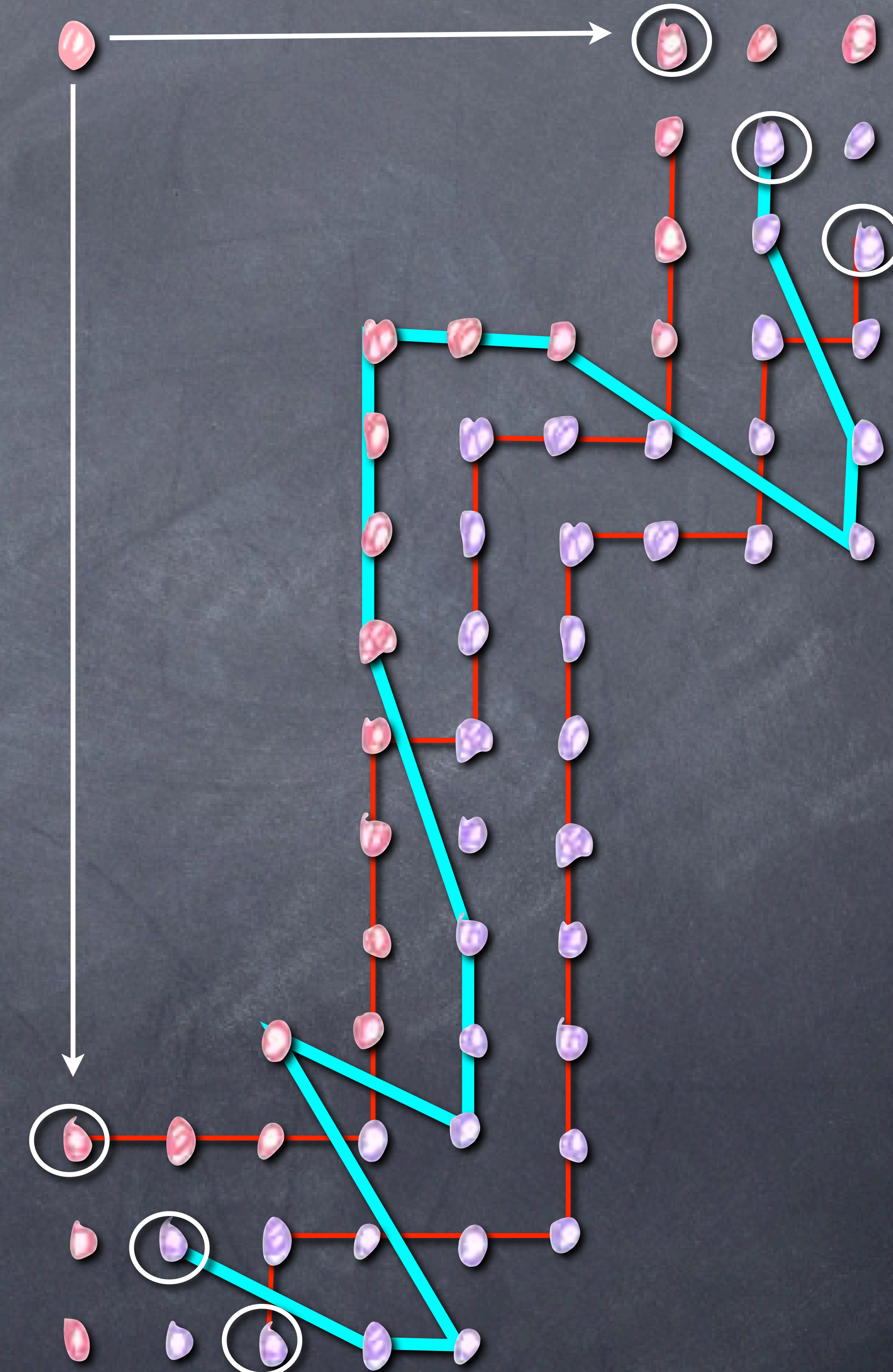
DUAL = $(n-1)$ NON

CROSSING CONFIGURATIONS

= BIG PATH



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WEIGHTED ENUMERATION OF PATHS

