On universal differential equations

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Outline

INTRODUCTION

Picard-Vessiot theory of ordinary differential equation

$(\mathbf{k}, \partial)$ a commutative differential ring without zero divisors.
$\mathrm{Const}(\mathbf{k}) = \{c \in \mathbf{k} | \partial c = 0\}$ is supposed to be a field.

$$(ODE) \quad (a_n\partial^n + a_{n-1}\partial^{n-1} + \ldots + a_0)y = 0, \quad a_0, \ldots, a_{n-1}, a_n \in \mathbf{k}.$$

a_n^{-1} is supposed to exist.

Definition 1

1. Let $y_1, \ldots, y_n$ be $\mathrm{Const}(\mathbf{k})$-linearly independent solutions of (ODE). Then $\{y_1, \ldots, y_n\}$ is called a fundamental set of solutions of (ODE) and it generates a $\mathrm{Const}(\mathbf{k})$-vector subspace of dimension at most n.
2. If[1] $M = \mathbf{k}\{y_1, \ldots, y_n\}$ and $\mathrm{Const}(M) = \mathrm{Const}(\mathbf{k})$ then M is called a Picard-Vessiot extension related to (ODE)
3. Let $\mathbf{k} \subset \mathbb{K}_1$ and $\mathbf{k} \subset \mathbb{K}_2$ be differential rings. An isomorphism of rings $\sigma : \mathbb{K}_1 \to \mathbb{K}_2$ is a differential $\mathbf{k}$-isomorphism if
$$\forall a \in \mathbb{K}_1, \quad \partial(\sigma(a)) = \sigma(\partial a) \text{ and, if } a \in \mathbf{k}, \ \sigma(a) = a.$$
If $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$, the differential galois group of $\mathbb{K}$ over $\mathbf{k}$ is by
$$\mathrm{Gal}_{\mathbf{k}}(\mathbb{K}) = \{\sigma | \sigma \text{ is a differential } \mathbf{k}\text{-automorphism of } \mathbb{K}\}.$$

1. Let R_1, R_2 be differential rings s.t. $R_1 \subset R_2$. Let S be a subset of R_2. $R_1\{S\}$ denotes the smallest differential subring of R_2 containing R_1. $R_1\{S\}$ is the ring (over R_1) generated by S and their derivatives of all orders.

Linear differential equations and Dyson series

Let $a_0, \ldots, a_n \in \mathbb{C}(z)$, $\quad (a_n(z)\partial^n + \ldots + a_1(z)\partial + a_0(z))y(z) = 0.$

$$(ED) \quad \left\{ \begin{array}{rcll} \partial q(z) & = & A(z)q(z), & A(z) \in \mathcal{M}_{n,n}(\mathbb{C}(z)), \\ q(z_0) & = & \eta, & \lambda \in \mathcal{M}_{1,n}(\mathbb{C}), \\ y(z) & = & \lambda q(z), & \eta \in \mathcal{M}_{n,1}(\mathbb{C}). \end{array} \right.$$

By successive Picard iterations, with the initial point $q(z_0) = \eta$, we get [2] $y(z) = \lambda U(z_0; z)\eta$, where $U(z_0; z)$ is the following functional expansion

$$U(z_0; z) = \sum_{k \geq 0} \int_{z_0}^{z} A(z_1)dz_1 \int_{z_0}^{z_1} A(z_2)dz_2 \ldots \int_{z_0}^{z_{k-1}} A(z_k)dz_k, \text{(Dyson series)}$$

and $(z_0, z_1 \ldots, z_k, z)$ is a subdivision of the path of integration $z_0 \rightsquigarrow z$.
In order to find the matrix $\Omega(z_0; z)$ s.t.

$$U(z_0; z) = \exp[\Omega(z_0; z)] = \top \exp \int_{z_0}^{z} A(s)ds, \qquad \text{(Feynman's notation)}$$

Magnus computed $\Omega(z_0; z)$ as limit of the following Lie-integral-functionals

$$\begin{array}{rcl} \Omega_1(z_0; z) & = & \displaystyle\int_{z_0}^{z} A(z)ds, \\ \Omega_k(z_0; z) & = & \displaystyle\int_{z_0}^{z} [A(z) + [A(z), \Omega_{k-1}(z_0; s)]/2 \\ & & +[[A(z), \Omega_{k-1}(z_0; s)], \Omega_{k-1}(z_0; s)]/12 + \ldots)ds. \end{array}$$

2. Subject to convergence.

Fuchsian linear differential equations

Let Ω be a simply connected domain and $\mathcal{H}(\Omega)$ be the ring of holomorphic functions over Ω (with $1_{\mathcal{H}(\Omega)}$ as neutral element). Let us consider, here, $\sigma = \{s_i\}_{i=0,..,m}$, $m \geq 1$, as set of simple poles of (ED) and $\Omega = \widetilde{\mathbb{C} \setminus \sigma}$.

$$A(z) = \sum_{i=0}^{m} M_i u_i(z), \quad \text{where} \quad \begin{cases} M_i & \in \mathcal{M}_{n,n}(\mathbb{C}), \\ u_i(z) = (z - s_i)^{-1} & \in \mathbb{C}(z). \end{cases}$$

$$(ED) \quad \begin{cases} \partial q(z) &= \left(\sum_{i=0}^{m} M_i u_i(z)\right) q(z), \\ q(z_0) &= \eta, \\ y(z) &= \lambda q(z). \end{cases}$$

Let X^* be the set of words over $X = \{x_0, \ldots, x_m\}$ and

$$\alpha_{z_0}^z \otimes \mathcal{M} : \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \to \mathcal{M}_{n,n}(\mathcal{H}(\Omega))$$

($z_0 \rightsquigarrow z$ is the path of integration previously introduced) s.t.

$$\mathcal{M}(1_{X^*}) = \mathtt{Id}_n \quad \text{and} \quad \mathcal{M}(x_{i_1} \cdots x_{i_k}) = M_{i_1} \ldots M_{i_k},$$

$$\alpha_{z_0}^z(1_{X^*}) = 1_{\mathcal{H}(\Omega)} \quad \text{and} \quad \alpha_{z_0}^z(x_{i_1} \cdots x_{i_k}) = \int_{z_0}^{z} \frac{dz_1}{z_1 - s_{i_1}} \ldots \int_{z_0}^{z_{k-1}} \frac{dz_k}{z_k - s_{i_k}}.$$

Then [3] $y(z) = \lambda U(z_0; z)\eta$ with

$$U(z_0; z) = \sum_{w \in X^*} \mathcal{M}(w) \alpha_{z_0}^z(w) = (\mathcal{M} \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.$$

3. Subject to convergence.

Examples of linear dynamical systems

Example 2 (Hypergeometric equation)

Let t_0, t_1, t_2 be parameters and

$$z(1-z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0 t_1 y(z) = 0.$$

Let $q_1(z) = -y(z)$ and $q_2(z) = (1-z)\dot{y}(z)$. Hence, one has

$$y(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} \dot{q}_1(z) \\ \dot{q}_2(z) \end{pmatrix} &= \left(\frac{M_0}{z} + \frac{M_1}{1-z} \right) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix} \\ &= \left(u_0(z) M_0 + u_1(z) M_1 \right) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}, \end{aligned}$$

where $u_0(z) = z^{-1}, u_1(z) = (1-z)^{-1}$ and

$$M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix} \quad \text{and} \quad M_1 = -\begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}.$$

Nonlinear differential equations

$$(NED) \quad \begin{cases} \partial q(z) &= \left(\sum_{i=0}^{m} T_i(q) u_i(z)\right)(q), \\ q(z_0) &= q_0, \\ y(z) &= f(q(z)), \end{cases}$$

where

- $u_i \in (\mathbf{k}, \partial)$,
- the state $q = (q_1, \ldots, q_n)$ belongs the complex analytic manifold Q of dimension n and q_0 is the initial state,
- the observation $f \in \mathcal{O}$, with $\mathcal{O}$ the ring of analytic functions over Q,
- for $i = 0..1$, $T_i = (T_i^1(q)\partial/\partial q_1 + \cdots + T_i^m(q)\partial/\partial q_m)$ is an analytic vector field over Q, with $T_i^j(q) \in \mathcal{O}$, for $j = 1, \ldots, n$.

