# An example of generalization of the Bernoulli numbers and polynomials to any dimension

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# Introduction

**Definition:**

The numbers $\mathcal{Z}e^{s_1,\cdots,s_r}$ defined by

$$\mathcal{Z}e^{s_1,\cdots,s_r} = \sum_{0<n_r<\cdots<n_1} \frac{1}{{n_1}^{s_1}\cdots{n_r}^{s_r}} \ ,$$

where $s_1,\cdots,s_r \in \mathbb{C}$ such that $\Re(s_1+\cdots+s_k) > k$, $k \in [\![1\,;\,r]\!]$, are called multiple zeta values.

**Fact:** There exist at least three different way to renormalize multiple zeta values at negative integers.

$$\mathcal{Z}e_{MP}^{0,-2}(0) = \frac{7}{720} \quad , \quad \mathcal{Z}e_{GZ}^{0,-2}(0) = \frac{1}{120} \quad , \quad \mathcal{Z}e_{FKMT}^{0,-2}(0) = \frac{1}{18} \ .$$

**Question:** Is there a group acting on the set of all possible multiple zeta values renormalisations?

**Main goal:** Define multiple Bernoulli numbers in relation with this.

# Outline


1 Reminders
- Reminders on Bernoulli Polynomials and Numbers
- Reminders on Hurwitz Zeta Function and Hurwitz multiple zeta functions
- Reminders on Quasi-Symmetric Functions

2 Algebraic reformulation of the problem

3 The Structure of a Multiple Bernoulli Polynomial

4 The General Reflexion Formula of Multiple Bernoulli Polynomial

5 An Example of Multiple Bernoulli Polynomial

6 An algorithm to compute the double Bernoulli Numbers

7 Properties satisfied by Bernoulli polynomials and numbers

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# Two Equivalent Definitions of Bernoulli Polynomials / Numbers

| **Bernoulli numbers:** | **Bernoulli polynomials:** |
|---|---|
| By a generating function: $$\frac{t}{e^t - 1} = \sum_{n \geq 0} b_n \frac{t^n}{n!} \ .$$ | By a generating function: $$\frac{te^{xt}}{e^t - 1} = \sum_{n \geq 0} B_n(x) \frac{t^n}{n!} \ .$$ |
| By a recursive formula: $$\begin{cases} b_0 = 1 \ , \\ \forall n \in \mathbb{N} \ , \ \sum_{k=0}^{n} \binom{n+1}{k} b_k = 0 \ . \end{cases}$$ | By a recursive formula: $$\begin{cases} B_0(x) = 1 \ , \\ \forall n \in \mathbb{N} \ , \ B'_{n+1}(x) = (n+1) B_n(x) \ , \\ \forall n \in \mathbb{N}^* \ , \ \int_0^1 B_n(x) \ dx = 0 \ . \end{cases}$$ |
| First examples: $$b_n = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \cdots$$ | First examples: $$\begin{aligned} B_0(x) &= 1 \ , \\ B_1(x) &= x - \frac{1}{2} \ , \\ B_2(x) &= x^2 - x + \frac{1}{6} \ , \\ &\vdots \end{aligned}$$ |

# Elementary properties satisfied by the Bernoulli polynomials and numbers

**P1** $b_{2n+1} = 0 \text{ if } n > 0.$

**P2** $B_n(0) = B_n(1) \text{ if } n > 1.$

**P3** $$\sum_{k=0}^{m} \binom{m+1}{k} b_k = 0, \ \ m > 0.$$

**P4** $$\begin{cases} B_n'(z) = nB_{n-1}(z) \text{ if } n > 0. \\ B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k} \text{ for all } n. \end{cases}$$

**P5** $B_n(x+1) - B_n(x) = nx^{n-1}$, for all $n$.

**P6** $(-1)^n B_n(1-x) = B_n(x)$, for all $n$.

**P7** $$\sum_{k=0}^{N-1} k^n = \frac{B_{n+1}(N) - B_{n+1}(0)}{n+1} .$$

**P8** $$\int_a^x B_n(t)\, dt = \frac{B_{n+1}(x) - B_{n+1}(a)}{n+1} .$$

**P9** $$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) \text{ for all } m > 0 \text{ and } n \geq 0.$$

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- Reminders on Hurwitz Zeta Function and Hurwitz multiple zeta functions
- Reminders on Quasi-Symmetric Functions

**Definition:**

The Hurwitz Zeta Function is defined, for $\Re e\ s > 1$, and $z \in \mathbb{C} - \mathbb{N}_{<0}$, by:

$$\zeta(s,z) = \sum_{n>0} \frac{1}{(n+z)^s} .$$

**Property:**

**H1** $$\begin{cases} \dfrac{\partial \zeta}{\partial z}(s,z) = -s\zeta(s+1,z). \\ \zeta(s,x+y) = \displaystyle\sum_{n\geq 0} \binom{-s}{n} \zeta(s+n,x) y^n. \end{cases}$$

**H2** $\zeta(s,z-1) - \zeta(s,z) = z^{-s}$.

**H3** $\zeta(s,mz) = m^{-s} \displaystyle\sum_{k=0}^{m-1} \zeta\left(s, z+\frac{k}{m}\right)$ if $m \in \mathbb{N}^*$.

# Link between the Hurwitz Zeta Function and the Bernoulli polynomials

**Property:**

$s \longmapsto \zeta(s,z)$ can be analytically extend to a meromorphic function on $\mathbb{C}$, with a simple pole located at 1.