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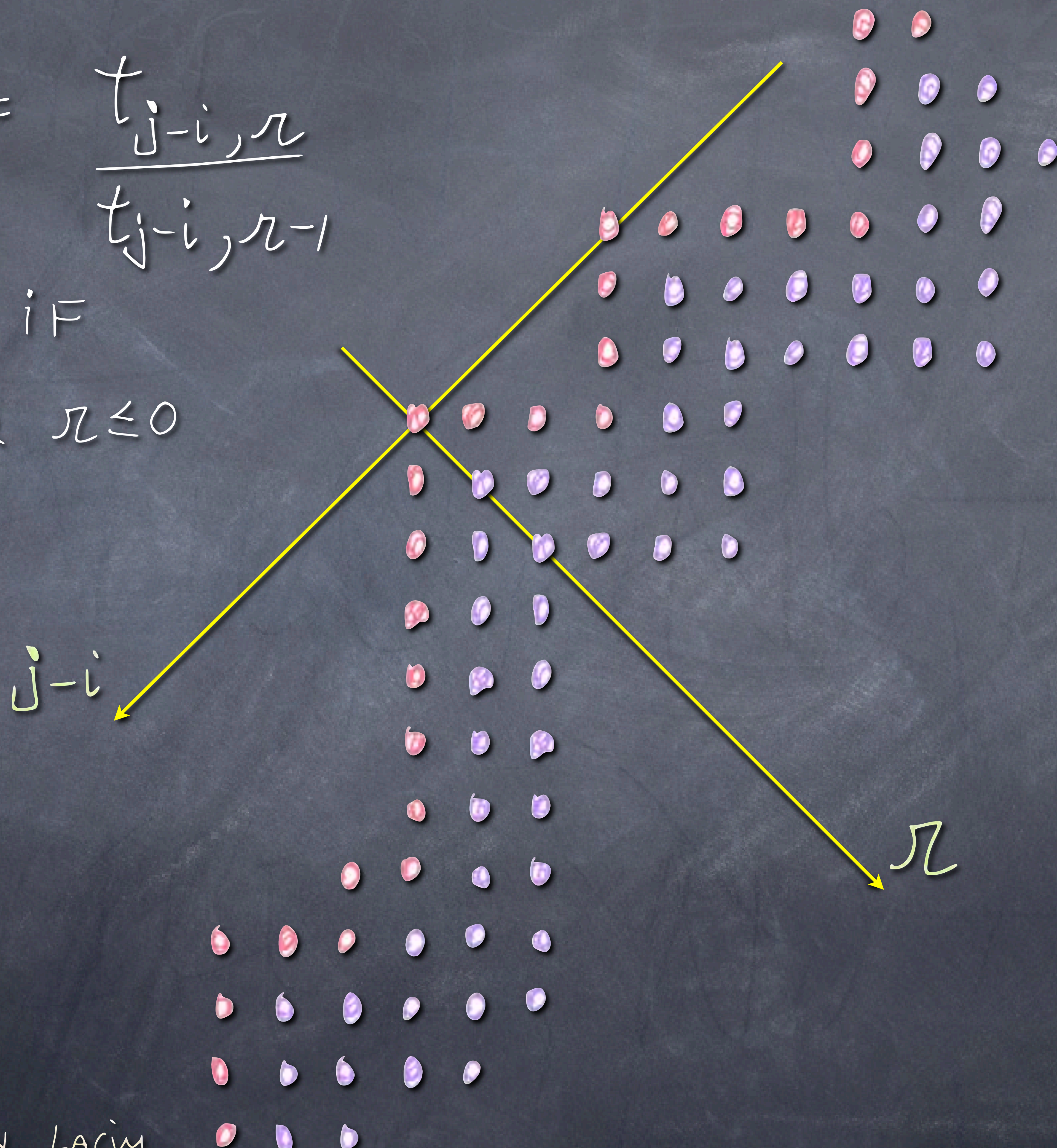
$$v(i,j) = \frac{t_{j-i,r}}{t_{j-i,r-1}}$$

