Mathematical renormalization in quantum electrodynamics via noncommutative generating series (Part I)

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Summary

1. Introduction.
2. Algebraic combinatorics of formal power series on noncommutative variables.
3. Algebraic combinatorics of polylogarithms, multiple harmonic sums and polyzetas.
4. Nonlinear differential equations.

INTRODUCTION

(Il était une fois le rêve d'Icare)

Nonlinear dynamical systems

Let ∂_z denotes d/dz.

$$(NDS)\begin{cases} y(z) & = & f(q(z)), \\ \partial_z q(z) & = & A_0(q)u_0(z) + A_1(q)u_1(z), \\ q(z_0) & = & q_0, \end{cases}$$

where :

- $u_0(z)$ and $u_1(z)$ are "controles", or "inputs",
- the state $q = (q_1, \ldots, q_n)$ belongs the complex analytic manifold Q of dimension n and q_0 is the initial state,
- the observation $f \in \mathcal{O}$, with $\mathcal{O}$ the ring of holomorphic functions over Q,
- For $i = 0..1$, $A_i(q) = \sum_{j=1}^{n} A_i^j(q)\frac{\partial}{\partial q_j}$ is an analytic vector field over Q, with $A_i^j(q) \in \mathcal{O}$, for $j = 1, \ldots, n$,
- $y(z)$ is the "output" of (NDS).

Examples of Nonlinear Dynamical System

Example (harmonic oscillator)

Let k_1, k_2 be parameters and $\partial_z y(z) + k_1 y(z) + k_2 y^2(z) = u_1(z)$ which can be represented by the following state equations

$$\begin{aligned}
\partial_z q(z) &= A_0(q)u_0(z) + A_1(q)u_1(z), \\
A_0 &= -(k_1 q + k_2 q^2)\frac{\partial}{\partial q} \quad \text{and} \quad A_1 = \frac{\partial}{\partial q}, \\
y(z) &= q(z).
\end{aligned}$$

Example (Duffing's equation)

Let a, b, c be parameters and $\partial_z^2 y(z) + a\partial_z y(z) + by(z) + cy^3(z) = u_1(z)$ which can be represented by the following state equations

$$\begin{aligned}
\partial_z q(z) &= A_0(q)u_0(z) + A_1(q)u_1(z), \\
A_0 &= -(aq_2 + b^2 q_1 + cq_1^3)\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q_1} \quad \text{and} \quad A_1 = \frac{\partial}{\partial q_2}, \\
y(z) &= q_1(z).
\end{aligned}$$

Nonlinear differential equations with three sigularities

$$
(NDE) \begin{cases} y(z) & = & f(q(z)), \\ \partial_z q(z) & = & A_0(q)u_0(z) + A_1(q)u_1(z), \\ q(z_0) & = & q_0, \end{cases}
$$

where :

- $u_0(z) = \dfrac{1}{z}$, $u_1(z) = \dfrac{1}{1-z}$,
- the state $q = (q_1, \ldots, q_n)$ belongs the complex analytic manifold Q of dimension n and q_0 is the initial state,
- the observation $f \in \mathcal{O}$, with $\mathcal{O}$ the ring of holomorphic functions over Q,
- For $i = 0..1$, $A_i(q) = \displaystyle\sum_{j=1}^{n} A_i^j(q) \frac{\partial}{\partial q_j}$ is an analytic vector field over Q, with $A_i^j(q) \in \mathcal{O}$, for $j = 1, \ldots, n$.

Particular cases : Fuchsian differential equations (FDE)

$$\partial_z q(z) = [M_0 u_0(z) + M_1 u_1(z)]\, q(z), \quad y(z) = \lambda q(z), \quad q(z_0) = \eta,$$

where $M_0, M_1 \in \mathcal{M}_{n,n}(\mathbb{C}), \lambda \in \mathcal{M}_{1,n}(\mathbb{C}), \eta \in \mathcal{M}_{n,1}(\mathbb{C})$, and $u_0(z) = z^{-1}, u_1(z) = (1-z)^{-1}$.