**Remark:**

$$\zeta(-n,z) = -\frac{B_{n+1}(z)}{n+1} \text{ for all } n \in \mathbb{N} .$$

$$\zeta(-n,0) = -\frac{b_{n+1}}{n+1} \text{ for all } n \in \mathbb{N} .$$

**Related properties:**

| | Hurwitz zeta function | Bernoulli polynomials |
|---|---|---|
| *Derivative property* | **H1** | **P4** |
| *Difference equation* | **H2** | **P5** |
| *Multiplication theorem* | **H3** | **P9** |

**Consequence:**

*The extension from Bernoulli to multiple Bernoulli polynomials will be done using a generalization of the Hurwitz zeta function: the* **Hurwitz multiple zeta functions***.*

# On Hurwitz Multiple Zeta Functions

## Definition of Hurwitz Multiple Zeta Functions

$$\mathcal{H}e^{s_1,\cdots,s_r}(z) = \sum_{0<n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}$$ , if $z \in \mathbb{C} - \mathbb{N}_{<0}$ and $(s_1,\cdots,s_r) \in (\mathbb{N}^*)^r$, such that $s_1 \geq 2$ .

## Lemma 1: (B., J. Ecalle, 2012)

For all sequences $(s_1,\cdots,s_r) \in (\mathbb{N}^*)^r$, $s_1 \geq 2$, we have:

$$\mathcal{H}e^{s_1,\cdots,s_r}(z-1) - \mathcal{H}e^{s_1,\cdots,s_r}(z) = \mathcal{H}e^{s_1,\cdots,s_{r-1}}(z) \cdot z^{-s_r} \ .$$

## Lemma 2:

The Hurwitz Multiple Zeta Functions multiply by the stuffle product (of $\mathbb{N}^*$).

**Reminder:** If $(\Omega,+)$ is a semi-group, the stuffle $⧢$ is defined over $\Omega^\star$ by:

$$\begin{cases} \varepsilon \uplus u &= u \uplus \varepsilon = u \ . \\ ua \uplus vb &= (u \uplus vb)\, a + (ua \uplus v)\, b + (u \uplus v)(a+b) \ . \end{cases}$$

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**Definition:**

Let $\mathrm{x} = \{x_1, x_2, x_3, \cdots\}$ be an infinite commutative alphabet.
A series is said to be quasi-symmetric when its coefficient of $x_1^{s_1} \cdots x_r^{s_r}$ is equals to this of $x_{i_1}^{s_1} \cdots x_{i_r}^{s_r}$ for all $i_1 < \cdots < i_r$.

**Example :** $M_{2,1}(x_1, x_2, x_3, \ldots) = x_1^2 x_2 + x_1^2 x_3 + \cdots + x_1^2 x_n + \cdots + x_2^2 x_3 + \cdots$
$x_1 x_2^2$ is not in $M_{2,1}$ but in $M_{1,2}$.

**Fact 1:**
- Quasi-symmetric functions span a vector space: *QSym*.
- A basis of *QSym* is given by the monomials $M_I$, for composition $I = (i_1, \cdots, i_r)$:

$$M_{i_1,\cdots,i_r}(X) = \sum_{0<n_1<\cdots<n_r} x_{n_1}^{i_1} \cdots x_{n_r}^{i_r}$$

**Fact 2:**
- *QSym* is an algebra whose product is the stuffle product.
- *QSym* is also a Hopf algebra whose coproduct $\Delta$ is given by:

$$\Delta\big(M_{i_1,\cdots,i_r}(\mathrm{x})\big) = \sum_{k=0}^{r} M_{i_1,\cdots,i_k}(\mathrm{x}) \otimes M_{i_{k+1},\cdots,i_r}(\mathrm{x}) \ .$$

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# Main Goal

**Heuristic:**

$$B^{s_1,\cdots,s_r}(z) = \text{Multiple (Divided) Bernoulli Polynomials} = \mathcal{H}e^{-s_1,\cdots,-s_r}(z)\ .$$

$$b^{s_1,\cdots,s_r} = \text{Multiple (Divided) Bernoulli Numbers} = \mathcal{H}e^{-s_1,\cdots,-s_r}(0)\ .$$

We want to define $B^{s_1,\cdots,s_r}(z)$ such that:

- their properties are similar to Hurwitz Multiple Zeta Functions' properties.
- their properties generalize these of Bernoulli polynomials.

**Main Goal:**

Find some polynomials $B^{s_1,\cdots,s_r}$ such that:

$$\begin{cases} B^{n}(z) = \dfrac{B_{n+1}(z)}{n+1}\ , \text{ where } n \geq 0\ , \\ B^{n_1,\cdots,n_r}(z+1) - B^{n_1,\cdots,n_r}(z) = B^{n_1,\cdots,n_{r-1}}(z)z^{n_r}\ , \text{ for } \ n_1,\cdots,n_r \geq 0\ , \\ \text{the } B^{n_1,\cdots,n_r} \text{ multiply by the stuffle product.} \end{cases}$$

# An algebraic construction

## Notation 1:

Let $\mathrm{X} = \{X_1, \cdots, X_n, \cdots\}$ be a (commutative) alphabet of indeterminates.
We denotes:

$$\mathcal{B}^{Y_1, \cdots, Y_r}(z) = \sum_{n_1, \cdots, n_r \geq 0} B^{n_1, \cdots, n_r}(z) \frac{Y_1^{n_1}}{n_1!} \cdots \frac{Y_r^{n_r}}{n_r!} \ ,$$

for all $r \in \mathbb{N}^*$, $Y_1, \cdots, Y_r \in \mathrm{X}$.

**Remark:** $\mathcal{B}^{Y_1, \cdots, Y_r}(z+1) - \mathcal{B}^{Y_1, \cdots, Y_r}(z) = \mathcal{B}^{Y_1, \cdots, Y_{r-1}}(z) e^{zY_r}$ .

