Sequential Selection of Long Increasing Subsequences and Related Problems

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Ulam's problem

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Ulam'61: *What is the length* $L_n^{\max}$ *of the longest increasing subsequence of a random permutation of n integers?*

Baer and Brock '68 conjectured that

$$\mathbb{E}L_n^{\max} \sim 2\sqrt{n}, \quad n \to \infty,$$

which was proved by Hammersley '72 ($c\sqrt{n}$), and then Logan-Shepp '77 and Vershik-Kerov '77 ($c = 2$).

Baik, Deift and Johansson '99:

$$\frac{L_n^{\max} - 2\sqrt{n}}{c_0 n^{1/6}} \stackrel{d}{\to} \quad \text{Tracy–Widom distribution.}$$

One can equally consider a sequence of i.i.d. random marks $X_1, \dots, X_n$ sampled from the uniform-$[0,1]$ (or any other continuous distribution), or exchangeable sequence without 'ties'.

The online selection problem

Baer and Brock '68: *What is the maximum expected length* $\mathbb{E}L_n$ *of an increasing subsequence which can be selected by a nonanticipating online strategy*, i.e. under the constraints that

- elements of the background sequence are revealed one at a time,
- a selection/rejection decision must be made straight at the time of observation.

Samuels and Steele '81: for $X_1, \cdots, X_n$ iid uniform-$[0,1]$,

$$\mathbb{E}L_n \sim \sqrt{2n}, \quad n \to \infty.$$

A (suboptimal) strategy achieving the asymptotics has constant acceptance window: observation x is selected if and only if

$$x < y < x + \sqrt{\frac{2}{n}},$$

where x is the last selection so far.

Extensions of the $\sqrt{2n}$ asymptotics

The $\sqrt{2n}$ asymptotics is still valid

- when the input data is a permutation of n known values (Davies '81),
- or when the selector learns only the rank of observation relative to the predecessors (G. 2000).

The asymptotically optimal value $\sqrt{2n}$ should be replaced by

- $$\sqrt{2\sum_{n=1}^{\infty}[\,\mathbb{P}(N \geq n)\,]^2},$$

  in the problem where the number of observations is a random variable N with given distribution (G. '99),
- $$\frac{d+1}{(d+1)!^{1/(d+1)}}n^{1/(d+1)},$$

  in the problem where $X_1, \cdots, X_n$ are sampled from the uniform distribution in $[0,1]^d$, and the online selected subsequence must be increasing coordinate-wise (Baryshnikov and G. '00).

A bin-packing connection

In this talk, we discuss finer results under the assumption that the input data has the uniform distribution and is exactly observable. In that case, the increasing subsequence problem can be cast as the problem with a sum constraint, that

the selected subsequence X_{τ_i} *must satisfy* $\sum_i X_{\tau_i} \leq 1$.

This connection leads to the upper bound

$$\mathbb{E} L_n < \sqrt{2n}, \quad n = 1, 2, \cdots .$$

Proof: under a weaker mean-value constraint $\mathbb{E}\left(\sum_i X_{\tau_i}\right) \leq 1$, the bin-packing strategy of choosing all $X_i \leq \sqrt{2/n}$ is *exactly* optimal.

On-line vs off-line mean, a comparison

- Increasing subsequence

$$\sqrt{2n} \text{ vs } 2\sqrt{n}$$

with gaps $O(\log n)$ resp. $O(n^{1/6})$.

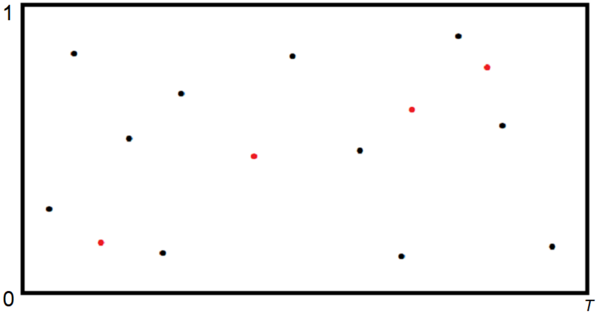
- Bin-packing

$$\sqrt{2n} \text{ vs } \sqrt{2n}$$

with gaps $O(\log n)$ resp. $O(1)$ (Coffman et al '87, G. '21).