Example (hypergeometric equation)

Let t_0, t_1, t_2 be parameters and

$$z(1-z)\partial_z^2 y(z) + [t_2 - (t_0 + t_1 + 1)z]\partial_z y(z) - t_0 t_1 y(z) = 0.$$

Let $q_1(z) = y(z)$ and $q_2(z) = z(1-z)\partial_z y(z)$. One has

$$\begin{pmatrix} \partial_z q_1 \\ \partial_z q_2 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ -t_0 t_1 & -t_2 \end{pmatrix} u_0(z) - \begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix} u_1(z) \right] \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

$$\lambda = \begin{pmatrix} 1 & 0 \end{pmatrix} M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix} M_1 = \begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix} \eta = \begin{pmatrix} q_1(z_0) \\ q_2(z_0) \end{pmatrix}.$$

$$A_0(q) = -(t_0 t_1 q_1 + t_2 q_2)\frac{\partial}{\partial q_2} \quad \text{and} \quad A_1(q) = -q_1 \frac{\partial}{\partial q_1} - (t_2 - t_0 - t_1) q_2 \frac{\partial}{\partial q_2}.$$

Present work

By successive Picard iterations, one get

$$
\begin{aligned}
y(z) &= \sum_{k\geq 0} \sum_{i_1,\ldots,i_k=0,1} A_{i_1}\circ\ldots\circ A_{i_k}(f(q_0)) \\
&\quad \int_{z_0}^{z} u_{i_1}(z_1)dz_1 \int_{z_0}^{z_1} u_{i_2}(z_2)dz_2 \ldots \int_{z_0}^{z_{k-1}} u_{i_k}(z_k)dz_k
\end{aligned}
$$

Let $X=\{x_0,x_1\}$ and for any $w=x_{i_1}\cdots x_{i_k}\in X^*$,

$$
\begin{aligned}
\mathcal{A}(w) &= A_{i_1}\circ\ldots\circ A_{i_k}, \\
\alpha_{z_0}^{z}(w) &= \int_{z_0}^{z} u_{i_1}(z_1)dz_1 \int_{z_0}^{z_1} u_{i_2}(z_2)dz_2 \ldots \int_{z_0}^{z_{k-1}} u_{i_k}(z_k)dz_k.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y(z) &= \Big[\sum_{w\in X^*} \mathcal{A}(w)\alpha_{z_0}^{z}(w)\Big](f(q_0)) \\
&= [(\mathcal{A}\otimes\alpha_{z_0}^{z})\mathcal{D}](f(q_0)),
\end{aligned}
$$

where, $\mathcal{D}_X = \sum_{w\in X^*} w\otimes w.$

Noncommutative generating series

- Fliess generating series and Chen series

$$\sigma f_{|_{q_0}} = \sum_{w \in X^*} \mathcal{A}(w)(f(q_0))\, w \quad \text{and} \quad S_{z_0 \rightsquigarrow z} = \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\, w.$$

- The duality between $\sigma f_{|_{q_0}}$ and $S_{z_0 \rightsquigarrow z}$ consists on the *convergence* of

$$\langle \sigma f_{|_{q_0}} \| S_{z_0 \rightsquigarrow z} \rangle = \sum_{w \in X^*} \langle \mathcal{A}(w)(f(q_0)) | w \rangle \langle S_{z_0 \rightsquigarrow z} | w \rangle.$$

- Divergence :

$$\left(\frac{d}{dz}\right)^n y(z) \underset{z \to 1}{\widetilde{\ }} ? \quad \text{and} \quad \left(z\frac{d}{dz}\right)^n y(z) \underset{z \to 1}{\widetilde{\ }} ?$$

- If theTaylor expansions of $(d/dz)^n y(z)$ and $(zd/dz)^n y(z)$ exist :

$$\left(\frac{d}{dz}\right)^n y(z) = \sum_{k \geq 0} d_k z^n \quad \text{and} \quad \left(z\frac{d}{dz}\right)^n y(z) = \sum_{k \geq 0} t_k z^k$$

then

$$d_k \underset{k \to \infty}{\widetilde{\ }} ? \quad \text{and} \quad t_k \underset{k \to \infty}{\widetilde{\ }} ?$$