$t_{k,r} = 1$ iF

$n \le r$ OR $r \le 0$



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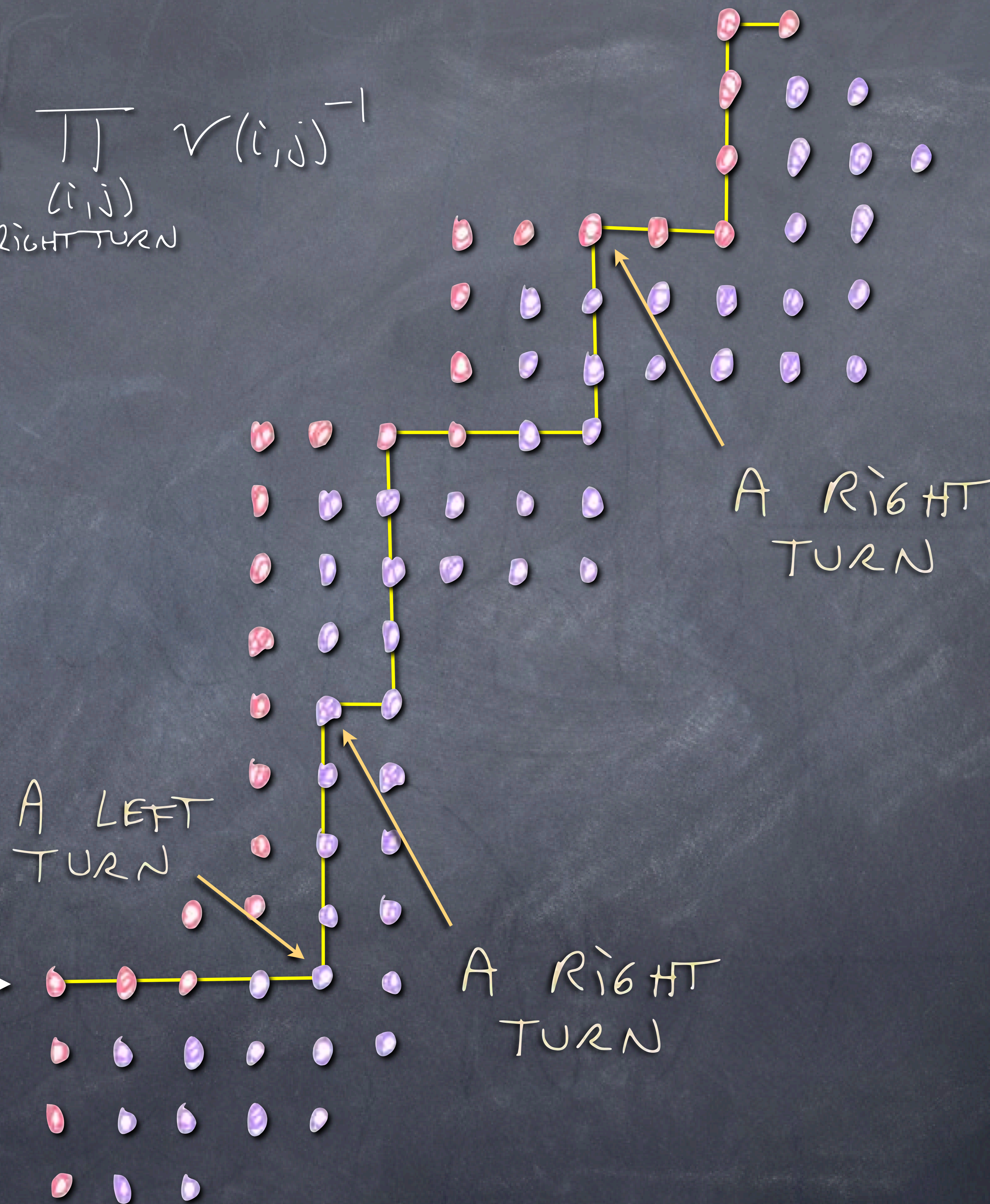
$$\omega(\theta) := \prod_{\substack{(i,j)\\ \text{LEFT TURN}}} r(i,j) \prod_{\substack{(i,j)\\ \text{RIGHT TURN}}} r(i,j)^{-1}$$