Poissonisation

Suppose the marks arrive by a Poisson process on $[0, T]$, so the number of observations is random with Poisson(T)-distribution. A increasing subsequence $(x_1, t_1), \dots, (x_k, t_k)$ of marks/arrival times is a chain in two dimensions: $x_1 < \cdots < x_k, \;\; t_1 < \cdots < t_k$ where each (x_i, t_i) is an atom of the planar Poisson process in $[0, T] \times [0, 1]$

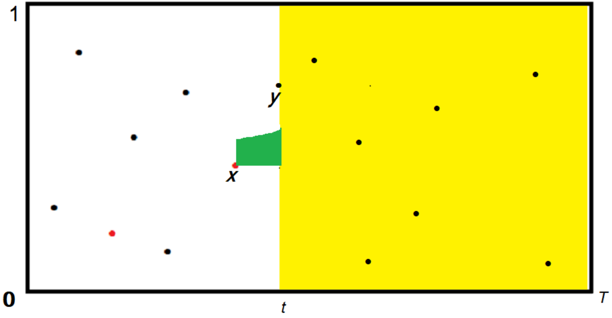


Selfsimilarity

It is sufficient to consider strategies of the kind: if the maximum so far selected mark is x, and (y, t) is observed then y is chosen iff

$$0 < y - x \leq \psi(t, T, x)$$

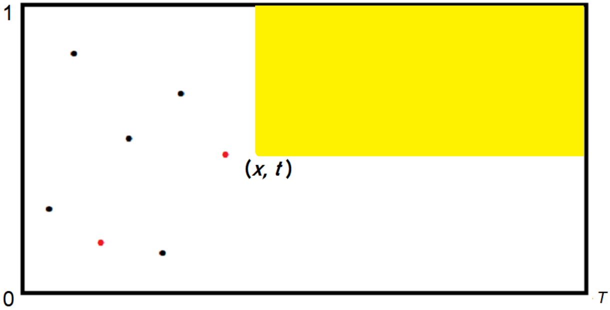
where the function ψ determines the size of the moving acceptance window. (Discrete-time analogues are found in Arlotto et al '18.)



In the case

$$\psi(t, T, x) = (1 - x)\varphi((T - t)(1 - x))$$

for some $\varphi : \mathbb{R} \to \mathbb{R}_+$ the strategy is called *self-similar*.
Then the acceptance window depends on both state x (running maximum) and time through $(T - t)(1 - x)$.

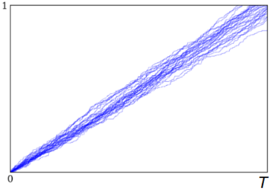


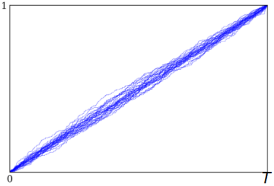
The running maximum under strategies

$$\psi(t, T, x) = \sqrt{2/T},$$

resp.

$$\psi(t, T, x) = \sqrt{\frac{2(1-x)}{T-t}}.$$





The (Poissonised) optimality equation

The expected length $F(T)$ under the optimal online strategy satisfies the dynamic programming equation

$$F'(T) = \int_0^1 \{F(T(1-x)) + 1 - F(T)\}_+ \, dx, \qquad F(0) = 0.$$

Under the optimal strategy (y, t) is accepted iff

$$0 < \frac{y-x}{1-x} < \varphi^*((T-t)(1-x))$$

where x is the last selection so far (the running maximum), and $\varphi^*(z)$ is the solution to

$$F(z(1-x)) + 1 - F(z) = 0$$

(for $z > F^{\leftarrow}(1) = 1.345\ldots$).