Chen's iterated integral along a path and polylogarithms

Let $\omega_0(z) = u_0(z)dz$ and $\omega_1(z) = u_1(z)dz$. The iterated integral over $\omega_0(z)$ and $\omega_1(z)$ associated to $w = x_{i_1} \cdots x_{i_k}$ is defined by

$$\alpha_{z_0}^{z}(1_{X^*}) = 1 \quad \text{and} \quad \alpha_{z_0}^{z}(x_{i_1} \ldots x_{i_k}) = \int_{z_0}^{z} \omega_{i_1}(z_1) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k).$$

For any $w = x_0^{s_1-1} x_1 \ldots x_0^{s_r-1} x_1 \in X^* x_1$,

$$\alpha_0^z(w) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} = \mathrm{Li}_{s_1,\ldots,s_r}(z).$$

Example $\alpha_0^z(x_0 x_1) = \mathrm{Li}_2(z)$

$$\begin{aligned} &= \int_0^z \frac{ds}{s} \int_0^s \frac{dt}{1-t} \\ &= \int_0^z \frac{ds}{s} \int_0^s dt \sum_{k \geq 0} t^k \\ &= \sum_{k \geq 1} \int_0^z ds \frac{s^{k-1}}{k} \\ &= \sum_{k \geq 1} \frac{z^k}{k^2}. \end{aligned}$$

Polylogarithms, multiple harmonic sums and polyzetas

$$\mathrm{H}_{\mathbf{s}}(N)=\sum_{N\geq n_1>\ldots>n_r>0}\frac{1}{n_1^{s_1}\ldots n_r^{s_r}}\quad\text{and}\quad\mathrm{Li}_{\mathbf{s}}(z)=\sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}.$$

If $s_1>1$ **then** by an Abel's theorem, one has

$$\lim_{N\to\infty}\mathrm{H}_{\mathbf{s}}(N)=\lim_{z\to1}\mathrm{Li}_{\mathbf{s}}(z)=\zeta(\mathbf{s})=\sum_{n_1>\ldots>n_r>0}\frac{1}{n_1^{s_1}\ldots n_r^{s_r}}$$

else to determine the asymptotic expansion of

$$\mathrm{H}_{\{1\}^r}(N)=\sum_{N\geq n_1>\ldots>n_r>0}\frac{1}{n_1\ldots n_r},$$

$$\mathrm{H}_{\{1\}^k,\underbrace{s_{k+1}}_{>1},\ldots,s_r}(N)=\sum_{N\geq n_1>\ldots>n_r>0}\frac{1}{n_1\ldots n_k n_{k+1}^{s_{k+1}}\ldots n_r^{s_r}}.$$

Fact : one has $\displaystyle\sum_{n\geq0}\mathrm{H}_{\mathbf{s}}(n)z^n=\frac{1}{1-z}\mathrm{Li}_{\mathbf{s}}(z)=\mathrm{P}_{\mathbf{s}}(z)$.

Let $\mathcal{Z}$ be the $\mathbb{Q}$-algebra generated by convergent polyzetas.

Encoding the multi-indices by words

$Y = \{y_k | k \in \mathbb{N}_+\}$ and $X = \{x_0, x_1\}$.
Y^* (resp. X^*) : set of words over Y (resp. X).