θ A PATH $\longrightarrow$



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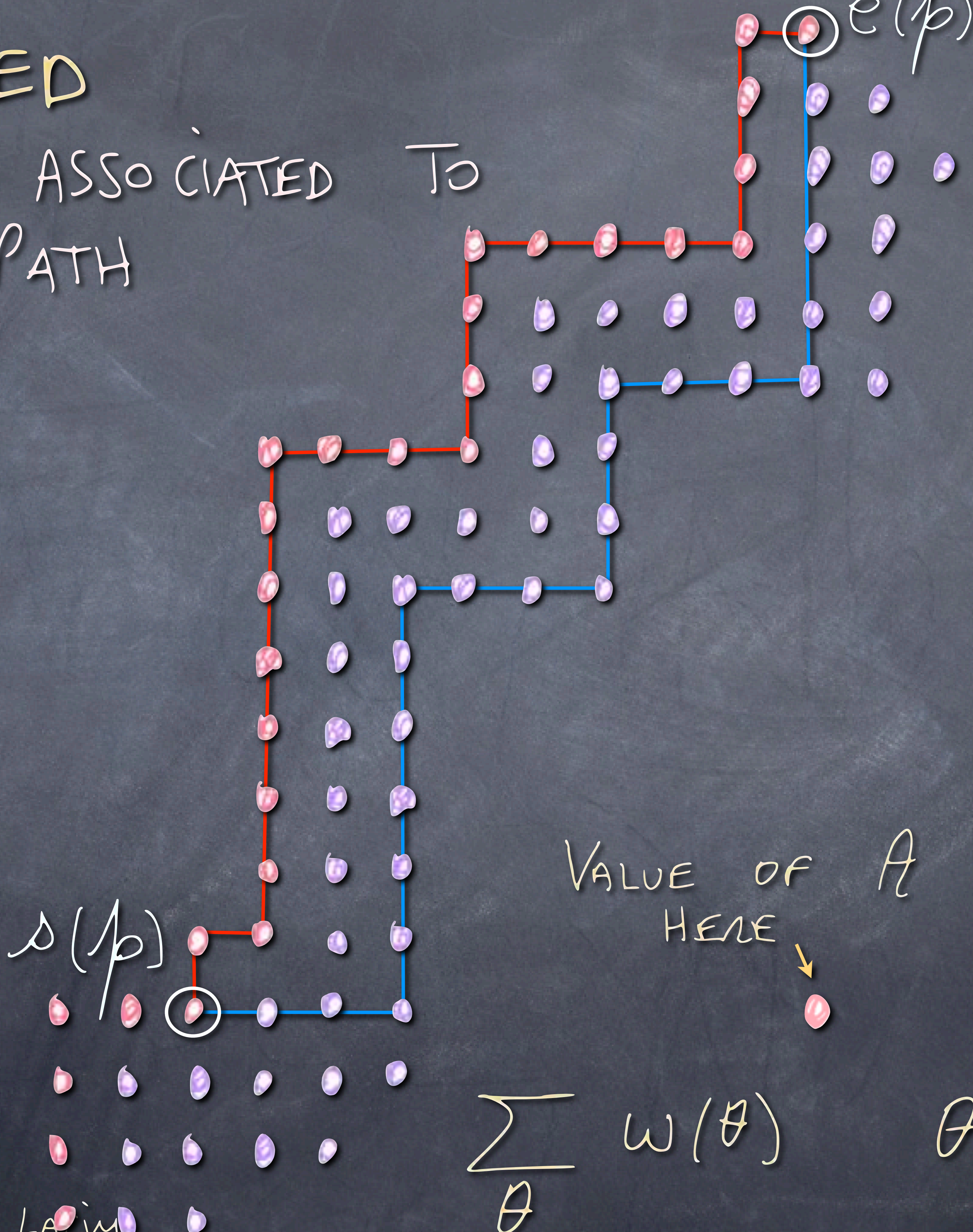
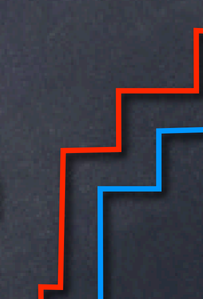
Weighted

SL_n-Tiling associated to a path



$$\sum_{\theta} w(\theta)$$ θ in 

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LAURENT PHENOMENON

THM

FOR ANY ADMISSIBLE PATH, THERE IS A UNIQUE TAME SL_m-TILING WITH ENTRIES POSITIVE INTEGER COEFF. LAURENT POLYNOMIALS IN THE VARIABLES $t_{k,r}$.