The exact solution is only known for small T:

$$F(T) = \int_0^T \frac{1 - e^{-t}}{t} dt, \qquad \nu < 1.345 \cdots ,$$

when the optimal strategy is *greedy*, that is selects every consecutive *record* (i.e. a mark bigger than all seen so far). But for large T the latter only yields about $\log T$ elements.
Bruss-Delbaen '01 bounds:

$$\sqrt{2T} - \log(1 + \sqrt{2T}) + c_0 < F(T) < \sqrt{2T}.$$

Asymptotic expansion of the mean length

G. and Seksenbayev '20: for every self-similar strategy with $\varphi(T) = \sqrt{2/T} + c/T$ (for any constant c) the expected length $F(T)$ of selected increasing subsequence is

$$F(T) = \sqrt{2T} - \frac{1}{12}\log T + c_1 + \frac{\sqrt{2}}{144\sqrt{T}} + O(T^{-1})$$

(c_1 is unknown).
Steps of proof:

- a formal matching of expansion coefficients, and justification using monotonicity (as in Baryshnikov-G. '00),
- renewal theory arguments to justify the constant term.

Linearisation

With $z = \sqrt{T}$ as a linearised size parameter and a change of variables, the equation for expected length under the optimal strategy becomes

$$u'(z) = 4\int_0^1 (u(z-y) + 1 - u(z))_+ (1 - y/z)dy.$$

This is a special case of a nonlinear renewal-type equation

$$u'_{r,\theta}(z) = 4\int_0^{\theta(z)} (u_{r,\theta}(z-y) + r(z) - u_{r,\theta}(z))(1 - y/z)dy$$

with given 'reward' function $r(z)$ and control function $0 < \theta(z) \leq z$ related to a self-similar strategy via

$$\varphi(z^2) = 1 - \left(1 - \frac{\theta(z)}{z}\right)^2.$$

Asymptotic comparison method (Baryshnikov and G. '00)

The operator

$$\mathcal{I}_{\theta,r}g(z) := 4\int_0^{\theta(z)} \left(g(z-y) + r(z) - g(z)\right)(1 - y/z)dy$$

has shift and monotonicity properties that imply

Lemma *If for large enough z,*

(a) $g'(z) > \mathcal{I}_{\theta,r}g(z)$ *then* $\limsup_{z\to\infty}(u_{\theta,r}(z) - g(z)) < \infty$,

(b) $g'(z) < \mathcal{I}_{\theta,r}g(z)$ *then* $\liminf_{z\to\infty}(u_{\theta,r}(z) - g(z)) > -\infty$.

Example For $g(z) = \alpha z$, in the optimality equation (with $r = 1$), (a) holds for $\alpha > \sqrt{2}$, and (b) holds for $\alpha < \sqrt{2}$, whence $u(z) \sim \sqrt{2}\, z$.

Iterating with the test function $g(z) = \sqrt{2}z + \alpha \log z$ yields

$$u(z) \sim \sqrt{2}\, z - \frac{1}{6}\log z + O(1), \quad z \to \infty.$$

Further expansion terms can be computed, but the method does not capture the constant term.

Piecewise deterministic Markov process

For given control function $0 < \theta(z) \leq z$, a PDMP process Z on $[0, \infty)$ is defined by the rules

(i) drifts with unit speed towards 0,

(ii) jumps at probability rate $4\lambda(z)$, where

$$\lambda(z) := \theta(z) - \frac{\theta^2(z)}{2z},$$

(iii) if jumps occurs, then from z to $z - y$, with y having density $(1 - y/z)/\lambda(z)$ for $y \in [0, \theta(z)]$,

(iv) terminates in 0.

The number of jumps $N_\theta(z)$ of Z starting from $z = \sqrt{T}$ is equal to L_T, the length of increasing subsequence under a self-similar strategy.