$$\mathbf{s} = (s_1, \ldots, s_r) \leftrightarrow w = y_{s_1} \ldots y_{s_r} \leftrightarrow w = x_0^{s_1 - 1} x_1 \ldots x_0^{s_r - 1} x_1.$$

w is said convergent if $s_1 > 1$. A divergent word is of the form

$$(1^k, s_{k+1}, \ldots, s_r) \leftrightarrow y_1^k y_{s_{k+1}} \ldots y_{s_r} \leftrightarrow x_1^k x_0^{s_{k+1} - 1} x_1 \ldots x_0^{s_r - 1} x_1.$$

$$\forall w \in Y^*, \quad \mathrm{Li}_w : w \mapsto \mathrm{Li}_w(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}},$$

$$\zeta_w : w \mapsto \zeta(w) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}},$$

$$\mathrm{H}_w : w \mapsto \mathrm{H}_w(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}},$$

$$\mathrm{P}_w : w \mapsto \mathrm{P}_w(z) = \sum_{N \geq 0} \mathrm{H}_w(N) z^N = \frac{1}{1 - z} \mathrm{Li}_w(z).$$

ALGEBRAIC COMBINATORICS OF FORMAL POWER SERIES ON NONCOMMUTATIVE VARIABLES

(La conquête de Mars ...)

Shuffle bialgebra and Schützenberger's factorization

Let $\mathcal{L}ynX$ be the set of Lyndon words over X.

$$
\begin{aligned}
P_l &= l && \text{for } l \in Y, \\
P_l &= [P_s, P_r] && \text{for } l \in \mathcal{L}ynY \setminus Y, \\
& && \text{standard factorization of } l = (s, r), \\
P_w &= P_{l_1}^{i_1} \ldots P_{l_k}^{i_k} && \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, \\
& && l_1 > \ldots > l_k, l_1 \ldots, l_k \in \mathcal{L}ynY. \\
S_l &= 1 && \text{for } l = 1_{X^*}, \\
S_l &= xS_u, && \text{for } l = xu \in \mathcal{L}ynY, \\
S_w &= \frac{S_{l_1}^{\sqcup\!\sqcup\, i_1} \,\sqcup\!\sqcup\, \ldots \,\sqcup\!\sqcup\, S_{l_k}^{\sqcup\!\sqcup\, i_k}}{i_1! \ldots i_k!} && \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, \\
& && l_1 > \ldots > l_k, , l_1 \ldots, l_k \in \mathcal{L}ynY.
\end{aligned}
$$

Theorem (Schützenberger, 1958)

$$\sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynY}^{\rightarrow} \exp(S_l \otimes P_l).$$

Example

l	P_l	S_l
x_0	x_0	x_0
x_1	x_1	x_1
$x_0 x_1$	$[x_0, x_1]$	$x_0 x_1$
$x_0^2 x_1$	$[x_0, [x_0, x_1]]$	$x_0^2 x_1$
$x_0 x_1^2$	$[[x_0, x_1], x_1]$	$x_0 x_1^2$
$x_0^3 x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3 x_1$
$x_0^2 x_1^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0^2 x_1^2$
$x_0 x_1^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0 x_1^3$
$x_0^4 x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0^4 x_1$
$x_0^3 x_1^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0^3 x_1^2$
$x_0^2 x_1 x_0 x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0^3 x_1^2 + x_0^2 x_1 x_0 x_1$
$x_0^2 x_1^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0^2 x_1^3$
$x_0 x_1 x_0 x_1^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0^2 x_1^3 + x_0 x_1 x_0 x_1^2$
$x_0 x_1^4$	$[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0 x_1^4$
$x_0^5 x_1$	$[x_0, [x_0, [x_0, [x_0, [x_0, x_1]]]]]$	$x_0^5 x_1$
$x_0^4 x_1^2$	$[x_0, [x_0, [x_0, [[x_0, x_1], x_1]]]]$	$x_0^4 x_1^2$
$x_0^3 x_1 x_0 x_1$	$[x_0, [[x_0, [x_0, x_1]], [x_0, x_1]]]$	$2x_0^4 x_1^2 + x_0^3 x_1 x_0 x_1$
$x_0^3 x_1^3$	$[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$	$x_0^3 x_1^3$
$x_0^2 x_1 x_0 x_1^2$	$[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$	$3x_0^3 x_1^3 + x_0^2 x_1 x_0 x_1^2$
$x_0^2 x_1^2 x_0 x_1$	$[[x_0, [[x_0, x_1], x_1]], [x_0, x_1]]$	$6x_0^3 x_1^3 + 3x_0^2 x_1 x_0 x_1^2 + x_0^2 x_1^2 x_0 x_1$
$x_0^2 x_1^4$	$[x_0, [[[[x_0, x_1], x_1], x_1], x_1]]$	$x_0^2 x_1^4$
$x_0 x_1 x_0 x_1^3$	$[[x_0, x_1], [[[x_0, x_1], x_1], x_1]]$	$4x_0^2 x_1^4 + x_0 x_1 x_0 x_1^3$
$x_0 x_1^5$	$[[[[[x_0, x_1], x_1], x_1], x_1], x_1]$	$x_0 x_1^5$