THESE ARE OBTAINED BY WEIGHTED ENUMERATION FOR POINTS LYING BELOW THE PATH.

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Example

A SL_3-tiling

$$
\begin{array}{ccccc}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \dfrac{1}{t_{02}}+\dfrac{t_{11}t_{\bar{1}1}}{t_{01}}+\dfrac{t_{12}t_{\bar{1}2}}{t_{01}t_{02}} & t_{11} & t_{21} & \cdots \\
\cdots & t_{\bar{1}1} & t_{01} & \dfrac{t_{12}}{t_{11}}+\dfrac{t_{01}t_{21}}{t_{11}} & \cdots \\
\cdots & t_{\bar{2}1} & \dfrac{t_{\bar{1}2}}{t_{\bar{1}1}}+\dfrac{t_{01}t_{\bar{2}1}}{t_{\bar{1}1}} & \dfrac{t_{02}}{t_{01}}+\dfrac{t_{21}t_{01}t_{\bar{2}1}}{t_{11}t_{\bar{1}1}}+\dfrac{t_{12}t_{\bar{2}1}}{t_{11}t_{\bar{1}1}}+\dfrac{t_{21}t_{\bar{1}2}}{t_{11}t_{\bar{1}1}}+\dfrac{t_{12}t_{\bar{1}2}}{t_{11}t_{1}t_{\bar{1}1}} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

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FRIEZE PATTERNS

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2 Triangulated Polygons and Frieze Patterns

	0		0		0		0		0		0		0		0		0		0		0		0	
1		1		1		1		1		1		1		1		1		1		1		1		1
	1		2		2		3		1		2		4		1		2		2		3		1	
3		1		3		5		2		1		7		3		1		3		5		2		1
	2		1		7		3		1		3		5		2		1		7		3		1	
3		1		2		4		1		2		2		3		1		2		4		1		2
	1		1		1		1		1		1		1		1		1		1		1		1	
0		0		0		0		0		0		0		0		0		0		0		0		0

Table 1.

The first author once asked an audience of a hundred students of mathematics to look at the pattern in Table 1 and find the simple rule connecting each number with its neighbors and allowing the pattern to be extended indefinitely to the right and left. After an embarrassingly long time it was necessary to break the suspense by explaining that any four numbers forming a diamond, such as

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfy the relation $ad - bc = 1$, which may also be written $c = (ad - 1)/b$; this is called the **unimodular rule.** Later, to test the effect of a brilliant brain, the same pattern was shown to Paul Erdős; he needed only a few seconds!

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0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1 1

1 2 2 3 1 2 4 1 2 2 3 1

3 1 3 5 2 1 7 3 1 3 5 2 1

2 1 7 3 1 3 5 2 1 7 3 1

3 1 2 4 1 2 2 3 1 2 4 1 2

1 1 1 1 1 1 1 1 1 1 1 1

0 0 0 0 0 0 0 0 0 0 0 0 0

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0 1 1 1 3 2 1 0

0 1 2 $\boxed{\begin{matrix} 7 & 5 \\ 4 & 3 \end{matrix}}$ 3 1 0

0 1 4 3 2 1 1 0

0 1 1 1 1 2 1 0

0 1 2 3 7 4 1 0

0 1 2 5 3 1 1 0

0 1 3 2 1 2 1 0

0 1 1 1 3 2 1 0

0 1 2 7 5 3 1 0

0 1 4 3 2 1 1

0 1 1 1 1 2

0 1 2 3 7

$$\det \boxed{\begin{matrix} a & b \\ c & d \end{matrix}} = 1$$

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0	1	1	1	3	2	1	0								
	0	1	2	7	5	3	1	0							
		0	1	4	3	2	1	1	0						
			0	1	1	1	1	2	1	0					
				0	1	2	3	7	4	1	0				
					0	1	2	5	3	1	1	0			
						0	1	3	2	1	2	1	0		
							0	1	1	1	3	2	1	0	
								0	1	2	7	5	3	1	0
									0	1	4	3	2	1	1
										0	1	1	1	1	2
											0	1	2	3	7