Let $U(z_0, dz)$, be the occupation measure on $[0, z_0]$, for the sequence of jump points of Z starting from z_0, and controlled by the optimal $\theta^*(z)$. The density is

$$U(z_0, dz) = 4\lambda(z)p(z_0, z)dz,$$

where $p(z_0, z)$ is the probability that z is a drift point.
Lemma *There exists a pointwise limit* $p(z) := \lim_{z_0 \to \infty} p(z_0, z)$, *such that* $\lim_{z \to \infty} p(z) = 1/2$ *and for some* $a, b > 0$

$$|p(z_0, z) - p(z)| < ae^{-b(z_0 - z)}, \quad 0 < z < z_0.$$

The proof is by coupling: two independent Z-processes starting with z_1 and z_2 (where $z_1 < z_2$) with high probability visit the same drift point close to z_1.

The 'mean reward' for process Z starting with $z > 0$ has representation

$$u_{\theta^*,r}(z) = \int_0^z r(y) U(z, dy).$$

A 'renewal reward theorem':

Corollary *For integrable $r(z)$,*

$$u_{\theta^*,r}(z) \to \int_0^\infty r(y)\lambda(y)p(y)dy, \quad z \to \infty.$$

If $r(z) = O(z^{-\beta})$ with $\beta > 1$ then the convergence rate is $O(z^{-\beta+1})$.

This allows us to obtain the asymptotic expansions of the moments of $N_\theta(t)$ and of the length of selected sequence $L_\varphi(t)$ under self-similar strategies. In particular, $w(z) = \mathbb{E}[N_{\theta^*}(z)]^2$ satisfies

$$w'(z) = 4\int_0^{\theta^*(z)} (w(z-y) - w(z) + (1 + 2u(z-y))(1 - y/z)dy,$$

with $w(0) = 0$.

The asymptotic expansion of moments

For L_T the length of selected subsequence under self-similar strategy with given control function φ, and $F(T) = \mathbb{E}L_T$ it holds:

Theorem (G. and Seksenbayev '20) *The expected length under the optimal strategy is*

$$F(T) \sim \sqrt{2T} - \frac{1}{12}\log T + c_1 + \frac{\sqrt{2}}{144\sqrt{T}} + O(T^{-1})$$

and the variance is

$$\mathrm{Var} L_T = \frac{\sqrt{2T}}{3} + \frac{1}{72}\log T + c_2 + O(T^{-1/2}\log T).$$

The optimal strategy is self-similar with

$$\varphi^*(T) \sim \sqrt{\frac{2}{T}} - \frac{1}{3T} + O(T^{-3/2}).$$

Constants c_1, c_2 are unknown.

A renewal process approximation to Z

The range of Z (starting at location z) is an alternating sequence of drift intervals and gaps skipped by jumps. Let D_z be the length of the generic drift interval and J_z that of jump. From

$$\theta^*(z) = \frac{1}{\sqrt{2}} + \frac{1}{12z} + O(z^{-2})$$

follows that for $z \to \infty$ that $4\lambda(z) \to 2\sqrt{2}$ and

$$D_z \stackrel{d}{\to} \frac{E}{2\sqrt{2}}, \quad J_z \stackrel{d}{\to} \frac{U}{\sqrt{2}},$$

where E and U are independent Exponential(1) and Uniform-$[0,1]$ random variables. At large distance from 0, the generic jump of Z is approximable by *decreasing* renewal process with cycle-size

$$D_z + J_{z-D_z} \stackrel{d}{\to} \frac{E}{2\sqrt{2}} + \frac{U}{\sqrt{2}} =: H$$

CLT by stochastic comparison

Cutsem and Ycart '94, Haas and Miermont '11, Alsmeyer and Marynych '16 obtained limit theorems for absorption times (or jump-counts) for decreasing Markov chains on $\mathbb{N}$.