q-stuffle bialgebra

The q-stuffle product defined by $u \sqcup\!\sqcup_q 1_{Y^*} = 1_{Y^*} \sqcup\!\sqcup_q u = u$ and

$$\begin{aligned} y_i u \sqcup\!\sqcup_q y_j v &= y_i(u \sqcup\!\sqcup_q y_j v) + y_j(y_i u \sqcup\!\sqcup_q v) \\ &+ q y_{i+j}(u \sqcup\!\sqcup_q v), \end{aligned}$$

and its associated coproduct is defined respectively by

$$\forall y_k \in Y, \Delta_{\sqcup\!\sqcup_q}(y_k) = y_k \otimes 1 + 1 \otimes y_k + q \sum_{i+j=k} y_i \otimes y_j$$

satistying $\langle \Delta_{\sqcup\!\sqcup_q}(w) | u \otimes v \rangle = \langle w | u \sqcup\!\sqcup_q v \rangle$ and if $\pi_1^{(q)}(y_k)$ is a homogenous polynomial of $\deg y_k = k$ and is given by

$$\pi_1^{(q)}(y_k) = y_k + \sum_{i \geq 2} \frac{(-q)^{i-1}}{i} \sum_{\substack{j_1,\ldots,j_i \geq 1 \\ j_1+\ldots+j_i=k}} y_{j_1} \ldots y_{j_i}.$$

then $\Delta_{\sqcup\!\sqcup_q}(\pi_1^{(q)}(y_k)) = \pi_1^{(q)}(y_k) \otimes 1 + 1 \otimes \pi_1^{(q)}(y_k)$.

Examples, with $q = +1, 0, -1$, lead respectively to stuffle, shuffle, minus-stuffle products.

Extended Schützenberger's factorization ($q = 1$)

$$\begin{cases} \Pi_y = \pi_1(y) & \text{for } y \in Y, \\ \Pi_l = [\Pi_s, \Pi_r] & \text{for } l \in \mathcal{L}ynY, \\ & \text{standard factorization of } l = (s, r), \\ \Pi_w = \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k} & \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, \\ & l_1 > \ldots > l_k, l_1 \ldots, l_k \in \mathcal{L}ynY, \end{cases}$$

$$\begin{cases} \Sigma_y = y & \text{for } y \in Y, \\ \Sigma_l = \displaystyle\sum_{\substack{\{s'_1,\cdots,s'_i\}\subset\{s_1,\cdots,s_k\}, l_1\geq\cdots\geq l_n\in\mathcal{L}ynY \\ (y_{s_1}\cdots y_{s_k}) \overset{*}{\Leftarrow} (y_{s'_1},\cdots,y_{s'_n}, l_1,\cdots,l_n)}} \frac{y_{s'_1+\cdots+s'_i}}{i!} \Sigma_{l_1\cdots l_n} & \text{for } \substack{l\in\mathcal{L}ynY \\ l=y_{s_1}\cdots y_{s_k},} \\ \Sigma_w = \dfrac{1}{i_1! \ldots i_k!} \Sigma_{l_1}^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \Sigma_{l_k}^{\sqcup\!\sqcup\, i_k} & \text{for } \substack{l_1>\ldots>l_k \\ w=l_1^{i_1}\ldots l_k^{i_k},} \end{cases}$$

$$\mathcal{D}_Y = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\Sigma_l \otimes \Pi_l) \in \mathcal{H}_{\sqcup\!\sqcup}^{\vee} \hat{\otimes} \mathcal{H}_{\sqcup\!\sqcup}.$$