## Notation 2:

Let $\mathrm{A} = \{a_1, \cdots, a_n, \cdots\}$ be a non-commutative alphabet.
We denotes:

$$\mathfrak{B}(z) = 1 + \sum_{r > 0} \sum_{k_1, \cdots, k_r > 0} \mathcal{B}^{X_{k_1}, \cdots, X_{k_r}}(z) a_{k_1} \cdots a_{k_r} \in \mathbb{C}[z][\![\mathrm{X}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle \ .$$

**Remark:** $\mathfrak{B}(z+1) = \mathfrak{B}(z) \cdot \left(1 + \sum_{k>0} e^{zX_k} a_k\right)$

# The abstract construction in the case of quasi-symmetric functions

Let see an analogue of $\mathfrak{B}(z)$ where the multiple Bernoulli polynomials are replaced with the monomial functions $M_I(\mathrm{x})$ of $QSym$:

$$M^{Y_1,\cdots,Y_r}(\mathrm{x}) := \sum_{n_1,\cdots,n_r\geq 0} M_{n_1+1,\cdots,n_r+1}(\mathrm{x}) \frac{Y_1^{n_1}}{n_1!}\cdots\frac{Y_r^{n_r}}{n_r!} \text{ , for all } Y_1,\cdots,Y_r \in \mathrm{X} \text{ .}$$

$$\begin{aligned}
\mathfrak{M} &:= 1+\sum_{r>0}\sum_{k_1,\cdots,k_r>0} M^{X_{k_1},\cdots,X_{k_r}}(\mathrm{x})\; a_{k_1}\cdots a_{k_r} \\
&= 1+\sum_{r>0}\sum_{0<p_1<\cdots<p_r}\prod_{i=1}^{r}\Big(1+\sum_{k>0} x_n e^{x_n X_k} a_k\Big) M^{X_{k_1},\cdots,X_{k_r}}(\mathrm{x}) \\
&= \prod_{n>0}^{\longrightarrow}\Big(1+\sum_{k>0} x_n e^{x_n X_k} a_k\Big) \in \mathbb{C}[\![\mathrm{x}]\!][\![\mathrm{X}]\!]\langle\!\langle \mathrm{A}\rangle\!\rangle \text{ .}
\end{aligned}$$

**Computation of the coproduct of $\mathfrak{M}$:** (which does not act on the $X$'s)

$$\Delta M^{Y_1,\cdots,Y_r}(\mathrm{x}) = \sum_{k=0}^{r} M^{Y_1,\ldots,Y_k}(\mathrm{x}) \otimes M^{Y_{k+1},\ldots,Y_r}(\mathrm{x}) \text{ .}$$

$$\Delta\mathfrak{M} = \mathfrak{M}\otimes\mathfrak{M} \text{ .}$$

# Transcription of the multiplication by the stuffle

**Property:** (J. Y. Thibon, F. Chapoton, J. Ecalle, F. Menous, D. Sauzin, ...)

A family of objects $(B^{n_1,\cdots,n_r})_{n_1,n_2,n_3,\cdots\geq 0}$ multiply by the stuffle product if, and only if, there exists a character $\chi_z$ of *QSym* such that

$$\chi_z\big(M_{n_1+1,\cdots,n_r+1}(\mathrm{x})\big) = B^{n_1,\cdots,n_r}(z) \tag{1}$$

**Consequences:**

1. $\chi_z$ can be extended to $QSym[\![\mathrm{X}]\!]$, applying it terms by terms.

$$\chi_z\left(M^{Y_1,\cdots,Y_r}(\mathrm{x})\right) = \mathcal{B}^{Y_1,\cdots,Y_r}(z) \text{ , for all } Y_1,\cdots,Y_r \in \mathrm{X} \text{ .}$$

2. If $B^{n_1,\cdots,n_r}$ multiply the stuffle, $\mathfrak{B} = \chi_z(\mathfrak{M})$ is “group-like” in $\mathbb{C}[z][\![\mathrm{X}]\!]\langle\!\langle \mathrm{A}\rangle\!\rangle$.

## Reformulation of the main goal

Find some polynomials $B^{n_1,\cdots,n_r}$ such that:

$$\begin{cases} \langle \mathfrak{B}(z) | a_k \rangle = \dfrac{e^{zX_k}}{e^{X_k} - 1} - \dfrac{1}{X_k} \ , \\ \mathfrak{B}(z+1) = \mathfrak{B}(z) \cdot \mathfrak{E}(z) \ , \text{ where } \mathfrak{E}(z) = 1 + \sum_{k>0} e^{zX_k} \, a_k \ , \\ \mathfrak{B} \text{ is a "group-like" element of } \mathbb{C}[z][\![\mathrm{X}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle \ . \end{cases}$$

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2 Algebraic reformulation of the problem

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# A singular solution

**Remainder:** $\mathfrak{E}(z) = 1 + \sum_{k>0} e^{zX_k}\, a_k$.

**From a false solution to a singular solution...**

$$\mathcal{S}(z) = \overleftarrow{\prod_{n>0}} \mathfrak{E}(z-n) = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} \frac{e^{z(X_{k_1}+\cdots+X_{k_r})}}{\prod_{i=1}^{r}(e^{X_{k_1}+\cdots+X_{k_i}}-1)} a_{k_1}\cdots a_{k_r} \text{ is a}$$

false solution to system
$$\begin{cases} \langle \mathfrak{B}(z) | a_k \rangle = \dfrac{e^{zX_k}}{e^{X_k}-1} - \dfrac{1}{X_k}, \\ \mathfrak{B}(z+1) = \mathfrak{B}(z)\cdot\mathfrak{E}(z), \\ \mathfrak{B} \text{ is a "group-like" element of } \mathbb{C}[z][\![X]\!]\langle\!\langle A\rangle\!\rangle . \end{cases}$$