With X and $\alpha_{z_0}^z$ given as previously, let the morphism τ be defined by $\tau(1_{X^*}) = \mathtt{Id}$ and $\tau(x_{i_1} \cdots x_{i_k}) = T_{i_1} \ldots T_{i_k}$. Then [4] $y(z) = \mathcal{T} \circ f_{|_{q_0}}$ with

$$\mathcal{T} = \sum_{w \in X^*} \tau(w) \alpha_{z_0}^z(w) = (\tau \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.$$

4. Subject to convergence.

Examples of nonlinear dynamical systems (1/2)

Example 3 (Harmonic oscillator)

Let k_1, k_2 be parameters and $\partial^2 y(z) + k_1 y(z) + k_2 y^2(z) = u_1(z)$ which can be represented by the following state equations (with $n = 1$)

$$\begin{array}{rcl} y(z) & = & q(z), \\ \partial q(z) & = & A_0(q)u_0(z) + A_1(q)u_1(z), \\ \text{where} \quad A_0 & = & -(k_1 q + k_2 q^2)\dfrac{\partial}{\partial q} \quad \text{and} \quad A_1 \; = \; \dfrac{\partial}{\partial q}. \end{array}$$

Example 4 (Duffing equation)

Let a, b, c be parameters and $\partial^2 y(z) + a\partial y(z) + by(z) + cy^3(z) = u_1(z)$ which can be represented by the following state equations (with $n = 2$)

$$\begin{array}{rcl} y(z) & = & q_1(z), \\ \begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} & = & \begin{pmatrix} q_2 \\ -(aq_2 + b^2 q_1 + cq_1^3) \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z) \\ & = & A_0(q)u_0(z) + A_1(q)u_1(z), \\ \text{where} \quad A_0 & = & -(aq_2 + b^2 q_1 + cq_1^3)\dfrac{\partial}{\partial q_2} + q_2 \dfrac{\partial}{\partial q_1} \quad \text{and} \quad A_1 \; = \; \dfrac{\partial}{\partial q_2}. \end{array}$$

Examples of nonlinear dynamical systems (2/2)

Example 5 (Van der Pol oscillator)

Let γ, g be parameters and

$$\partial^2 x(z) - \gamma[1 + x(z)^2]\partial x(z) + x(z) = g\cos(\omega z)$$

which can be tranformed into (with C is some constant of integration)

$$\partial x(z) = \gamma[1 + x(z)^2/3]x(z) - \int_{z_0}^{z} x(s)ds + \frac{g}{\omega}\sin(\omega z) + C.$$

Supposing $x = \partial y$ and $u_1(z) = g\sin(\omega z)/\omega + C$, it leads then to

$$\partial^2 y(z) = \gamma[\partial y(z) + (\partial y(z))^3/3] + y(z) + u_1(z)$$

which can be represented by the following state equations (with $n = 2$)

$$\begin{aligned} y(z) &= q_1(z), \\ \begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} &= \begin{pmatrix} q_2 \\ \gamma(q_2 + q_2^3/3) + q_1 \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z) \\ &= A_0(q)u_0(z) + A_1(q)u_1(z), \end{aligned}$$

$$\text{where} \quad A_0 = [\gamma(q_2 + q_2^3/3) + q_1]\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q_1} \quad \text{and} \quad A_1 = \frac{\partial}{\partial q_2}.$$

DUAL LAWS AND REPRESENTATIVE SERIES

Dual law in bialgebra

Startting with a $\mathbf{k}-\mathbf{AAU}$ ($\mathbf{k}$ is a ring) $\mathcal{A}$. Dualizing $\mu : \mathcal{A}\otimes_{\mathbf{k}}\mathcal{A}\to\mathcal{A}$, we get the transpose ${}^t\mu : \mathcal{A}^\vee \to (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^\vee$ so that we do not get a co-multiplication in general.

- Remark that when $\mathbf{k}$ is a field, the following arrow is into (due to the fact that $\mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee$ is torsionfree)
$$\Phi : \mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee \to (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^\vee.$$
- One restricts the codomain of ${}^t\mu$ to $\mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee$ and then the domain to $({}^t\mu)^{-1}\Phi(\mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee) =: \mathcal{A}^\circ$.

$$\begin{tikzcd}
\mathcal{A}^\vee \arrow[r, "{}^t\mu"] & (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^\vee \\
\mathcal{A}^\circ \arrow[u, hook, "can"] \arrow[r, "\Delta_\mu"] & \mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee \arrow[u, hook, "\Phi"'] \\
\mathcal{A}^{\circ\circ} \arrow[u, hook, "can"] \arrow[r, "\Delta_\mu"] & \mathcal{A}^\circ\otimes_{\mathbf{k}}\mathcal{A}^\circ \arrow[u, hook, "j\otimes j"']
\end{tikzcd}$$

The descent can stop at first step for a field $\mathbf{k}$ and then $\mathcal{A}^{\circ\circ}=\mathcal{A}^\circ$. The coalgebra $(\mathcal{A}^\circ,\Delta_\mu)$ is called the Sweedler's dual of $(\mathcal{A},\mu)$.

Case of algebras noncommutative series

- $\mathcal{X}$ denotes the ordered alphabets $Y := \{y_k\}_{k\geq 1}$ or $X := \{x_0, x_1\}$. On the free monoid $(\mathcal{X}^*, \mathrm{conc}, 1_{\mathcal{X}^*})$, we use the correspondences
$$x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1 \underset{\pi_X}{\overset{\pi_Y}{\rightleftharpoons}} y_{s_1} \ldots y_{s_r} \in Y^* \leftrightarrow (s_1, \ldots, s_r) \in \mathbb{N}_+^r.$$
Let $\mathcal{L}yn\mathcal{X}$ denote the set of Lyndon words generated by $\mathcal{X}$.
- Let $(\mathcal{L}ie_A\langle\langle\mathcal{X}\rangle\rangle, [.])$ and $(A\langle\langle\mathcal{X}\rangle\rangle, \mathrm{conc})$ (resp. $(\mathcal{L}ie_A\langle\mathcal{X}\rangle, [.])$ and $(A\langle\mathcal{X}\rangle, \mathrm{conc})$) denote the algebras of (Lie) series (resp. polynomials) with coefficients in the ring A, over $\mathcal{X}$.

  $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY}$) is a basis of Lie algebra of primitive elements and $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$) is a transcendence basis of $(A\langle\mathcal{X}\rangle, ⧢, 1_{\mathcal{X}^*})$ (resp. $(A\langle Y\rangle, \uplus, 1_{Y^*})$).
- $\mathcal{H}_{⧢}(\mathcal{X}) := (A\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{\mathcal{X}^*}, \Delta_{⧢}, \mathrm{e})$ and $\mathcal{H}_{\uplus}(Y) := (A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\uplus}, \mathrm{e})$ with [5] (for $x \in \mathcal{X}, y_i \in Y$)
$$\begin{aligned}\Delta_{⧢}\, x &= x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x,\\ \Delta_{\uplus}\, y_i &= y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \textstyle\sum_{k+l=i} y_k \otimes y_l.\end{aligned}$$
- The dual law associated to conc is defined, for $w \in \mathcal{X}^*$, by
$$\Delta_{\mathrm{conc}}(w) = \sum_{u,v\in\mathcal{X}^*, uv=w} u \otimes v.$$

5. Or equivalently, for $x, y \in \mathcal{X}, y_i, y_j \in Y$ and $u, v \in \mathcal{X}^*$ (resp. Y^*), $u ⧢ 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} ⧢ u = u$ and $xu ⧢ yv = x(u ⧢ yv) + y(xu ⧢ v)$, $u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u$ and $x_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v)$.

Dualizable laws in conc-shuffle bialgebras (1/2)

We can exploit the basis of words as follows

1. Any bilinear law (shuffle, stuffle or any) $\mu : A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \to A\langle\mathcal{X}\rangle$ can be decribed through its structure constants wrt to the basis of words, *i.e.* for $u, v, w \in \mathcal{X}^*$, $\Gamma^w_{u,v} := \langle \mu(u \otimes v) | w \rangle$ so that
$$\mu(u \otimes v) = \sum_{w \in \mathcal{X}^*} \Gamma^w_{u,v} w.$$
2. In the case when $\Gamma^w_{u,v}$ is locally finite in w, we say that the given law is dualizable, the arrow ${}^t\mu$ restricts nicely to $A\langle\mathcal{X}\rangle \hookrightarrow A\langle\langle\mathcal{X}\rangle\rangle$ and one can define on the polynomials a comultiplication by
$$\Delta_\mu(w) := \sum_{u,v \in \mathcal{X}^*} \Gamma^w_{u,v} u \otimes v.$$
3. When the law μ is dualizable, we have

$$\begin{tikzcd}
A\langle\langle\mathcal{X}\rangle\rangle \arrow[r, "{}^t\mu"] & A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \\
A\langle\mathcal{X}\rangle \arrow[u, hook, "can"] \arrow[r, "\Delta_\mu"] & A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \arrow[u, hook, "\Phi|_{A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle}"']
\end{tikzcd}$$

The arrow Δ_μ is unique to be able to close the rectangle and $\Delta_\mu(P)$ is defined as above.