Example (for $q = 1$)

$$\begin{aligned}
\Pi_{y_4} &= y_4 - \frac{1}{2} y_1 y_3 - \frac{1}{2} y_2 y_2 - \frac{1}{2} y_3 y_1 + \frac{1}{3} y_1^2 y_2 + \frac{1}{3} y_1 y_2 y_1 \\
&+ \frac{1}{3} y_2 y_1^2 - \frac{1}{4} y_1^4, \\
\Pi_{y_3 y_1} &= y_3 y_1 - \frac{1}{2} y_2 y_1^2 - y_1 y_3 + \frac{1}{2} y_1^2 y_2, \\
\Pi_{y_2 y_2} &= y_2 y_2 - \frac{1}{2} y_2 y_1^2 - \frac{1}{2} y_1^2 y_2 + \frac{1}{4} y_1^4, \\
\Pi_{y_2 y_1^2} &= y_2 y_1^2 - 2\, y_1 y_2 y_1 + y_1^2 y_2, \\
\Pi_{y_1 y_3} &= y_1 y_3 - \frac{1}{2} y_1^2 y_2 - \frac{1}{2} y_1 y_2 y_1 + \frac{1}{3} y_1^4, \\
\Pi_{y_1 y_2 y_1} &= y_1 y_2 y_1 - y_1^2 y_2, \\
\Pi_{y_1^2 y_2} &= y_1^2 y_2 - \frac{1}{2} y_1^4, \\
\Pi_{y_1^4} &= y_1^4.
\end{aligned}$$

Example (for $q = 1$)

$$
\begin{aligned}
\Sigma_{y_4} &= y_4, \\
\Sigma_{y_3 y_1} &= \frac{1}{2} y_4 + y_3 y_1, \\
\Sigma_{y_2^2} &= \frac{1}{2} y_4 + y_2^2, \\
\Sigma_{y_2 y_1^2} &= \frac{1}{6} y_4 + \frac{1}{2} y_3 y_1 + \frac{1}{2} y_2 y_2 + y_2 y_1^2, \\
\Sigma_{y_1 y_3} &= y_4 + y_3 y_1 + y_1 y_3, \\
\Sigma_{y_1 y_2 y_1} &= \frac{1}{2} y_4 + \frac{1}{2} y_3 y_1 + y_2^2 + + y_2 y_1^2 + \frac{1}{2} y_1 y_3 + y_1 y_2 y_1, \\
\Sigma_{y_1^2 y_2} &= \frac{1}{2} y_4 + y_3 y_1 + y_2^2 + y_2 y_1^2 + y_1 y_3 + y_1 y_2 y_1 + y_1^2 y_2, \\
\Sigma_{y_1^4} &= \frac{1}{24} y_4 + \frac{1}{6} y_3 y_1 + \frac{1}{4} y_2^2 + \frac{1}{2} y_2 y_1^2 + \frac{1}{6} y_1 y_3 \\
&+ \frac{1}{2} y_1 y_2 y_1 + \frac{1}{2} y_1^2 y_2 + y_1^4.
\end{aligned}
$$

ALGEBRAIC COMBINATORICS OF POLYLOGARITMS, HARMONIC SUMS AND POLYZETAS

(La vie sur Mars ...)

Noncommutative generating series of polyzetas

Let $X = \{x_0, x_1\}$ and $Y = \{Y_i\}_{i\geq 1}$.