**Explanations:** 1.
$$\begin{aligned} \mathfrak{B}(z) &= \cdots = \mathfrak{B}(z-n)\cdot\mathfrak{E}(z-n)\cdots\mathfrak{E}(z-1) \\ &= \cdots = \left(\lim_{n\longrightarrow+\infty} \mathfrak{B}(z-n)\right)\cdot \overleftarrow{\prod_{n>0}} \mathfrak{E}(z-n) . \end{aligned}$$

2. $\mathcal{S}(z) \in \mathbb{C}[z](\!(X)\!)\langle\!\langle A\rangle\!\rangle$, $\mathcal{S}(z) \notin \mathbb{C}[z][\![X]\!]\langle\!\langle A\rangle\!\rangle$.

**Heuristic:** Find a correction of $\mathcal{S}$, to send it into $\mathbb{C}[z][\![X]\!]\langle\!\langle A\rangle\!\rangle$.

# Another solution

**Fact:** If $\Delta(f)(z) = f(z-1) - f(z)$, $\ker \Delta \cap z\mathbb{C}[z] = \{0\}$.

**Consequence:** There exist a unique family of polynomials such that:

$$\begin{cases} B_0^{n_1,\cdots,n_r}(z+1) - B_0^{n_1,\cdots,n_r}(z) = B_0^{n_1,\cdots,n_{r-1}}(z)z^{n_r} \ . \\ B_0^{n_1,\cdots,n_r}(0) = 0 \ . \end{cases}$$

This produces a series $\mathfrak{B}_0 \in \mathbb{C}[z][\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle$ defined by:

$$\mathfrak{B}_0(z) = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} \mathcal{B}_0^{X_{k_1},\cdots,X_{k_r}}(z)\ a_{k_1}\cdots a_{k_r} \ .$$

## Lemma: (B., 2013)

1. The noncommutative series $\mathfrak{B}_0$ is a "group-like" element of $\mathbb{C}[z][\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle$.
2. The coefficients of $\mathfrak{B}_0(z)$ satisfy a recurence relation:

$$\begin{cases} \mathcal{B}_0^{Y_1}(z) = \dfrac{e^{zY_1}-1}{e^{Y_1}-1} \ , \ Y_1 \in \mathrm{X} \ . \\ \mathcal{B}_0^{Y_1,\cdots,Y_r}(z) = \dfrac{\mathcal{B}_0^{Y_1+Y_2,Y_3,\cdots,Y_r}(z) - \mathcal{B}_0^{Y_2,Y_3,\cdots,Y_r}(z)}{e^{Y_1}-1} \ , \ Y_1,\cdots,Y_r \in \mathrm{X} \ . \end{cases}$$

3. The series $\mathfrak{B}_0$ can be expressed in terms of $\mathcal{S}$: $\mathfrak{B}_0(z) = \big(\mathcal{S}(0)\big)^{-1} \cdot \mathcal{S}(z)$.

# Characterization of the set of solutions

**Reminder:** A family of multiple Bernoulli polynomials produces a series $\mathfrak{B}$ such that:

$$\begin{cases} \mathfrak{B}(z+1) = \mathfrak{B}(z) \cdot \mathfrak{E}(z) \text{ , where } \mathfrak{E}(z) = 1 + \sum_{k>0} e^{zX_k} a_k \text{ ,} \\ \mathfrak{B} \text{ is a "group-like" element of } \mathbb{C}[z][\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle \text{ ,} \\ \langle \mathfrak{B}(z) | a_k \rangle = \dfrac{e^{zX_k}}{e^{X_k} - 1} - \dfrac{1}{X_k} \text{ .} \end{cases}$$

**Proposition: (B. 2013)**

Any familly of polynomials which are solution of the previous system comes from a noncommutative series $\mathfrak{B} \in \mathbb{C}[z][\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle$ such that there exists $\mathfrak{b} \in \mathbb{C}[\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle$ satisfying:

1. $\langle \mathfrak{b} | A_k \rangle = \dfrac{1}{e^{X_k} - 1} - \dfrac{1}{X_k}$
2. $\mathfrak{b}$ is "group-like"
3. $\mathfrak{B}(z) = \mathfrak{b} \cdot \mathfrak{B}_0 = \mathfrak{b} \cdot \big(\mathcal{S}(0)\big)^{-1} \cdot \mathcal{S}(z)$ .

**Theorem: (B., 2013)**

The subgroup of "group-like" series of $\mathbb{C}[z][\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle$, with vanishing coefficients in length 1, acts on the set of all possible multiple Bernoulli polynomials, *i.e.* on the set of all possible *algebraic* renormalization.

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**New Goal:**

From $\mathfrak{B}(z) = \mathfrak{b} \cdot \mathfrak{B}_0$, determine a suitable series $\mathfrak{b}$ such that the reflexion formula

$$(-1)^n B_n(1-z) = B_n(z) \ , n \in \mathbb{N}$$

has a nice generalization.