1 2 4 1 2 2 3 1 2 4 1 2

Frises à valeurs dans $\mathbb{N}^+$

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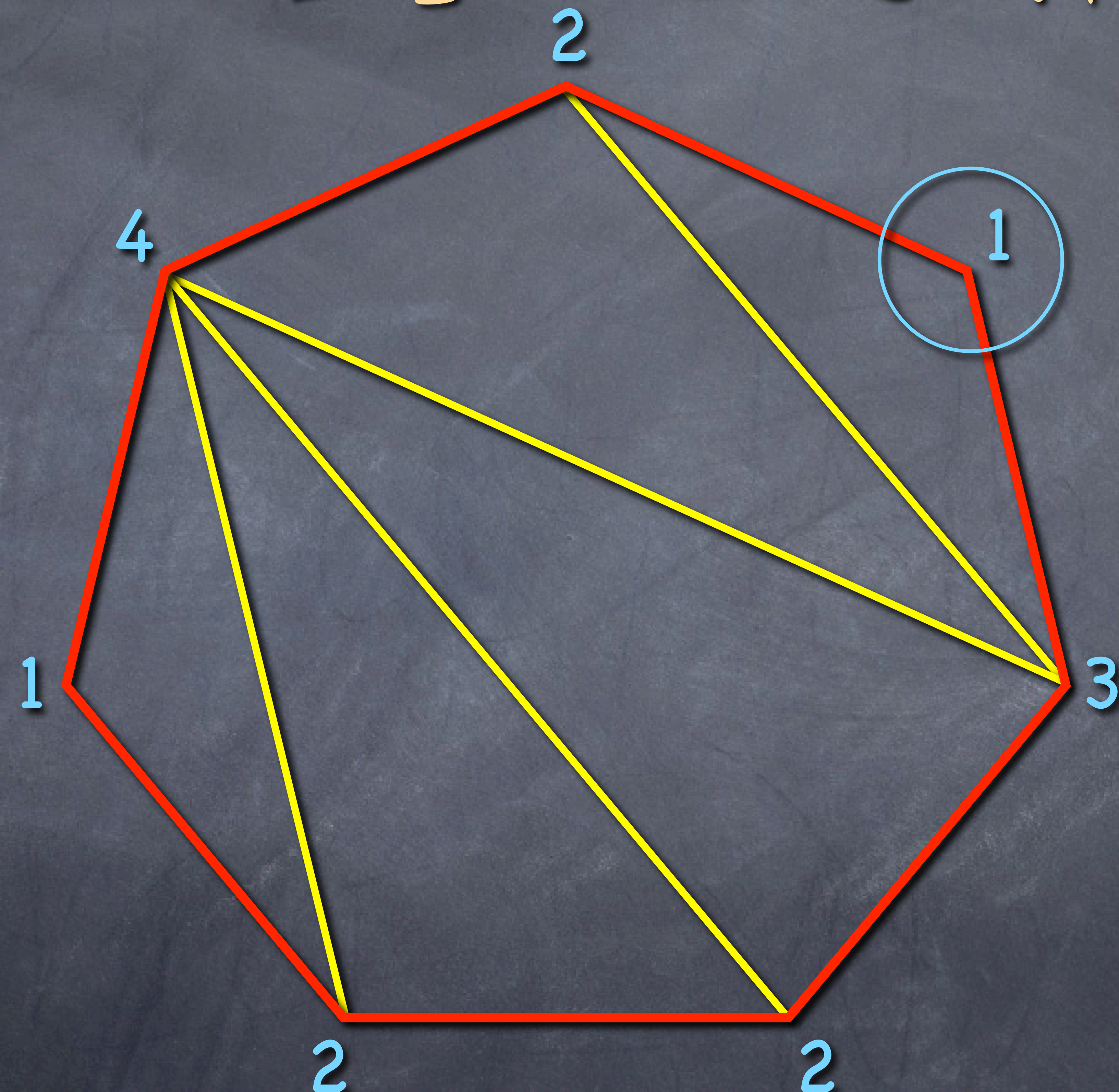
SEULES SOLUTIONS