Adapting (and filling in a gap in) the stochastic comparison method of Cutsem and Ycart, we squeeze

$$(1 + c/\underline{z})^{-1} H <_{\mathrm{st}} D_z + J_{z-D_z} <_{\mathrm{st}} (1 - c/\underline{z})^{-1} H$$

for $z > \underline{z}$, where $\underline{z} = \omega\sqrt{z}$ and ω large parameter.
Accordingly, the number of jumps of Z within $[\underline{z}, z]$ is squeezed between two renewal processes which satisfy the CLT.

It is important that the cycle-size of Z is within $O(z^{-1})$ from the limit, since by slower convergence rate $O(z^{-1/2+\epsilon})$ the normal approximation may fail.

Theorem *For every self-similar selection strategy with*

$$\varphi(T) = \sqrt{\frac{2}{T}} + O(T^{-1})$$

the expected length of selected increasing subsequence is within $O(1)$ *from the optimum, and the CLT holds*

$$\sqrt{3}\,\frac{L_T - \sqrt{2T}}{(2T)^{1/4}} \xrightarrow{d} \mathcal{N}(0,1).$$

Compare with the Tracy-Widom limit for the length of the LIS.

Bruss and Delbaen '04, Arlotto et al '15, proved analogous CLT in the fixed-n setting, for the optimal strategy using concavity of the value function $F(T)$ and martingale methods.

Transversal fluctuations of the online selected subsequence

Q: *What is the behaviour of the optimal online selected subsequence viewed as a function of time?*

Standardised framework: Poisson point process of intensity T in the square $[0,1] \times [0,1]$.
$L_T(t)$ number of selections by time $t \in [0,1]$,
$X_T(t)$ the running maximum.

Bruss and Delbaen '04 have shown: The processes (X_T, L_T) centred by their compensators (C_X, C_L) (and properly scaled) converge weakly in Skorokhod's space $D[0,1]$ to a bivariate BM $\boldsymbol{W} = (W_1, W_2)$ with covariance

$$\mathbb{E}[\boldsymbol{W}(1)^T \boldsymbol{W}(1)] = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{2}}{3} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

For given ψ, an acceptance window (in the Poissonised setting), the compensators themselve are unknown adapted processes

$$C_X(t) = \frac{\nu}{2} \int_0^t \psi^2(s, \nu, X(s)) \mathrm{d}s,$$

$$C_L(t) = \nu \int_0^t \psi(s, \nu, X(s)) \mathrm{d}s,$$

which account for a substantial fluctuation component.

We shall assume that the employed selection strategy is optimal or is any other self-similar with

$$\varphi(T) = \sqrt{\frac{2}{T}} + O(T^{-1}),$$

so with the acceptance window

$$\psi(t, T, x) = \sqrt{\frac{2(1-x)}{T(1-t)}} + O\left(\frac{1}{T(1-t)}\right).$$

We scale the running maximum and running length as

$$\widetilde{X}_T(t) := T^{1/4}(X_T(t) - t),$$
$$\widetilde{L}_T(t) := T^{1/4}\left(\frac{L_T(t)}{\sqrt{2T}} - t\right)$$

To compare: scaling for the longest increasing subsequence is $T^{1/6}$ resp. $T^{1/3}$, and yields a non-Gaussian functional limit (Duverne, Nica, Virag '19).

G. and Seksenbayev '20:

Theorem *As $T \to \infty$, in $D[0,1]$ the weak convergence holds:*

$$(\widetilde{X}_T, \widetilde{L}_T) \Rightarrow (Y_1, Y_2),$$

where the limit satisfies SDE

$$\mathrm{d}Y_1(t) = \frac{-Y_1(t)}{1-t}\mathrm{d}t + \mathrm{d}W_1(t),$$
$$\mathrm{d}Y_2(t) = \frac{-Y_1(t)}{2(1-t)}\mathrm{d}t + \mathrm{d}W_2(t).$$

with the initial condition 0*. Explicitly,*

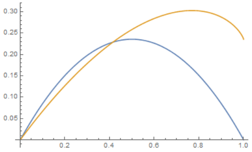
$$Y_1(t) = (1-t)\int_0^t \frac{\mathrm{d}W_1(s)}{1-s}$$

is a Brownian bridge with $\sigma_1^2 = 2\sqrt{2}/3$*, and*

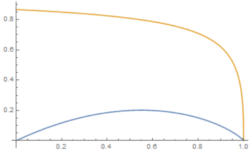
$$Y_2(t) = \frac{Y_1(t)}{2} - \frac{W_1(t)}{2} + W_2(t).$$