Dualizable laws in conc-shuffle bialgebras (2/2)

4. Proof that the arrow $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \longrightarrow A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ is into :
Let $T = \sum_{i=1}^n P_i \otimes_A Q_i$ such that $\Phi(T) = 0$. Rewriting T as a finitely supported sum $T = \sum_{u,v\in\mathcal{X}^*} c_{u,v}\, u \otimes v$ (this is indeed the iso between $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle$ and $A[\mathcal{X}^* \times \mathcal{X}^*]$), $\Phi(T)$ is by definition of Φ the double series (here a polynomial) s.t. $\langle\Phi(T)|u \otimes v\rangle = c_{u,v}$. If $\Phi(T) = 0$, then for all $(u, v) \in \mathcal{X}^* \times \mathcal{X}^*$, $c_{u,v} = 0$ entailing $T = 0$.

We extend by linearity and infinite sums, for $S \in A\langle\langle Y\rangle\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$), by

$$\begin{aligned}
\Delta_{\uplus} S &= \sum_{w\in Y^*} \langle S|w\rangle \Delta_{\uplus} w &&\in A\langle\langle Y^* \otimes Y^*\rangle\rangle,\\
\Delta_{\mathtt{conc}} S &= \sum_{w\in\mathcal{X}^*} \langle S|w\rangle \Delta_{\mathtt{conc}} w &&\in A\langle\langle \mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle,\\
\Delta_{\sqcup\!\sqcup} S &= \sum_{w\in\mathcal{X}^*} \langle S|w\rangle \Delta_{\sqcup\!\sqcup} w &&\in A\langle\langle \mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle.
\end{aligned}$$

$A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$ embeds injectively in [6] $A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \cong [A\langle\langle\mathcal{X}\rangle\rangle]\langle\langle\mathcal{X}\rangle\rangle$.

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6. $A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$ contains the elements of the form $\sum_{i\in I \text{ finite}} G_i \otimes D_i$, for $(G_i, D_i) \in A\langle\langle\mathcal{X}\rangle\rangle \times A\langle\langle\mathcal{X}\rangle\rangle$. But since elements of $M \otimes N$ are finite combination of $m_i \otimes n_i, m_i \in M, n_i \in N$ then $\sum_{i\geq 0} u^i \otimes v^i$ belongs to $A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ and does not belong to $A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$, for $u, v \in \mathcal{X}^{\geq 1}$.

Extended Ree's theorem

Let $S \in A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle$), A is a commutative ring containing $\mathbb{Q}$. The series S is said to be

1. a $⧢$ (resp. conc, $ш$)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w\rangle\langle S|v\rangle = \langle S|w ⧢ v\rangle$ (resp. $\langle S|wv\rangle, \langle S|w ш v\rangle$) and $\langle S|1\rangle = 1$.
2. an infinitesimal $⧢$ (resp. conc, $ш$)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w ⧢ v\rangle = \langle S|w\rangle\langle v|1_{Y^*}\rangle + \langle w|1_{Y^*}\rangle\langle S|v\rangle$ (resp. $\langle S|wv\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$, $\langle S|w ш v\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$).
3. a group-like series iff $\langle S|1_{\mathcal{X}^*}\rangle = 1$ and $\Delta_{⧢} S = \Phi(S \otimes S)$ (resp. $\Delta_{\mathtt{conc}} S = \Phi(S \otimes S), \Delta_{⧢} S = \Phi(S \otimes S)$).
4. a primitive series iff $\Delta_{⧢} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$ (resp. $\Delta_{\mathtt{conc}} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}, \Delta_{ш} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}$).

Then the following assertions are equivalent

1. S is a $⧢$ (resp. conc and $ш$)-character.
2. $\log S$ an infinitesimal $⧢$ (resp. conc and $ш$)-character.
3. S is group-like, for $\Delta_{⧢}$ (resp. $\Delta_{\mathtt{conc}}$ and $\Delta_{ш}$).
4. $\log S$ is primitive, for $\Delta_{⧢}$ (resp. $\Delta_{\mathtt{conc}}$ and $\Delta_{ш}$).

Extension by continuity (infinite sums)

Now, suppose that the ring A (containing $\mathbb{Q}$) is a field $\mathbf{k}$. Then

$$\Delta_{\sqcup\!\sqcup} : \mathbf{k}\langle\mathcal{X}\rangle \to \mathbf{k}\langle\mathcal{X}\rangle \otimes \mathbf{k}\langle\mathcal{X}\rangle \text{ and } \Delta_{\uplus} : \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle Y\rangle \otimes \mathbf{k}\langle Y\rangle$$

are graded for the multidegree. Then $\Delta_{\uplus}$ is graded for the length. Their extension to the completions (*i.e.* $\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle$ and $\mathbf{k}\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$) are continuous and then, when exist, commute with infinite sums. Hence [7, 8],

$$\forall c \in \mathbf{k}, \quad \Delta_{\sqcup\!\sqcup}(cx)^* = \sum_{n\geq 0} c^n \Delta_{\sqcup\!\sqcup} x^n = \sum_{n\geq 0} c^n \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}.$$

For $c \in \mathbb{N}_{\geq 2}$ which is neither a field nor a ring (containing $\mathbb{Q}$), we also get

$$(cx)^* = (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \sqcup\!\sqcup (bx)^* \quad \in \mathbb{N}_{\geq 2}\langle\langle\mathcal{X}\rangle\rangle,$$

$$\Delta_{\sqcup\!\sqcup}(cx)^* \neq (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \otimes (bx)^* \quad \in \mathbb{Q}\langle\langle\mathcal{X}\rangle\rangle \otimes \mathbb{Q}\langle\langle\mathcal{X}\rangle\rangle,$$

because

$$\langle \mathrm{LHS} | x \otimes 1_{\mathcal{X}^*}\rangle = c \quad \text{and} \quad \langle \mathrm{RHS} | x \otimes 1_{\mathcal{X}^*}\rangle = (c-1)^{-1} \sum_{a=1}^{c-1} a = \frac{c}{2}.$$

For $c \in \mathbb{Z}$ (or even $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), the such decomposition is not finite.

7. For $S \in A\langle\langle\mathcal{X}\rangle\rangle$ s.t. $\langle S | 1_{\mathcal{X}^*}\rangle = 0$, $S^* = \sum_{n\geq 0} S^n$ is called Kleene star of S.
8. $\Delta_{\sqcup\!\sqcup} x^n = (\Delta_{\sqcup\!\sqcup} x)^n = (1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*})^n = \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}$.

Case of rational series and of $\Delta_{\mathtt{conc}}$

$A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ denotes the algebraic closure by[9] $\{\mathtt{conc},+,*\}$ of $\widehat{A.\mathcal{X}}$ in $A\langle\langle\mathcal{X}\rangle\rangle$.

$$\begin{tikzcd}
A\langle\langle\mathcal{X}\rangle\rangle \arrow[rr, "{}^t\mathtt{conc}"] & & A\langle\langle\mathcal{X}^*\otimes\mathcal{X}^*\rangle\rangle \\
A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \arrow[u, hook, "can"] \arrow[rr, dashed] & & A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle\otimes_A A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \arrow[u, "\Phi|_{A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle\otimes_A A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle}"']
\end{tikzcd}$$

The dashed arrow may not exist in general, but for any $R\in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ admitting (λ,μ,η) as linear representation of dimension n, we can get

$$ {}^t\mathtt{conc}(R)=\Phi(\textstyle\sum_{i=1}^n G_i\otimes D_i).$$

Indeed, since $\langle R|xy\rangle=\lambda\mu(xy)\eta=\lambda\mu(x)\mu(y)\eta$ $(x,y\in\mathcal{X})$ then, letting e_i is the vector such that ${}^te_i=\begin{pmatrix}0&\ldots&0&1&0&\ldots&0\end{pmatrix}$, one has

$$\langle R|xy\rangle=\sum_{i=1}^n\lambda\mu(x)e_i{}^te_i\mu(y)\eta=\sum_{i=1}^n\langle G_i|x\rangle\langle D_i|y\rangle=\sum_{i=1}^n\langle G_i\otimes D_i|x\otimes y\rangle.$$

G_i (resp. D_i) admits then (λ,μ,e_i) (resp. $({}^te_i,\mu,\eta)$) as linear representation. If $A=\mathbf{k}$ being a field then, due to the injectivity of Φ, all expressions of the type $\sum_{i=1}^n G_i\otimes D_i$, of course, coincide. Hence, the dashed arrow (a restriction of $\Delta_{\mathtt{conc}}$) in the above diagram is well-defined.

9. $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ is closed under $\sqcup\!\sqcup$. $A^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ is also closed under $\uplus$.

Representative series and Sweedler's dual

Theorem 6 (representative series)

Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$. The following assertions are equivalent

1. *The series S belongs to $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$.*
2. *There exists a linear representation (ν, μ, η), of rank n, for S with $\nu \in M_{1,n}(A), \eta \in M_{n,1}(A)$ and a morphism of monoids $\mu : \mathcal{X}^* \to M_{n,n}(A)$ s.t., for any $w \in \mathcal{X}^*$, $\langle S|w\rangle = \nu\mu(w)\eta$.*
3. *The shifts[10] $\{S \triangleleft w\}_{w\in\mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w\in\mathcal{X}^*}$) lie within a finitely generated shift-invariant A-module.*

Moreover, if A is a field $\mathbf{k}$, the previous assertions are equivalent to

4. *There exist $(G_i, D_i)_{i\in F\,finite}$ s.t. $\Delta_{\mathtt{conc}}(S) = \sum_{i\in F\,finite} G_i \otimes D_i$.*

Hence, $\mathcal{H}^{\circ}_{\shuffle}(\mathcal{X}) = (\mathbf{k}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \shuffle, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$ and $\mathcal{H}^{\circ}_{\stuffle}(Y) = (\mathbf{k}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \stuffle, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$.