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EXTENSION PÉRIODIQUE DOCILE

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Extension périodique docile

1	1	3	2	1	0	-1	-1	-1	-3	-2	-1
1	2	7	5	3	1	0	-1	-2	-7	-5	-3
0	1	4	3	2	1	1	0	-1	-4	-3	-2
-1	0	1	1	1	1	2	1	0	-1	-1	-1
-4	-1	0	1	2	3	7	4	1	0	-1	-2
-3	-1	-1	0	1	2	5	3	1	1	0	-1
-2	-1	-2	-1	0	1	3	2	1	2	1	0
-1	-1	-3	-2	-1	0	1	1	1	3	2	1
-1	-2	-7	-5	-3	-1	0	1	2	7	5	3
0	-1	-4	-3	-2	-1	-1	0	1	4	3	2
1	0	-1	-1	-1	-1	-2	-1	0	1	1	1
4	1	0	-1	-2	-3	-7	-4	-1	0	1	2

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$$c = \frac{1+b}{a} \qquad d = \frac{1+a+b}{a\,b} \qquad e = \frac{1+a}{b}.$$

a	b	1	0	-1	$-a$	$-b$	-1	0	1	a	b
1	c	d	1	0	-1	$-c$	$-d$	-1	0	1	c
0	1	e	a	1	0	-1	$-e$	$-a$	-1	0	1
-1	0	1	b	c	1	0	-1	$-b$	$-c$	-1	0
$-e$	-1	0	1	d	e	1	0	-1	$-d$	$-e$	-1
$-a$	$-b$	-1	0	1	a	b	1	0	-1	$-a$	$-b$
-1	$-c$	$-d$	-1	0	1	c	d	1	0	-1	$-c$
0	-1	$-e$	$-a$	-1	0	1	e	a	1	0	-1
1	0	-1	$-b$	$-c$	-1	0	1	b	c	1	0
e	1	0	-1	$-d$	$-e$	-1	0	1	d	e	1
a	b	1	0	-1	$-a$	$-b$	-1	0	1	a	b
1	c	d	1	0	-1	$-c$	$-d$	-1	0	1	c

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$$
\begin{array}{ccccccccccc}
a & 1 & 0 & 0 & 1 & a & 1 & 0 & 0 & 1 & a \\
1 & \frac{1+b}{a} & 1 & 0 & 0 & 1 & \frac{1+b}{a} & 1 & 0 & 0 & 1 \\
0 & 1 & \frac{a+1}{b} & 1 & 0 & 0 & 1 & \frac{a+1}{b} & 1 & 0 & 0 \\
0 & 0 & 1 & b & 1 & 0 & 0 & 1 & b & 1 & 0 \\
1 & 0 & 0 & 1 & \frac{1+b+a}{ab} & 1 & 0 & 0 & 1 & \frac{1+b+a}{ab} & 1 \\
a & 1 & 0 & 0 & 1 & a & 1 & 0 & 0 & 1 & a \\
1 & \frac{1+b}{a} & 1 & 0 & 0 & 1 & \frac{1+b}{a} & 1 & 0 & 0 & 1 \\
0 & 1 & \frac{a+1}{b} & 1 & 0 & 0 & 1 & \frac{a+1}{b} & 1 & 0 & 0 \\
0 & 0 & 1 & b & 1 & 0 & 0 & 1 & b & 1 & 0 \\
1 & 0 & 0 & 1 & \frac{1+b+a}{ab} & 1 & 0 & 0 & 1 & \frac{1+b+a}{ab} & 1 \\
a & 1 & 0 & 0 & 1 & a & 1 & 0 & 0 & 1 & a
\end{array}
$$

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SL_3-Tiling: $\mathcal{A}$

a	b	1	0	0	1	a	b
1	$\frac{x+b}{a}$	$\frac{ay+x+b}{ab}$	1	0	0	1	$\frac{x+b}{a}$
0	1	$\frac{ay+x+b}{bx}$	$\frac{ay+x+b}{yx}$	1	0	0	1
0	0	1	$\frac{x+b}{y}$	x	1	0	0
1	0	0	1	y	$\frac{ay+x+b}{ax}$	1	0
$\frac{ay+x+b}{by}$	1	0	0	1	$\frac{(ay+x+b)(x+b)}{abyx}$	$\frac{ay+x+b}{by}$	1
a	b	1	0	0	1	a	b
1	$\frac{x+b}{a}$	$\frac{ay+x+b}{ab}$	1	0	0	1	$\frac{x+b}{a}$
0	1	$\frac{ay+x+b}{bx}$	$\frac{ay+x+b}{yx}$	1	0	0	1
0	0	1	$\frac{x+b}{y}$	x	1	0	0