For $0 \leq s < t < 1$ the cross-covariance matrix is

$$\mathbb{E}\{\boldsymbol{Y}(s)^T \boldsymbol{Y}(t)\} = \begin{pmatrix} \frac{2\sqrt{2}\, s(1-t)}{3} & \frac{2s(1-t)-(1-s)\log(1-s)}{3\sqrt{2}} \\ \frac{(1-t)(2s-\log(1-s))}{3\sqrt{2}} & \frac{2s(2-t)-(2-s-t)\log(1-s)}{6\sqrt{2}} \end{pmatrix},$$



$\mathrm{Var}\, Y_1$, $\mathrm{Var}\, Y_2$



$\mathrm{Cov}(Y_1(t), Y_2(t))$, $\mathrm{Corr}(Y_1(t), Y_2(t))$

Generators

The generator of $(\widetilde{X}, \widetilde{L}, t)$ is

$$\widetilde{\mathcal{L}}_T f = f_t - T^{-1/4}(f_x + f_\ell) + T^{3/4} \int_0^{\widetilde{\psi}} \{f(x+u, \ell+1, t) - f(x, \ell, t)\} \mathrm{d}u,$$

where $\widetilde{\psi} = T^{1/4}\psi(t, T, xT^{-1/4} + t)$, converges to

$$\widetilde{\mathcal{L}} f := f_t - \frac{x}{1-t} f_x - \frac{x}{2(1-t)} f_\ell + \frac{\sigma_1^2}{2} f_{xx} + \frac{\sigma_2^2}{2} f_{\ell\ell} + \sigma_1 \sigma_2 \rho f_{x\ell}$$

for functions f on $[0,1] \times \mathbb{R}_+ \times [0, 1-h]$ in a core of the operator, for every fixed $h \in (0,1)$.

This implies convergence in $D[0, 1-h]$, for any given $h > 0$.

End of proof: tightness in $D[0,1]$

Let $X_T^{\uparrow}$ be the process driven by the control function

$$\psi^{\uparrow}(t) := \sqrt{\frac{2}{T}} + \frac{\beta}{T(1-t)}\,1(t \le 1 - KT^{-1/2}),$$

and a reflection near the diagonal $t = x$, and $X_T^{\downarrow}$ a process with control

$$\psi^{\downarrow}(t,x) = \begin{cases} \left(\sqrt{\frac{2}{T}} - \frac{\beta}{T(1-t)}\right) \wedge (t-x), & \text{for} \quad 0 \le t \le 1 - K/\sqrt{T}, \\ 0, \quad \text{for} \quad 1 - K/\sqrt{T} < t \le 1, \end{cases}$$

where $K, \beta > 0$ are large enough constants. Both $X^{\uparrow}$ and $X^{\downarrow}$ converge in $D[0,1]$ to reflected Brownian motions. The tightness of $\widetilde{X}_T$'s on the whole $[0,1]$ follows, since

$$X_T^{\downarrow} \le_{\mathrm{st}} X_T \le_{\mathrm{st}} X_T^{\uparrow}.$$

Moment bounds

If $\varphi(\nu) = \sqrt{2/T} + O(T^{-1})$ then the mean running maximum and the mean running length satisfy uniformly in $t \in [0, 1]$

$$|\mathbb{E}X_T(t) - t| = O(T^{-1/2}),$$

$$\sqrt{2T}\, t + \frac{1}{6}\log(1 - t) + C_1 \leq \mathbb{E}L_T(t) \leq \sqrt{2T}\, t + C_2 t.$$

For the optimal strategy, $C_2 = 0$.