For a generic series $s \in \mathbb{C}[z][\![\boldsymbol{X}]\!]\langle\!\langle \boldsymbol{A} \rangle\!\rangle$,

$$s(z) = \sum_{r \in \mathbb{N}} \sum_{k_1, \cdots, k_r > 0} s^{X_{k_1}, \cdots, X_{k_r}}(z)\, a_{k_1} \cdots a_{k_r} \ ,$$

we consider:

$$\begin{aligned}
\overline{s}(z) &= \sum_{r \in \mathbb{N}} \sum_{k_1, \cdots, k_r > 0} s^{X_{k_r}, \cdots, X_{k_1}}(z)\, a_{k_1} \cdots a_{k_r} \\
\widetilde{s}(z) &= \sum_{r \in \mathbb{N}} \sum_{k_1, \cdots, k_r > 0} s^{-X_{k_1}, \cdots, -X_{k_r}}(z)\, a_{k_1} \cdots a_{k_r}
\end{aligned}$$

# The reflection equation for $\mathfrak{B}_0(z)$

**Proposition: (B. 2014)**

Let $sg = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} (-1)^r a_{k_1} \cdots a_{k_r} = \left(1 + \sum_{n>0} a_n\right)^{-1}$. Then,

$$\widetilde{\mathcal{S}}(0) = \left(\overline{\mathcal{S}}(0)\right)^{-1} \cdot sg \quad \text{and} \quad \widetilde{\mathcal{S}}(1-z) = \left(\overline{\mathcal{S}}(z)\right)^{-1} .$$

**Corollary 1: (B. 2014)**

For all $z \in \mathbb{C}$, we have: $sg \cdot \widetilde{\mathfrak{B}}_0(1-z) = \left(\overline{\mathfrak{B}}_0(z)\right)^{-1}$ .

**Example:**

$$\begin{aligned} \mathcal{B}_0^{-X,-Y,-Z}(1-z) &= -\mathcal{B}_0^{X,Y,Z}(z) - \mathcal{B}_0^{X+Y,Z}(z) - \mathcal{B}_0^{X,Y+Z}(z) \\ & \quad -\mathcal{B}_0^{X+Y+Z}(z) + \mathcal{B}_0^{Y,Z}(z) + \mathcal{B}_0^{Y+Z}(z) . \end{aligned}$$

**Corollary 2: (B. 2014)**

$$\widetilde{\mathfrak{B}}(1-z)\cdot\overline{\mathfrak{B}}(z)=\widetilde{\mathfrak{b}}\cdot sg^{-1}\cdot\overline{\mathfrak{b}}\ . \qquad (2)$$

**Remark:** $\widetilde{\mathcal{S}}(0)\cdot sg^{-1}\cdot\overline{\mathcal{S}}(0)=1.$

**Heuristic:**

A reasonable candidate for a multi-Bernoulli polynomial comes from the coefficients of a series $\mathfrak{B}(z)=\mathfrak{b}\cdot\mathfrak{B}_0(z)$ where $\mathfrak{b}$ satisfies:

1. $\langle\mathfrak{b}|a_k\rangle=\dfrac{1}{e^{X_k}-1}-\dfrac{1}{X_k}$
2. $\mathfrak{b}$ is "group-like"
3. $\widetilde{\mathfrak{b}}\cdot sg^{-1}\cdot\overline{\mathfrak{b}}=1$ .

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# Resolution of an equation

**Goal:** Characterise the solutions of $\begin{cases} \widetilde{\mathfrak{u}} \cdot sg^{-1} \cdot \overline{\mathfrak{u}} = 1 \ . \\ \mathfrak{u} \text{ is "group-like"} \ . \end{cases}$

## Proposition: (B., 2014)

Let us denote $\sqrt{sg^{-1}} = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} \frac{(-1)^r}{2^{2r}} \binom{2r}{r} a_{k_1} \cdots a_{k_r}$ ..

Any "group-like" solution $\mathfrak{u}$ of $\widetilde{\mathfrak{u}} \cdot sg^{-1} \cdot \overline{\mathfrak{u}} = 1$ comes from a "primitive" series $\mathfrak{v}$ satisfying

$$\overline{\mathfrak{v}} + \widetilde{\mathfrak{v}} = 0 \ ,$$

and is given by:

$$\mathfrak{u} = exp(\mathfrak{v}) \cdot \sqrt{sg} \ .$$

If moreover $\langle \mathfrak{u} | a_k \rangle = \frac{1}{e^{X_k} - 1} - \frac{1}{X_k}$, then necessarily, we have:

$$\langle \mathfrak{v} | a_k \rangle = \frac{1}{e^{X_k} - 1} - \frac{1}{X_k} + \frac{1}{2} := f(X_k) \ .$$

**New goal:** Find a nice series $\mathfrak{v}$ satisfying:

1. $\mathfrak{v}$ is "primitive". 2. $\overline{\mathfrak{v}} + \widetilde{\mathfrak{v}} = 0$. 3. $\langle \mathfrak{v} | a_k \rangle = \dfrac{1}{e^{X_k} - 1} - \dfrac{1}{X_k} + \dfrac{1}{2} = f(X_k)$ .

**Remark:** $\langle \mathfrak{v} | a_k \rangle$ is an odd formal series in $X_k \in \mathrm{X}$.

**Generalization:** $\widetilde{\mathfrak{v}} = -\mathfrak{v}$ , so $\overline{\mathfrak{v}} = \mathfrak{v}$ .

$\Longrightarrow \langle \mathfrak{v} | a_{k_1} a_{k_2} \rangle = -\frac{1}{2} f(X_{k_1} + X_{k_2})$, but does not determine $\langle \mathfrak{v} | a_{k_1} a_{k_2} a_{k_3} \rangle$ .

**A restrictive condition:**

A natural condition is to have:

there exists $\alpha_r \in \mathbb{C}$ such that $\langle \mathfrak{v} | a_{k_1} \cdots a_{k_r} \rangle = \alpha_r f(X_{k_1} + \cdots + X_{k_r})$ .