Now, let $A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$) be the set of exchangeable[11] series (resp. series admitting a linear representation with commuting matrices).

10. The *left* (resp. *right*) shift of S by P is $P \triangleright S$ (resp. $S \triangleleft P$) defined by, for $w \in \mathcal{X}^*$, $\langle P \triangleright S|w\rangle = \langle S|wP\rangle$ (resp. $\langle S \triangleleft P|w\rangle = \langle S|Pw\rangle$).

11. *i.e.* if $S \in A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ then $(\forall u, v \in \mathcal{X}^*)((\forall x \in \mathcal{X})(|u|_x = |v|_x) \Rightarrow \langle S|u\rangle = \langle S|v\rangle)$.

Kleene stars of the plane and conc-characters

For any $S \in A\langle\langle\mathcal{X}\rangle\rangle$, let ∇S denotes $S - 1_{\mathcal{X}^*}$.

Theorem 7 (rational exchangeable series)

1. $A_{\rm exc}^{\rm rat}\langle\langle\mathcal{X}\rangle\rangle \subset A^{\rm rat}\langle\langle\mathcal{X}\rangle\rangle \cap A_{\rm exc}\langle\langle\mathcal{X}\rangle\rangle$. *If A is a field then the equality holds and* $A_{\rm exc}^{\rm rat}\langle\langle X\rangle\rangle = A^{\rm rat}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup A^{\rm rat}\langle\langle x_1\rangle\rangle$ *and, for the algebra of series over subalphabets* $A_{\rm fin}^{\rm rat}\langle\langle Y\rangle\rangle := \cup_{F\subset_{finite} Y} A^{\rm rat}\langle\langle F\rangle\rangle$, *we get*[12] $A_{\rm exc}^{\rm rat}\langle\langle Y\rangle\rangle \cap A_{\rm fin}^{\rm rat}\langle\langle Y\rangle\rangle = \cup_{k\geq 0} A^{\rm rat}\langle\langle y_1\rangle\rangle \sqcup\!\sqcup \ldots \sqcup\!\sqcup A^{\rm rat}\langle\langle y_k\rangle\rangle \subsetneq A_{\rm exc}^{\rm rat}\langle\langle Y\rangle\rangle$.
2. $\forall x \in \mathcal{X}, A^{\rm rat}\langle\langle x\rangle\rangle = \{P(1-xQ)^{-1}\}_{P,Q\in A[x]}$. *If* $\mathbf{k}$ *is an algebraically closed field then* $\mathbf{k}^{\rm rat}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbf{k}}\{(ax)^* \sqcup\!\sqcup \mathbf{k}\langle x\rangle | a \in K\}$.
3. *If A is a $\mathbb{Q}$-algebra without zero divisors,* $\{x^*\}_{x\in\mathcal{X}}$ *(resp.* $\{y^*\}_{y\in Y}$*) are conc-character and algebraically independent over* $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup)$ *(resp.* $(A\langle Y\rangle, \uplus)$*) within* $(A^{\rm rat}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup)$ *(resp.* $(A^{\rm rat}\langle\langle Y\rangle\rangle, \sqcup\!\sqcup)$*).*
4. *Let* $S \in A\langle\langle\mathcal{X}\rangle\rangle$. *If* $A = \mathbf{k}$, *a field, then t.f.a.e.*
   a) S *is groupe-like, for* $\Delta_{\rm conc}$.
   b) *There exists* $M := \sum_{x\in\mathcal{X}} c_x x \in \widehat{\mathbf{k}.\mathcal{X}}$ *s.t.* $S = M^*$.
   c) *There exists* $M := \sum_{x\in\mathcal{X}} c_x x \in \widehat{\mathbf{k}.\mathcal{X}}$ *s.t.* $\nabla S = MS = SM$.

---

12. The following identity lives in $A_{\rm exc}^{\rm rat}\langle\langle Y\rangle\rangle$ but not in $A_{\rm exc}^{\rm rat}\langle\langle Y\rangle\rangle \cap A_{\rm fin}^{\rm rat}\langle\langle Y\rangle\rangle$,
$(y_1 + \ldots)^* = \lim_{k\to+\infty}(y_1 + \ldots + y_k)^* = \lim_{k\to+\infty} y_1^* \sqcup\!\sqcup \ldots \sqcup\!\sqcup y_k^* = \sqcup\!\sqcup_{k\geq 1} y_k^*$.

Triangular sub bialgebras of $(A^{\mathrm{rat}}\langle\langle X\rangle\rangle, ⧢, 1_{X^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$

Let (ν, μ, η) be a linear representation of $R \in A^{\mathrm{rat}}\langle\langle X\rangle\rangle$ and $\mathcal{L}$ be the Lie algebra generated by $\{\mu(x)\}_{x\in X}$.
Let $M(x) := \mu(x)x$, for $x \in X$. Then $R = \nu M(X^*)\eta$. If $\{\mu(x)\}_{x\in X}$ are triangular then let $D(X)$ (resp. $N(X)$) be the diagonal (resp. nilpotent) letter matrix s.t. $M(X) = D(X) + N(X)$ then
$M(X^*) = ((D(X^*)T(X))^* D(X^*))$. Moreover, if $X = \{x_0, x_1\}$ then
$M(X^*) = (M(x_1^*)M(x_0))^* M(x_1^*) = (M(x_0^*)M(x_1))^* M(x_0^*)$.

If A is an algabraically closed field, the modules generated by the following families are closed by $\mathtt{conc}$, ⧢ and coproducts :

$$
\begin{array}{llll}
(F_0) & E_1x_1\ldots E_jx_1E_{j+1}, & \text{where} & E_k \in A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle,\\
(F_1) & E_1x_0\ldots E_jx_0E_{j+1}, & \text{where} & E_k \in A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle,\\
(F_2) & E_1x_{i_1}\ldots E_jx_{i_j}E_{j+1}, & \text{where} & E_k \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle, x_{i_k} \in X.
\end{array}
$$

It follows then that

1. R is a linear combination of expressions in the form (F_0) (resp. (F_1)) iff $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is nilpotent,
2. R is a linear combination of expressions in the form (F_2) iff $\mathcal{L}$ is solvable. Thus, if $R \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle ⧢ A\langle X\rangle$ then $\mathcal{L}$ is nilpotent.

CONTINUITY OVER CHEN SERIES

Iterated integrals over $\omega_i(z) = u_{x_i}(z)dz$ and along $z_0 \rightsquigarrow z$

Let Ω be a simply connected domain admitting $1_{\mathcal{H}(\Omega)}$ as neutral element.
Let $\mathcal{A} := \mathcal{H}(\Omega)$ and let $\mathcal{C}_0$ be a differential subring of $\mathcal{A}$ ($\partial(\mathcal{C}_0) \subset \mathcal{C}_0$) which is an integral domain containing $\mathbb{C}$.
$\mathbb{C}\{\{(g_i)_{i\in I}\}\}$ denotes the differential subalgebra of $\mathcal{A}$ generated by $(g_i)_{i\in I}$, *i.e.* the $\mathbb{C}$-algebra generated by g_i's and their derivatives

$\{u_x\}_{x\in\mathcal{X}}$: elements in $\mathcal{C}_0 \cap \mathcal{A}^{-1}$ in correspondence with $\{\theta_x\}_{x\in\mathcal{X}}$ ($\theta_x = u_x^{-1}\partial$).