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DUAL: $\mathcal{A}^*$

x	y	1	0	0	1	x
1	$\frac{ay+x+b}{ax}$	$\frac{(ay+x+b)(x+b)}{abyx}$	1	0	0	1
0	1	$\frac{ay+x+b}{by}$	a	1	0	0
0	0	1	b	$\frac{x+b}{a}$	1	0
1	0	0	1	$\frac{ay+x+b}{ab}$	$\frac{ay+x+b}{bx}$	1
$\frac{x+b}{y}$	1	0	0	1	$\frac{ay+x+b}{yx}$	$\frac{x+b}{y}$
x	y	1	0	0	1	x
1	$\frac{ay+x+b}{ax}$	$\frac{(ay+x+b)(x+b)}{abyx}$	1	0	0	1
0	1	$\frac{ay+x+b}{by}$	a	1	0	0

PAVAGE SL_R DE $\mathbb{N} \times \mathbb{N}$

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SOIT $S \in SL_k$ ET

$$A = \begin{pmatrix} S & \Delta \\ \Gamma & \chi \end{pmatrix}$$

UNE MATRICE DE RANG k

ALORS

$$\chi = \Gamma S^{-1} \Delta$$

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Preuve :

$$\begin{pmatrix} S & \Delta \\ \Gamma & \chi \end{pmatrix} \begin{pmatrix} \mathrm{Id}_k & -S^{-1}\Delta \\ 0 & I_n \end{pmatrix} =$$

$$\begin{pmatrix} S & 0 \\ \Gamma & \chi - \Gamma S^{-1}\Delta \end{pmatrix}$$

$$\Longrightarrow \boxed{\chi - \Gamma S^{-1}\Delta = 0}$$

François Bergeron, LaCIM

FONCTION GÉNÉRATRICE DANS LE CAS DOCILE

$$A(x,y) = X\,\Gamma\,S^{-1}\,\Delta\,Y$$

où

$$X = (x_i^i)_{0\leq i}$$

$$Y = (y^j)_{0\leq j}^{TR}$$

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FONCTION GÉNÉRATRICE

DANS LE CAS DOCILE

$$A(x,y) = \begin{pmatrix} C_1(x) & \dots & C_k(x) \end{pmatrix} S^{-1} \begin{pmatrix} L_1(y) \\ \vdots \\ L_k(y) \end{pmatrix}$$

$C_i(x)$ FONCTION GÉNÉRATRICE DE LA i-IÈME COLONE DE A

(SEMBLABLE POUR $L_j(y)$)

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Exemple

$$C_i(x) := \frac{1}{(1-x)^i},$$

$$L_j(y) := \frac{1}{(1-y)^j},$$

$$S := \mathrm{Id}_k.$$

EXEMPLE

$$A(x,y) = \sum_{m=1}^{k} \frac{x^{m-1}\, y^{m-1}}{(1-x)^m\,(1-y)^m}$$

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EXEMPLE

$$A(x,y) = \sum_{m=1}^{k} \frac{x^{m-1}\, y^{m-1}}{(1-x)^m\,(1-y)^m}$$

$$a_{ij} = \sum_{l=0}^{k-1} \binom{i}{l}\binom{j}{l}$$

EXEMPLE $\boxed{k=3}$

1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10
1	3	6	10	15	21	28	36	45	55
1	4	10	19	31	46	64	85	109	136
1	5	15	31	53	81	115	155	201	253
1	6	21	46	81	126	181	246	321	406
1	7	28	64	115	181	262	358	469	595
1	8	36	85	155	246	358	491	645	820
1	9	45	109	201	321	469	645	849	1081
1	10	55	136	253	406	595	820	1081	1378