*Now*, there is a unique "primitive" series $\mathfrak{v}$ satisfying this condition and the new goal:

$$\langle \mathfrak{v} | a_{k_1} \cdots a_{k_r} \rangle = \frac{(-1)^{r-1}}{r} f(X_{k_1} + \cdots + X_{k_r}) .$$

## Definition : (B., 2014)

The series $\mathfrak{B}(z)$ and $\mathfrak{b}$ defined by

$$\begin{cases} \mathfrak{B}(z) & = & exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \cdot (\mathcal{S}(0))^{-1} \cdot \mathcal{S}(z) \\ \mathfrak{b} & = & exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \end{cases}$$

are noncommutative series of $\mathbb{C}[z][\![X]\!]\langle\langle A\rangle\rangle$ whose coefficients are respectively the exponential generating functions of *multiple Bernoulli polynomials* and *multiple Bernoulli numbers*.

## Example:

The exponential generating function of bi-Bernoulli polynomials and numbers are respectively:

$$\begin{aligned} \sum_{n_1,n_2\geq 0} B^{n_1,n_2}(z)\frac{X^{n_1}}{n_1!}\frac{Y^{n_2}}{n_2!} &= -\frac{1}{2}f(X+Y)+\frac{1}{2}f(X)f(Y)-\frac{1}{2}f(X)+\frac{3}{8} \\ &+f(X)\ \frac{e^{zY}-1}{e^Y-1}-\frac{1}{2}\ \frac{e^{zY}-1}{e^Y-1} \\ &+\frac{e^{z(X+Y)}-1}{(e^X-1)(e^{X+Y}-1)}-\frac{e^{zY}-1}{(e^X-1)(e^Y-1)}\ . \end{aligned}$$

## Examples of explicit expression for multiple Bernoulli numbers:

Consequently, we obtain explicit expressions like, for $n_1, n_2, n_3 > 0$:

$$b^{n_1,n_2} = \frac{1}{2}\left(\frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+1}}{n_2+1} - \frac{b_{n_1+n_2+1}}{n_1+n_2+1}\right).$$

$$\begin{aligned} b^{n_1,n_2,n_3} = & +\frac{1}{6}\frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+1}}{n_2+1}\frac{b_{n_3+1}}{n_3+1} \\ & -\frac{1}{4}\left(\frac{b_{n_1+n_2+1}}{n_1+n_2+1}\frac{b_{n_3+1}}{n_3+1} + \frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+n_3+1}}{n_2+n_3+1}\right) \\ & +\frac{1}{3}\frac{b_{n_1+n_2+n_3+1}}{n_1+n_2+n_3+1}. \end{aligned}$$

**Remark:** If $n_1 = 0$, $n_2 = 0$ or $n_3 = 0$, the expressions are not so simple...

# Table of Multiple Bernoulli Numbers in length 2

| $b^{p,q}$ | $p=0$ | $p=1$ | $p=2$ | $p=3$ | $p=4$ | $p=5$ | $p=6$ |
|---|---|---|---|---|---|---|---|
| $q=0$ | $\frac{3}{8}$ | $-\frac{1}{12}$ | $0$ | $\frac{1}{120}$ | $0$ | $-\frac{1}{252}$ | $0$ |
| $q=1$ | $-\frac{1}{24}$ | $\frac{1}{288}$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ |
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| $q=4$ | $0$ | $-\frac{1}{504}$ | $0$ | $\frac{1}{480}$ | $0$ | $-\frac{1}{264}$ | $0$ |
| $q=5$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ | $-\frac{1}{60480}$ | $-\frac{1}{264}$ | $\frac{1}{127008}$ | $\frac{691}{65520}$ |

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- one out of four Multiple Bernoulli Numbers is null.

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- one out of four Multiple Bernoulli Numbers is null.
- one out of two antidiagonals is “constant”.

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- one out of four Multiple Bernoulli Numbers is null.
- one out of two antidiagonals is “constant”.
- “symmetrie” relatively to $p = q$ .
- cross product around the zeros are equals : $28800 \cdot 127008 = 60480^2$ .

# Construction Table of Multiple Bernoulli Numbers in length 2

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

| $b^{p,q}$ | $p=0$ | $p=1$ | $p=2$ | $p=3$ | $p=4$ | $p=5$ | $p=6$ |
|---|---|---|---|---|---|---|---|
| $q=0$ | | | | | | | |
| $q=1$ | $-\frac{1}{24}$ | $\frac{1}{288}$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ |
| $q=2$ | | | | | | | |
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| $q=2$ | 0 | | 0 | | 0 | | 0 |
| $q=3$ | | | | | | | |
| $q=4$ | 0 | | 0 | | 0 | | 0 |
| $q=5$ | | | | | | | |
| $q=6$ | 0 | | 0 | | 0 | | 0 |

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| $q=2$ | 0 | $\frac{1}{240}$ | 0 | | 0 | | 0 |
| $q=3$ | | $-\frac{1}{2880}$ | | | | | |
| $q=4$ | 0 | $-\frac{1}{504}$ | 0 | | 0 | | 0 |
| $q=5$ | | $\frac{1}{6048}$ | | | | | |
| $q=6$ | 0 | $\frac{1}{480}$ | 0 | | 0 | | 0 |