The iterated integral associated to $x_{i_1}\ldots x_{i_k} \in \mathcal{X}^*$, over the differential forms $\omega_i(z) = u_{x_i}(z)dz$, and along a path $z_0 \rightsquigarrow z$ on Ω, is defined by

$$\begin{aligned}
\alpha_{z_0}^{z}(1_{\mathcal{X}^*}) &= 1_\Omega,\\
\alpha_{z_0}^{z}(x_{i_1}\ldots x_{i_k}) &= \int_{z_0}^{z}\omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k).\\
\partial\alpha_{z_0}^{z}(x_{i_1}\ldots x_{i_k}) &= u_{x_{i_1}}(z)\int_{z_0}^{z}\omega_{i_2}(z_2)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k).
\end{aligned}$$

$$\begin{aligned}
\mathrm{span}_{\mathbb{C}}\{\partial^l\alpha_{z_0}^{z}(w)\}_{w\in\mathcal{X}^*,l\geq 0} &\subset \mathrm{span}_{\mathbb{C}\{\{(u_x)_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^{z}(w)\}_{w\in\mathcal{X}^*}\\
&\subset \mathrm{span}_{\mathbb{C}\{\{(u_x^{\pm1})_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^{z}(w)\}_{w\in\mathcal{X}^*}\\
&\cong \mathbb{C}\{\{(u_x^{\pm1})_{x\in\mathcal{X}}\}\}\otimes_{\mathbb{C}}\mathrm{span}_{\mathbb{C}}\{\alpha_{z_0}^{z}(w)\}_{w\in\mathcal{X}^*}\ ?
\end{aligned}$$

Iterated integrals and integro differential operators

Let $\mathcal{C} = \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}$. One has $\theta_x \in \mathcal{C}\langle\partial\rangle$, for $x \in \mathcal{X}$, and

$$\forall x, y \in \mathcal{X}, \quad \forall w \in \mathcal{X}^*, \quad \theta_x \alpha_{z_0}^z(yw) = u_x^{-1}(z) u_y(z) \alpha_{z_0}^z(w).$$

Now, let Θ be the morphism $\mathbb{C}\langle\mathcal{X}\rangle \longrightarrow \mathcal{C}\langle\partial\rangle$ defined as follows

$$\Theta(w) = \begin{cases} \mathrm{Id} & \text{if} \quad w = 1_{\mathcal{X}^*}, \\ \Theta(u)\theta_x & \text{if} \quad w = ux \in \mathcal{X}^*\mathcal{X}. \end{cases}$$

One has, for any $w \in \mathcal{X}^*$,

1. $\Theta(\tilde{w})\alpha_{z_0}^z(w) = 1_\Omega$, and then $\partial(\Theta(\tilde{w})\alpha_{z_0}^z(w)) = 0$.
2. $L_w \alpha_{z_0}^z(\tilde{w}) = 0$, where $L_w := \partial\Theta(w) \in \mathcal{C}\langle\partial\rangle$.

For any $x_i \in \mathcal{X}$, let us consider a section of θ_{x_i} : $\theta_{x_i}\iota_{x_i}^{z_0} = \mathrm{Id}$, *i.e.*

$$\forall f \in \mathcal{H}(\Omega), \quad \iota_{x_i}^{z_0} f(z) = \int_{z_0}^{z} \omega_i(s) f(s).$$

The operator $\theta_y \iota_x^{z_0}$, for $x \neq y$, admits $u_y u_x^{-1}$ as eigenvalue, *i.e.*

$$\forall f \in \mathcal{H}(\Omega), \quad (\theta_y \iota_x^{z_0}) f = u_y u_x^{-1} f, \quad \text{in particular,} \quad (\theta_y \iota_x^{z_0}) 1_\Omega = u_y u_x^{-1}.$$

Now, let $\Im^{z_0}$ be the morphism defined as follows

$$\Im^{z_0}(w) = \begin{cases} \mathrm{Id} & \text{if} \quad w = 1_{\mathcal{X}^*}, \\ \Im^{z_0}(u)\iota_x^{z_0} & \text{if} \quad w = ux \in \mathcal{X}^*\mathcal{X}. \end{cases}$$

Hence, for any $w \in X^*, \Im^{z_0}(w) 1_\Omega = \alpha_{z_0}^z(w)$.

First properties

Proposition 1

The following assertions are equivalent

1. *The morphism* $(\mathcal{C}\langle\mathcal{X}\rangle, ⧢, 1_{\mathcal{X}^*}) \longrightarrow (\mathrm{span}_{\mathcal{C}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}, \times, 1_\Omega)$ *is injective.*
2. $\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}$ *is* $\mathcal{C}$*-linearly independent.*
3. $\{\alpha_{z_0}^z(l)\}_{l\in\mathcal{L}yn\mathcal{X}}$ *is* $\mathcal{C}$*-algebraically independent.*
4. $\{\alpha_{z_0}^z(x)\}_{x\in\mathcal{X}}$ *is* $\mathcal{C}$*-algebraically independent.*
5. $\{\alpha_{z_0}^z(x)\}_{x\in\mathcal{X}\cup\{1_{\mathcal{X}^*}\}}$ *is* $\mathcal{C}$*-linearly independent.*

If one of the above assertions holds then

1. $\mathcal{C}[\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}]$ forms the universal $\mathcal{C}$-module of solutions of all differential equations $Ly = 0$,
2. $\mathcal{C}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}$ forms the universal Picard-Vessiot extension related to all differential equations $Ly = 0$,

where[13] L's are linear differential operators belonging to $\mathcal{C}\langle\partial\rangle$.

13. For any $w \in X^*$, let $\mathcal{I}_w := \{L \in \mathcal{C}\langle\partial\rangle \text{ s.t. } L\alpha_{z_0}^z(w) = 0\}$. Then $\mathcal{I}_w$ is a left ideal.

Practical example (polylogarithms)

For $X = \{x_0, x_1\}$, let us consider

$$u_{x_0}(z) = z^{-1} \quad \text{and} \quad u_{x_1}(z) = (1-z)^{-1}.$$

Then, on the other hand,

$$\omega_0(z) = u_{x_0}(z)dz = z^{-1}dz \quad \text{and} \quad \omega_1(z) = u_{x_1}(z)dz = (1-z)^{-1}dz,$$
$$\theta_{x_0} = u_{x_0}^{-1}(z)\partial = z\partial \quad \text{and} \quad \theta_{x_1} = u_{x_1}^{-1}(z)\partial = (1-z)\partial.$$

On the other hand[14], $\mathcal{C} = \mathbb{C}\{\{(u_x^{\pm 1})_{x\in X}\}\} = \mathbb{C}[z, z^{-1}, (1-z)^{-1}]$ being closed by $\theta_{x_0}, \theta_{x_1}$ and then by $\partial = \theta_{x_0} + \theta_{x_1} = \Theta(x_0 + x_1)$.

One also has

1. $\Theta([x_1, x_0]) = [\theta_{x_1}, \theta_{x_0}] = \partial$.
2. $\forall w \in X^* x_1, \Im^0(w)1_\Omega = \alpha_0^z(w) = \mathrm{Li}_w(z)$.
3. $(\theta_{x_0}\iota_{x_1}^{z_0})1_\Omega = z(1-z)^{-1}$ and $(\theta_{x_1}\iota_{x_0}^{z_0})1_\Omega = z^{-1} - 1$.
4. $[\theta_{x_0}\iota_{x_1}^{z_0}, \theta_{x_1}\iota_{x_0}^{z_0}] = 0$.
5. $(\theta_{x_0}\iota_{x_1}^{z_0})(\theta_{x_1}\iota_{x_0}^{z_0}) = (\theta_{x_1}\iota_{x_0}^{z_0})(\theta_{x_0}\iota_{x_1}^{z_0}) = \mathrm{Id}$.

For any $L \in \mathcal{C}\langle\partial\rangle$, there is $P \in \mathcal{C}\langle X\rangle$ s.t $L = \Theta(P)$, meaning that Θ is surjective and non injective. ker Θ ?

14. Any $p \in \mathcal{C}$ is polynomial on z, z^{-1} and $(1-z)^{-1}$ and admits 0 and 1 as poles.

Examples of linear differential equation

Example 8 (with $\mathcal{C} = \mathbb{C}(z)$)

$$(\partial - z)y = 0. \qquad (1)$$

1. $e^{z^2/2}$ is solution of (1).
2. $ce^{z^2/2} = e^{z^2/2}e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$).
3. $\{e^{z^2/2}\}$ is a fundamental set of solutions of (1).
4. $\mathcal{C}\{e^{z^2/2}\}$ is a Picard-Vessiot extension related to (1).

For $\theta_{x_0} = z\partial$ and $\theta_{x_1} = (1-z)\partial$, since $L_{x_1x_0} = \partial\theta_{x_1}\theta_{x_0} \in \mathcal{C}\langle\partial\rangle$ then let

$$L_{x_1x_0}y = (z(1-z)\partial^3 + (2-3z)\partial^2 - 1)y = 0. \qquad (2)$$

1. $L_{x_1x_0}\,\mathrm{Li}_2 = 0$ meaning that Li_2 is solution of (2).
2. $c\,\mathrm{Li}_2 = \mathrm{Li}_2\, e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$) but it is not independent to Li_2.
3. $\{\mathrm{Li}_2, \log, 1_\Omega\}$ is a fundamental set of solutions of (2).
4. $\mathcal{C}\{\mathrm{Li}_2, \log, 1_\Omega\}$ is a Picard-Vessiot extension[15] related to (2).

15. $\mathcal{C}\{\mathrm{Li}_2(z)\} = \mathcal{C} \otimes \mathbb{C}[\mathrm{Li}_2(z), \log(1-z), \log(z)]$.

Chen series of $\{\omega_i\}_{i\geq 1}$ and along $z_0 \rightsquigarrow z$

Let $\mathcal{C} = \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}$. For any $A \supset Q$, we get, on $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X})$ and $\mathcal{H}_{\uplus}(Y)$,

$$\mathcal{D}_{\mathcal{X}} := \sum_{w\in\mathcal{X}^*} w\otimes w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l\otimes P_l} \text{ and } \mathcal{D}_Y := \sum_{w\in Y^*} w\otimes w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\Sigma_l\otimes\Pi_l}.$$

Hence, since iterated integrals satisfy $\alpha_{z_0}^z(u \sqcup\!\sqcup v) = \alpha_{z_0}^z(u)\alpha_{z_0}^z(v)$ $(u, v \in \mathcal{X}^*)$ then the Chen series, $C_{z_0\rightsquigarrow z} \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, is given by

$$C_{z_0\rightsquigarrow z} := \sum_{w\in\mathcal{X}^*} \alpha_{z_0}^z(w)w = (\alpha_{z_0}^z \otimes \mathtt{Id})\mathcal{D}_{\mathcal{X}} = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{\alpha_{z_0}^z(S_l)P_l}$$

and then [16] $\Delta_{\sqcup\!\sqcup} C_{z_0\rightsquigarrow z} = C_{z_0\rightsquigarrow z} \otimes C_{z_0\rightsquigarrow z}$ and $\langle C_{z_0\rightsquigarrow z}|1_{\mathcal{X}^*}\rangle = 1$.

For any $n \geq 0$, one has $\mathbf{d}^n C_{z_0\rightsquigarrow z} = p_n C_{z_0\rightsquigarrow z}$, where [17]

$$p_n = \sum_{\mathrm{wgt}\mathbf{r}=n} \sum_{w\in\mathcal{X}^n} \prod_{i=1}^{\deg\mathbf{r}} \binom{\sum_{j=1}^i r_j + j - 1}{r_i} \tau_{\mathbf{r}}(w) \in \mathcal{C}\langle\mathcal{X}\rangle,$$

and, for $w = x_{i_1}\ldots x_{i_k} \in \mathcal{X}^*$ associated to the derivation multiindex $\mathbf{r} = (r_1,\ldots,r_k) \in \mathbb{N}^k$ of weight $\mathrm{wgt}\mathbf{r} = |w| + \sum_{i=1}^k r_i$ and of degree $\deg\mathbf{r} = |w|$, $\tau_{\mathbf{r}}(w) := \tau_{r_1}(x_{i_1})\ldots\tau_{r_k}(x_{i_k}) = (\partial^{r_1}u_{x_{i_1}})x_{i_1}\ldots(\partial^{r_k}u_{x_{i_k}})x_{i_k}$.

16. $\langle C_{z_0\rightsquigarrow z}|u \sqcup\!\sqcup v\rangle = \langle C_{z_0\rightsquigarrow z}|u\rangle\langle C_{z_0\rightsquigarrow z}|v\rangle$ and on the other hand, $\langle C_{z_0\rightsquigarrow z}|u \sqcup\!\sqcup v\rangle = \langle\Delta_{\sqcup\!\sqcup} C_{z_0\rightsquigarrow z}|u\otimes v\rangle, \langle C_{z_0\rightsquigarrow z}|u\rangle\langle C_{z_0\rightsquigarrow z}|v\rangle = \langle C_{z_0\rightsquigarrow z} \otimes C_{z_0\rightsquigarrow z}|u\otimes v\rangle$.
17. $\forall S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle, \mathbf{d}S = \sum_{w\in\mathcal{X}^*}(\partial\langle S|w\rangle)w \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$.

Continuity, indiscernability and growth condition

For $i = 0, 2$, let $(\mathbf{k}_i, \|.\|_i)$ be a semi-normed space and $g_i \in \mathbb{Z}$.

Definition 9

1. Let $\mathcal{C}l$ be a class of $\mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$. Let $S \in \mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle$ and it is said to be
   a) *continuous* over $\mathcal{C}l$ if, for $\Phi \in \mathcal{C}l$, the following sum is convergent
   $$\sum_{w\in\mathcal{X}^*} \|\langle S|w\rangle\|_2 \|\langle\Phi|w\rangle\|_1 .$$
   We will denote $\langle S\|\Phi\rangle$ the sum $\sum_{w\in\mathcal{X}^*}\langle S|w\rangle\langle\Phi|w\rangle$ and $\mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{\mathrm{cont}}$ the set of continuous power series over $\mathcal{C}l$.
   b) *indiscernable* over $\mathcal{C}l$ iff, for any $\Phi \in \mathcal{C}l$, $\langle S\|\Phi\rangle = 0$.
2. Let χ_1 and χ_2 be real positive functions over $\mathcal{X}^*$. Let $S \in \mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$.
   a) S satisfies the χ_1-*growth condition* of order g_1 if it satisfies
   $$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in \mathcal{X}^{\geq n}, \quad \|\langle S|w\rangle\|_1 \leq K\chi_1(w)\ |w|!^{g_1}.$$
   We denote by $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ the set of formal power series in $\mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$ satisfying the χ_1-growth condition of order g_1.
   b) If S is continuous over $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ then it will be said to be (χ_2, g_2)-*continuous*. The set of formal power series which are (χ_2, g_2)-continuous is denoted by $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle^{\mathrm{cont}}$.

Convergence condition

Proposition 2

Let χ_1 and χ_2 be real positive functions over $\mathcal{X}^*$.
Let g_1 and $g_2 \in \mathbb{Z}$ such that $g_1 + g_2 \leq 0$.

1. *Let $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ and let $P \in \mathbf{k}_1\langle\mathcal{X}\rangle$.*
   The right residual of S by P belongs to $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$.
2. *Let $R \in \mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ and let $Q \in \mathbf{k}_2\langle\mathcal{X}\rangle$.*
   The concatenation QR belongs to $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$.
3. *χ_1, χ_2 are morphisms over $\mathcal{X}^*$ satisfying $\sum_{x\in\mathcal{X}} \chi_1(x)\chi_2(x) < 1$.*
   If $F_1 \in \mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $F_2 \in \mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$) then F_1 (resp. F_2) is continuous over $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$).

Proposition 3

Let $\mathcal{C}l \subset \mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$ be a monoid containing $\{e^{tx}\}_{x\in\mathcal{X}}^{t\in\mathbf{k}_1}$. Let $S \in \mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{cont}$.

1. *If S is indiscernable over $\mathcal{C}l$ then for any $x \in \mathcal{X}$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{cont}$ and they are indiscernable over $\mathcal{C}l$.*
2. *S is indiscernable over $\mathcal{C}l$ iff $S = 0$.*

Chen series and differential equations

Let K be a compact on Ω. There is $c_K \in \mathbb{R}_{\geq 0}$ and a morphism M_K s.t.

$$\forall w \in \mathcal{X}^*, \quad \|\langle C_{z_0 \rightsquigarrow z}|w\rangle\|_K \leq c_K M_K(w)\,|w|!^{-1}.$$

Let $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ of minimal representation (λ, μ, η) of dimension n. Then

$$\forall w \in \mathcal{X}^*, \quad |\langle R|w\rangle| \leq \|\lambda\|_\infty^{1,n}\|\mu(w)\|_\infty^{n,n}\|\eta\|_\infty^{n,1}.$$

With these data, we have

Theorem 10

If $c_K\|\lambda\|_\infty^{1,n}\|\eta\|_\infty^{n,1}\sum_{x\in\mathcal{X}} M_K(x)\|\mu(x)\|_\infty^{n,n} < 1$ then $\alpha_{z_0}^z(R) = \langle R\| C_{z_0\rightsquigarrow z}\rangle$ and

$$\forall x \in \mathcal{X}, \quad \theta_x \alpha_{z_0}^z(R) = \sum_{x'\in\mathcal{X}} u_x^{-1}(z)u_{x'}(z)\alpha_{z_0}^z(R \triangleleft x').$$

Letting $y(z_0, z) := \langle R\| C_{z_0\rightsquigarrow z}\rangle$, the following assertions are equivalent :

1. *There is $p \in \mathcal{C}_0\langle\mathcal{X}\rangle$ s.t. $\langle R\|pC_{z_0\rightsquigarrow z}\rangle = \langle R \triangleleft p\| C_{z_0\rightsquigarrow z}\rangle = 0$.*
2. *There is $l = 0, .., n-1$ s.t. $\{\partial^k y\}_{0\leq k\leq l}$ is $\mathcal{C}_0$-linearly independent and $a_l, \ldots, a_1, a_0 \in \mathcal{C}_0$ s.t. $(a_l\partial^l + \ldots + a_1\partial + a_0)y = 0$.*

Proposition 4

Let $G \in \mathbb{C}\langle\langle X\rangle\rangle$ and $H \in \mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$ s.t. $\alpha_{z_0}^z(G) = \langle G\| C_{z_0\rightsquigarrow z}\rangle$ and $h(\alpha_{z_0}^z(x_0), \alpha_{z_0}^z(x_1)) := \alpha_{z_0}^z(H) = \langle H\| C_{z_0\rightsquigarrow z}\rangle$ exist ($X = \{x_0, x_1\}$). Then

$$\alpha_{z_0}^z(HG) = \langle G|1_{X^*}\rangle\alpha_{z_0}^z(H) + \int_{z_0}^z h(\alpha_s^z(x_0), \alpha_s^z(x_1))d\alpha_{z_0}^s(G).$$