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

| $be^{p,q}$ | $p=0$ | $p=1$ | $p=2$ | $p=3$ | $p=4$ | $p=5$ | $p=6$ |
|---|---|---|---|---|---|---|---|
| $q=0$ | | | | | | | |
| $q=1$ | $-\frac{1}{24}$ | $\frac{1}{288}$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ |
| $q=2$ | 0 | $\frac{1}{240}$ | 0 | $-\frac{1}{504}$ | 0 | $\frac{1}{480}$ | 0 |
| $q=3$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | | $\frac{1}{480}$ | | $-\frac{1}{264}$ |
| $q=4$ | 0 | $-\frac{1}{504}$ | 0 | $\frac{1}{480}$ | 0 | $-\frac{1}{264}$ | 0 |
| $q=5$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ | | $-\frac{1}{264}$ | | $\frac{691}{65520}$ |
| $q=6$ | 0 | $\frac{1}{480}$ | 0 | $-\frac{1}{264}$ | 0 | $\frac{691}{65520}$ | 0 |

# Construction Table of Multiple Bernoulli Numbers in length 2

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

| $be^{p,q}$ | $p=0$ | $p=1$ | $p=2$ | $p=3$ | $p=4$ | $p=5$ | $p=6$ |
|---|---|---|---|---|---|---|---|
| $q=0$ | | | | | | | |
| $q=1$ | $-\frac{1}{24}$ | $\frac{1}{288}$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ |
| $q=2$ | 0 | $\frac{1}{240}$ | 0 | $-\frac{1}{504}$ | 0 | $\frac{1}{480}$ | 0 |
| $q=3$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{28800}$ | $\frac{1}{480}$ | | $-\frac{1}{264}$ |
| $q=4$ | 0 | $-\frac{1}{504}$ | 0 | $\frac{1}{480}$ | 0 | $-\frac{1}{264}$ | 0 |
| $q=5$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ | | $-\frac{1}{264}$ | | $\frac{691}{65520}$ |
| $q=6$ | 0 | $\frac{1}{480}$ | 0 | $-\frac{1}{264}$ | 0 | $\frac{691}{65520}$ | 0 |

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

| $be^{p,q}$ | $p=0$ | $p=1$ | $p=2$ | $p=3$ | $p=4$ | $p=5$ | $p=6$ |
|---|---|---|---|---|---|---|---|
| $q=0$ | | | | | | | |
| $q=1$ | $-\frac{1}{24}$ | $\frac{1}{288}$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ |
| $q=2$ | 0 | $\frac{1}{240}$ | 0 | $-\frac{1}{504}$ | 0 | $\frac{1}{480}$ | 0 |
| $q=3$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{28800}$ | $\frac{1}{480}$ | $-\frac{1}{60480}$ | $-\frac{1}{264}$ |
| $q=4$ | 0 | $-\frac{1}{504}$ | 0 | $\frac{1}{480}$ | 0 | $-\frac{1}{264}$ | 0 |
| $q=5$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ | | $-\frac{1}{264}$ | | $\frac{691}{65520}$ |
| $q=6$ | 0 | $\frac{1}{480}$ | 0 | $-\frac{1}{264}$ | 0 | $\frac{691}{65520}$ | 0 |

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

| $be^{p,q}$ | $p = 0$ | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ | $p = 6$ |
|---|---|---|---|---|---|---|---|
| $q = 0$ | | | | | | | |
| $q = 1$ | $-\frac{1}{24}$ | $\frac{1}{288}$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ |
| $q = 2$ | 0 | $\frac{1}{240}$ | 0 | $-\frac{1}{504}$ | 0 | $\frac{1}{480}$ | 0 |
| $q = 3$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{28800}$ | $\frac{1}{480}$ | $-\frac{1}{60480}$ | $-\frac{1}{264}$ |
| $q = 4$ | 0 | $-\frac{1}{504}$ | 0 | $\frac{1}{480}$ | 0 | $-\frac{1}{264}$ | 0 |
| $q = 5$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ | $-\frac{1}{60480}$ | $-\frac{1}{264}$ | | $\frac{691}{65520}$ |
| $q = 6$ | 0 | $\frac{1}{480}$ | 0 | $-\frac{1}{264}$ | 0 | $\frac{691}{65520}$ | 0 |

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

| $be^{p,q}$ | $p=0$ | $p=1$ | $p=2$ | $p=3$ | $p=4$ | $p=5$ | $p=6$ |
|---|---|---|---|---|---|---|---|
| $q=0$ | | | | | | | |
| $q=1$ | $-\frac{1}{24}$ | $\frac{1}{288}$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ |
| $q=2$ | $0$ | $\frac{1}{240}$ | $0$ | $-\frac{1}{504}$ | $0$ | $\frac{1}{480}$ | $0$ |
| $q=3$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{28800}$ | $\frac{1}{480}$ | $-\frac{1}{60480}$ | $-\frac{1}{264}$ |
| $q=4$ | $0$ | $-\frac{1}{504}$ | $0$ | $\frac{1}{480}$ | $0$ | $-\frac{1}{264}$ | $0$ |
| $q=5$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ | $-\frac{1}{60480}$ | $-\frac{1}{264}$ | $\frac{1}{127008}$ | $\frac{691}{65520}$ |
| $q=6$ | $0$ | $\frac{1}{480}$ | $0$ | $-\frac{1}{264}$ | $0$ | $\frac{691}{65520}$ | $0$ |