Practical examples (1/2)

For any $x \in \mathcal{X}^*, t \in \mathbb{C}$ and $n \geq 0$, since $x^n = x^{\sqcup\!\sqcup\, n}/n!$ then

$$\alpha_{z_0}^z(x^n) = [\alpha_{z_0}^z(x)]^n/n! \quad \text{and} \quad \alpha_{z_0}^z((tx)^*) = e^{t\alpha_{z_0}^z(x)}.$$

Example 11 (extension of eulerian functions)

Hor any $z \in \mathbb{C}, |z| < 1$, let us consider

$$\ell_1(z) := \gamma z - \sum_{k\geq 2} \zeta(k) \frac{(-z)^k}{k}, \quad \forall r \geq 2, \ell_r(z) := -\sum_{k\geq 1} \zeta(kr) \frac{(-z^r)^k}{k}.$$

Then, for any $k \geq 1$, letting $\omega_k(z) = \partial \ell_k$, one has

$$\alpha_0^z(y_1^*) = e^{\gamma z - \sum_{k\geq 2} \zeta(k) \frac{(-z)^k}{k}}, \quad \forall r \geq 2, \alpha_0^z(y_k^*) = e^{-\sum_{k\geq 1} \zeta(kr) \frac{(-z^r)^k}{k}}.$$

Example 12 ($\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$)

For any $a, b \in \mathbb{C}$ and $n \geq 0$, one has

$$\begin{aligned} \mathrm{Li}_{x_0^n}(z) &= \alpha_1^z(x_0^n) = [\log(z)]^n/n!, & \mathrm{Li}_{x_1^n}(z) &= \alpha_0^z(x_1^n) = [-\log(1-z)]^n/n!, \\ \mathrm{Li}_{(ax_0)^*}(z) &= \alpha_1^z((ax_0)^*) = z^a, & \mathrm{Li}_{(bx_1)^*}(z) &= \alpha_0^z((bx_1)^*) = (1-z)^{-b}. \end{aligned}$$

Hence, for any $S \in \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X \rangle\rangle$ (resp. $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X \rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X \rangle$), letting $\mathcal{C} = \mathbb{C}[z^a, (1-z)^b]_{a,b\in\mathbb{C}}$, one has

$$\mathrm{Li}_S(z) \in \mathcal{C}[\log(z), \log(1-z)] \quad (\text{resp.} \quad \mathcal{C}[\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}]).$$

Practical examples (2/2)

Example 13 (Polylogarithms indexed by non positive integers)

Now, let us use the noncommutative multivariate exponential transforms, *i.e.*, for any rational exchangeable series, we get the following transform

$$\sum_{i_0,i_1\geq 0} s_{i_0,i_1} x_0^{i_0} \sqcup\!\sqcup\, x_1^{i_1} \quad\longmapsto\quad \sum_{i_0,i_1\geq 0} \frac{s_{i_0,i_1}}{i_0!\,i_1!}\log^{i_0}(z)\log^{i_1}((1-z)^{-1}).$$

In particular, for any $n\in\mathbb{N}$, we have $x_0^n\mapsto \log^n(z)/n!$ and $x_1^n\mapsto \log^n((1-z)^{-1})/n!$. Then $(tx_0)^*\mapsto z^t$ and $(tx_1)^*\mapsto(1-z)^{-t}$.
We then obtain the following polylogarithms indexed by rational series

$$\mathrm{Li}_{x_0^*}(z)=z,\quad \mathrm{Li}_{x_1^*}(z)=(1-z)^{-1},\quad \mathrm{Li}_{(ax_0+bx_1)^*}(z)=z^a(1-z)^{-b}$$

Thus, for any $(s_1,\ldots,s_r)\in\mathbb{N}_+^r$, there exists an unique series $R_{y_{s_1}\ldots y_{s_r}}$ belonging to $(\mathbb{Z}[x_1^*],\ \sqcup\!\sqcup\ ,1_{X^*})$ s.t. $\mathrm{Li}_{-s_1,\ldots,-s_r}=\mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}$. More precisely,

$$R_{y_{s_1}\ldots y_{s_r}}=\sum_{k_1=0}^{s_1}\ldots\sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\ldots\binom{\sum_{i=1}^r s_i-\sum_{i=1}^{r-1}k_i}{k_r}\rho_{k_1}\sqcup\!\sqcup\ \ldots\ \sqcup\!\sqcup\,\rho_{k_r},$$

where, for any $i=1,\ldots,r$, if $k_i=0$ then $\rho_{k_i}=x_1^*-1_{X^*}$ else

$$\rho_{k_i}=x_1^*\sqcup\!\sqcup\sum_{j=1}^{k_i}S_2(k_i,j)j!(x_1^*-1_{X^*})^{\sqcup\!\sqcup\, j}$$

the $S_2(k_i,j)$ being the Stirling numbers of second kind.

NONCOMMUTATIVE PV THEORY AND INDEPENDENCE VIA WORDS

First step of noncommutative PV theory

The Chen series $C_{z_0 \rightsquigarrow z}$ satisfies the following differential equation

$$(NCDE) \quad \mathbf{d}S = MS, \quad \text{with} \quad M = \sum_{x \in \mathcal{X}} u_x x.$$

$$\Delta_{\sqcup\!\sqcup} M = \sum_{x \in \mathcal{X}} u_x (1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*}) = 1_{\mathcal{X}^*} \otimes M + M \otimes 1_{\mathcal{X}^*}.$$

The space of solutions of $(NCDE)$ is a right free $\mathbb{C}\langle\langle X \rangle\rangle$-module of rank 1.
By a theorem of Ree, $C_{z_0 \rightsquigarrow z}$ is a $\sqcup\!\sqcup$ $-$group-like solution [18] of $(NCDE)$.
Moreover, if G and H are $\sqcup\!\sqcup$ $-$group-like solutions $(NCDE)$ there is a constant Lie series C such that $G = He^C$ (and conversely).
From this, it follows that

- the differential Galois group of $(NCDE)$+ $\sqcup\!\sqcup$ $-$group-like is the group [19] $\{e^C\}_{C \in \mathcal{L}ie_{\mathbb{C}.1_\Omega}\langle\langle \mathcal{X} \rangle\rangle}$.

Which leads us to the following definition

- the PV extension related to $(NCDE)$ is $\widehat{\mathcal{C}_0.\mathcal{X}}\{C_{z_0 \rightsquigarrow z}\}$.

It, of course, is such that $\mathrm{Const}(\mathcal{C}_0\langle\langle \mathcal{X} \rangle\rangle) = \ker \mathbf{d} = \mathbb{C}.1_\Omega\langle\langle \mathcal{X} \rangle\rangle$.

18. It can be obtained as the limit of a convergent Picard iteration, initialized at $\langle C_{z_0 \rightsquigarrow z} | 1_{\mathcal{X}^*} \rangle = 1_{\mathcal{H}(\Omega)} 1_{\mathcal{X}^*}$, for ultrametric distance.
19. In fact, the Hausdorff group (group of characters) of $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X})$.

Basic triangular theorem over a differential ring

Suppose that the $\mathbb{C}$-commutative ring $\mathcal{A}$ is without zero divisors and equipped with a differential operator ∂ such that $\mathbb{C} = \ker \partial$.
Let $S \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ be a group-like solution of $(NCDE)$ in the following form

$$S = \sum_{w \in \mathcal{X}^*} \langle S|w\rangle w = \sum_{w \in \mathcal{X}^*} \langle S|S_w\rangle P_w = \prod_{l \in \mathcal{L}yn\mathcal{X}}^{\searrow} e^{\langle S|S_l\rangle P_l}.$$

Then

1. If $H \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ is another grouplike solution then there exists $C \in \mathcal{L}ie_{\mathcal{A}}\langle\langle\mathcal{X}\rangle\rangle$ such that $S = He^C$ (and conversely).
2. The following assertions are equivalent
   a) $\{\langle S|w\rangle\}_{w \in \mathcal{X}^*}$ is $\mathcal{C}_0$-linearly independent,
   b) $\{\langle S|l\rangle\}_{l \in \mathcal{L}yn\mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,
   c) $\{\langle S|x\rangle\}_{x \in \mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,
   d) $\{\langle S|x\rangle\}_{x \in \mathcal{X} \cup \{1_{\mathcal{X}^*}\}}$ is $\mathcal{C}_0$-linearly independent,
   e) $\{u_x\}_{x \in \mathcal{X}}$ is such that, for $f \in \mathrm{Frac}(\mathcal{C}_0)$ and $(c_x)_{x \in \mathcal{X}} \in \mathbb{C}^{(\mathcal{X})}$,
   $$\sum_{x \in \mathcal{X}} c_x u_x = \partial f \quad \Longrightarrow \quad (\forall x \in \mathcal{X})(c_x = 0).$$
   f) $(u_x)_{x \in \mathcal{X}}$ is free over $\mathbb{C}$ and $\partial\mathrm{Frac}(\mathcal{C}_0) \cap \mathrm{span}_{\mathbb{C}}\{u_x\}_{x \in \mathcal{X}} = \{0\}$.