# Construction Table of Multiple Bernoulli Numbers in length 2

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

| $be^{p,q}$ | $p=0$ | $p=1$ | $p=2$ | $p=3$ | $p=4$ | $p=5$ | $p=6$ |
|---|---|---|---|---|---|---|---|
| $q=0$ | $\frac{3}{8}$ | $-\frac{1}{12}$ | $0$ | $\frac{1}{120}$ | $0$ | $-\frac{1}{252}$ | $0$ |
| $q=1$ | $-\frac{1}{24}$ | $\frac{1}{288}$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ |
| $q=2$ | $0$ | $\frac{1}{240}$ | $0$ | $-\frac{1}{504}$ | $0$ | $\frac{1}{480}$ | $0$ |
| $q=3$ | $\frac{1}{240}$ | $-\frac{1}{2880}$ | $-\frac{1}{504}$ | $\frac{1}{28800}$ | $\frac{1}{480}$ | $-\frac{1}{60480}$ | $-\frac{1}{264}$ |
| $q=4$ | $0$ | $-\frac{1}{504}$ | $0$ | $\frac{1}{480}$ | $0$ | $-\frac{1}{264}$ | $0$ |
| $q=5$ | $-\frac{1}{504}$ | $\frac{1}{6048}$ | $\frac{1}{480}$ | $-\frac{1}{60480}$ | $-\frac{1}{264}$ | $\frac{1}{127008}$ | $\frac{691}{65520}$ |
| $q=6$ | $0$ | $\frac{1}{480}$ | $0$ | $-\frac{1}{264}$ | $0$ | $\frac{691}{65520}$ | $0$ |

# Outline

1 Reminders

2 Algebraic reformulation of the problem

3 The Structure of a Multiple Bernoulli Polynomial

4 The General Reflexion Formula of Multiple Bernoulli Polynomial

5 An Example of Multiple Bernoulli Polynomial

6 An algorithm to compute the double Bernoulli Numbers

7 Properties satisfied by Bernoulli polynomials and numbers

# Properties satisfied by multiple Bernoulli polynomials 1

**Proposition: (B., 2013)**

The multiple Bernoulli polynomials $B^{n_1,\cdots,n_r}$ multiply the stuffle.

**Theorem: (B., 2014)**

**P'1** All multiple Bernoulli numbers satisfy : $b^{2p_1,\cdots,2p_r} = 0$ .

**P'2** If $n_r > 0$, $B^{n_1,\cdots,n_r}(0) = B^{n_1,\cdots,n_r}(1)$ .

**P'5** $B^{n_1,\cdots,n_r}(z+1) - B^{n_1,\cdots,n_r}(z) = B^{n_1,\cdots,n_{r-1}}(z) \cdot z^{n_r}$ .

**P'6** There exists a reflexion formula for multiple Bernoulli polynoms: $\widetilde{\mathfrak{B}}(1-z) \cdot \overline{\mathfrak{B}}(z) = 1$ .

**P'7** The truncated multiple sums of powers $S_N^{s_1,\cdots,s_r}$, defined by

$$S_N^{s_1,\cdots,s_r} = \sum_{0 \leq n_r < \cdots < n_1 < N} {n_1}^{s_1} \cdots {n_r}^{s_r}$$

are given by the coefficients of $\mathfrak{B}_0(N)$ .

# Properties satisfied by multiple Bernoulli polynomials 2

**Proposition: (B. 2014)**

For all positive integers $n_1, \cdots, n_r$, $b^{n_1,\cdots,n_r} = b^{n_r,\cdots,n_1}$ .

**Proposition: (B., 2014)**

For a series $s(z) \in \mathbb{C}[z][\![\boldsymbol{X}]\!]\langle\!\langle \boldsymbol{A} \rangle\!\rangle$, let us define $\Delta(s)(z)$ by:

$$\Delta(s)(z) = \sum_{r\in\mathbb{N}} \sum_{k_1,\cdots,k_r>0} X_{k_1}\cdots X_{k_r} \langle s(z)|a_{k_1}\cdots a_{k_r}\rangle \; a_{k_1}\cdots a_{k_r} \; .$$

$\Delta$ is a derivation, and :

**P'4** The derivation of multiple Bernoulli polynomials are given by:

$$\partial_z \mathfrak{B}(z) = \Delta\left(\mathfrak{b}\cdot\mathcal{S}(0)^{-1}\right)\cdot\left(\mathfrak{b}\cdot\mathcal{S}(0)^{-1}\right)^{-1}\cdot\mathfrak{B}(z) + \Delta(\mathfrak{B}(z)) \; .$$

# Properties satisfied by multiple Bernoulli polynomials 3

**Proposition: (B., 2015)**

**P'3** The recurrence relation of bi-Bernoulli numbers is (partially) given by:

$$2\left(\sum_{k=0}^{p}\sum_{l=0}^{q}\binom{p}{k}\binom{q}{l}be^{k,l} - be^{p,q}\right) = \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{1}{2}-\frac{1}{q+1}\right) + \left(\frac{1}{2}-\frac{1}{p+1}\right)be^{q} + be^{p}\left(\frac{1}{2}-\frac{1}{q+1}\right) - \left(\frac{1}{2}-\frac{1}{p+1}\right) - \left(\frac{1}{2}-\frac{1}{p+q+1}\right) - be^{p} + \frac{3}{4}$$

if $p, q > 0$ .

# Conclusion

1. We have respectively defined the Multiple (divided) Bernoulli Polynomials and Multiple (divided) Bernoulli Numbers by:

$$\begin{cases} \mathfrak{B}(z) = exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \cdot (\mathcal{S}(0))^{-1} \cdot \mathcal{S}(z) \\ \mathfrak{b} = exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \end{cases}$$

They both multiply the stuffle.

2. The Multiple Bernoulli Polynomials satisfy a nice generalization of:

- the nullity of $b_{2n+1}$ if $n > 0$.
- the symmetry $B_n(1) = B_n(0)$ if $n > 1$.
- the difference equation $\Delta(B_n)(x) = nx^{n-1}$.
- the reflection formula $(-1)^n B_n(1-x) = B_n(x)$.

**THANK YOU FOR YOUR ATTENTION !**