Examples of positive cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = 1_\Omega, \mathcal{C}_0 = \mathbb{C}\{\{u_x^{\pm 1}\}\} = \mathbb{C}$.
   $\alpha_0^z(x^n) = z^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = xS$ and

   $$S = \sum_{n \geq 0} \alpha_0^z(x^n) x^n = \sum_{n \geq 0} \frac{z^n}{n!} x^n = e^{zx}.$$

   Moreover, $\alpha_0^z(x) = z$ which is transcendent over $\mathcal{C}_0$
   and the family $\{\alpha_0^z(x^n)\}_{n \geq 0}$ is $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f = 0$. Thus,
   if $\partial f = c u_x$ then $c = 0$.

2. $\Omega = \mathbb{C} \setminus]-\infty, 0], u_x(z) = z^{-1}, \mathcal{C}_0 = \mathbb{C}\{\{z^{\pm 1}\}\} = \mathbb{C}[z^{\pm 1}] \subset \mathbb{C}(z)$.
   $\alpha_1^z(x^n) = \log^n(z)/n!$, for $n \geq 1$. Thus $\mathbf{d}S = z^{-1} x S$ and

   $$S = \sum_{n \geq 0} \alpha_1^z(x^n) x^n = \sum_{n \geq 0} \frac{\log^n(z)}{n!} x^n = z^x.$$

   Moreover, $\alpha_1^z(x) = \log(z)$ which is transcendent over $\mathbb{C}(z)$ then
   over $\mathbb{C}[z^{\pm 1}]$. The family the family $\{\alpha_1^z(x^n)\}_{n \geq 0}$ is $\mathbb{C}(z)$-free and
   then $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f \in \mathrm{span}_{\mathbb{C}}\{z^{\pm n}\}_{n \neq 1}$. Thus,
   if $\partial f = c u_x$ then $c = 0$.

Examples of negative cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = e^z, \mathcal{C}_0 = \mathbb{C}\{\{e^{\pm z}\}\} = \mathbb{C}[e^{\pm z}]$.
   $\alpha_0^z(x^n) = (e^z - 1)^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = e^z x S$ and
   $$S = \sum_{n \geq 0} \alpha_0^z(x^n) x^n = \sum_{n \geq 0} \frac{(e^z - 1)^n}{n!} x^n = e^{(e^z - 1)x}.$$
   Moreover, $\alpha_0^z(x) = e^z - 1$ which is not transcendent over $\mathcal{C}_0$ and and $\{\alpha_0^z(x^n)\}_{n \geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = ce^z \in \mathcal{C}_0$ $(c \neq 0)$ then $\partial f(z) = ce^z = cu_x(z)$.
2. $\Omega = \mathbb{C} \backslash]-\infty, 0], u_x(z) = z^a (a \notin \mathbb{Q})$,
   $\mathcal{C}_0 = \mathbb{C}\{\{z, z^{\pm a}\}\} = \mathrm{span}_{\mathbb{C}}\{z^{ka+l}\}_{k,l \in \mathbb{Z}}$.
   $\alpha_0^z(x^n) = (a+1)^{-n} z^{n(a+1)}/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = z^a x S$ and
   $$S = \sum_{n \geq 0} \alpha_0^z(x^n) x^n = \sum_{n \geq 0} \frac{z^{n(a+1)}}{(a+1)^n n!} x^n = e^{(a+1)^{-1} z^{(a+1)} x}.$$
   Moreover, $\alpha_0^z(x) = z^{a+1}/(a+1)$ which is not transcendent over $\mathcal{C}_0$ and $\{\alpha_0^z(x^n)\}_{n \geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = cz^{a+1}/(a+1) \in \mathcal{C}_0$ $(c \neq 0)$ then $\partial f(z) = cz^a = cu_x(z)$.

Chen series of $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$

Let $\gamma_0(\varepsilon)$ and $\gamma_1(\varepsilon)$ be the circular paths of radius ε encircling 0 and 1 clockwise, respectively. In particular, letting $\beta = \beta_1 - \beta_0$, one considers

$$\begin{array}{rcccl} \gamma_0(\varepsilon,\beta) & = & \varepsilon e^{\mathrm{i}\beta_0} \rightsquigarrow \varepsilon e^{\mathrm{i}\beta_1} & \subset & \gamma_0(\varepsilon), \\ \gamma_1(\varepsilon,\beta) & = & 1-\varepsilon e^{\mathrm{i}\beta_0} \rightsquigarrow 1-\varepsilon e^{\mathrm{i}\beta_1} & \subset & \gamma_1(\varepsilon). \end{array}$$

On the one hand, one has, for any $i = 0$ or 1 and $w \in X^+$,

$$|\langle C_{\gamma_i(\varepsilon,\beta)}|w\rangle| \leq \varepsilon^{|w|_{x_i}} \beta^{|w|} |w|!^{-1}.$$

It follows then

$$C_{\gamma_i(\varepsilon,\beta)} = e^{\mathrm{i}\beta x_i} + o(\varepsilon) \quad \text{and} \quad C_{\gamma_i(\varepsilon)} = e^{2\mathrm{i}\pi x_i} + o(\varepsilon).$$

Hence[20], for $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ of minimal representation (λ, μ, η), one has

$$\langle R \| C_{\gamma_i(\varepsilon,\beta)}\rangle = \lambda \Big(\prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\alpha_{\gamma_i(\varepsilon,\beta)}(S_l)\mu(P_l)} \Big)\eta,$$

$$\langle R \| C_{\gamma_i(\varepsilon)}\rangle = \lambda \Big(\prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\alpha_{\gamma_i(\varepsilon)}(S_l)\mu(P_l)} \Big)\eta.$$

20. Recall that the map $\alpha_{z_0}^z : \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle \to \mathcal{H}(\Omega)$ is not injective. For example, $\alpha_{z_0}^z(z_0 x_0^* + (1-z_0)(-x_1)^* - 1_{X^*}) = 0$.

$\mathrm{Dom}(\mathrm{Li}_\bullet)$

Let $\mathcal{C} := \mathbb{C}[z^a, (1-z)^b]_{a,b\in\mathbb{C}}$ and $\mathrm{Dom}(\mathrm{Li}_\bullet)$ be the set of $S = \sum_{n\geq 0} S_n$ with $S_n = \sum_{|w|=n}\langle S|w\rangle w$ s.t. $\sum_{n\geq 0}\mathrm{Li}_{S_n}$ converges uniformly on any compact of Ω.

Proposition 5

$\mathrm{Dom}(\mathrm{Li}_\bullet)$, containing $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle$, is closed by shuffle and then $\mathrm{Li}_{S \sqcup\!\sqcup\, T} = \mathrm{Li}_S\,\mathrm{Li}_T$ $(S, T \in \mathrm{Dom}(\mathrm{Li}_\bullet))$.

Proposition 6 ($\mathrm{L}(z) = C_{z_0\rightsquigarrow z}\mathrm{L}(z_0)$)

For $R \in \mathrm{Dom}(\mathrm{Li}_\bullet)$, let $\rho := \langle R\|\mathrm{L}\rangle$. Then, for $n \geq 0, \partial^n\rho = \langle R\|\mathbf{d}^n\mathrm{L}\rangle$ and $\mathbf{d}^n\mathrm{L} = p_n\mathrm{L}$, where p_n is given previously, with $\tau_r(x_0) = -r!(-z)^{-(r+1)}x_0$ and $\tau_r(x_1) = r!(1-z)^{-(r+1)}x_1$. The following assertions are equivalent :

1. *ρ satisfies a differential equation with coefficients in $(\mathcal{C}, \partial)$.*
2. *There exists $P \in \mathcal{C}\langle X\rangle$ such that $\langle R\|P\mathrm{L}\rangle = \langle R \triangleleft P\|\mathrm{L}\rangle = 0$.*

Example 14 ($\omega_0(z) = z^{-1}dz, \omega_1(z) = (1-z)^{-1}dz$ & $|c| < 1$)

$$\begin{aligned}\mathrm{Li}_{(cx_0)^* x_1}(z) = \alpha_0^z((cx_0)^* x_1) = &\int_0^z e^{c\log(z/s)}\omega_1(s) &&= z^c\int_0^z \sum_{n\geq 0} s^{n-c}ds\\ = &\; z^c\sum_{n\geq 0}\frac{z^{n-c+1}}{n-c+1} &&= \sum_{n\geq 1}\frac{z^n}{n-c}.\end{aligned}$$

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