Definition

$$\mathrm{L}(z) := \sum_{w\in X^*} \mathrm{Li}_w(z)\, w \quad \text{and} \quad \mathrm{H}(N) := \sum_{w\in Y^*} \mathrm{H}_w(N)\, w.$$

Theorem (HNM, 2009)

$$\Delta_{\sqcup\!\sqcup}\, \mathrm{L} = \mathrm{L}\otimes\mathrm{L} \quad \textit{and} \quad \Delta_{\uplus}\, \mathrm{H} = \mathrm{H}\otimes\mathrm{H},$$

$$\mathrm{L}(z) = e^{x_1 \log\frac{1}{1-z}} \mathrm{L}_{\mathrm{reg}}(z) e^{x_0 \log z} \quad \textit{and} \quad \mathrm{H}(N) = e^{\mathrm{H}_1(N)\, y_1} \mathrm{H}_{\mathrm{reg}}(N),$$

$$\textit{where } \mathrm{L}_{\mathrm{reg}}(z) = \prod_{\substack{l\in\mathcal{L}ynX \\ l\neq x_0, x_1}}^{\searrow} e^{\mathrm{Li}_{S_l}(z)\, P_l} \quad \textit{and} \quad \mathrm{H}_{\mathrm{reg}}(N) = \prod_{\substack{l\in\mathcal{L}ynY \\ l\neq y_1}}^{\searrow} e^{\mathrm{H}_{\Sigma_l}(N)\, \Pi_l}.$$

Definition

$$Z_{\sqcup\!\sqcup} := \mathrm{L}_{\mathrm{reg}}(1) \quad \text{and} \quad Z_{\uplus} := \mathrm{H}_{\mathrm{reg}}(\infty).$$

Global regularizations

$$Z_{\shuffle} = \prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}}^{\searrow} \exp[\zeta(S_l) P_l] \text{ and } Z_{\stuffle} = \prod_{\substack{l \in \mathcal{L}ynY \\ l \neq y_1}}^{\searrow} \exp[\zeta(\Sigma_l) \Pi_l].$$

$$\mathrm{L}(z) \underset{z \to 1}{\widetilde{\ }} \exp\left[x_1 \log \frac{1}{1-z}\right] Z_{\shuffle} \text{ and } \mathrm{H}(N) \underset{N \to \infty}{\widetilde{\ }} \exp\left[-\sum_{k \geq 1} \mathrm{H}_{y_k}(N) \frac{(-y_1)^k}{k}\right] \pi_Y Z_{\shuffle}$$

For any $w \in Y^*$ and for any $k \geq 1$, we have

$$\mathrm{H}_w(N) = \sum_{i=1}^{|w|} \alpha_i \log^i(N) + \gamma_w + \sum_{j=1}^{k} \sum_{i=0}^{|w|-1} \beta_{i,j} \frac{1}{N^j} \log^i(N) + \mathrm{O}\left(\frac{1}{N^k}\right),$$

where γ_w, α_i and $\beta_{i,j}$ belong to $\mathcal{Z}[\gamma]$.

$$Z_\gamma := \sum_{w \in Y^*} \gamma_w \, w.$$

Theorem (HNM, 2005)

Z_γ is group-like and $Z_\gamma = B(y_1) \pi_Y Z_{\shuffle} = e^{\gamma y_1} Z_{\stuffle}$, where

$$B(y_1) := \exp\left[-\gamma y_1 + \sum_{k > 1} \zeta(k) \frac{(-y_1)^k}{k}\right] \text{ and } B'(y_1) := e^{\gamma y_1} B(y_1).$$

Generalized Euler constants

By specializing at

$$t_1 = \gamma$$

and

$$\forall l \geq 2, \qquad t_l = (-1)^{l-1}(l-1)!\zeta(l)$$

in the Bell polynomials $b_{n,k}(t_1, \ldots, t_k)$, we get

Corollary

$$\gamma_{y_1^k} = \sum_{\substack{s_1,\ldots,s_k>0 \\ s_1+\ldots+ks_k=k}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

$$\gamma_{y_1^k w} = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} \sqcup\!\sqcup \pi_X w])}{i!}\left[\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right].$$

Generalized Euler constants by computer

$$
\begin{aligned}
\gamma_{1,1} &= \frac{\gamma^2 - \zeta(2)}{2}, \\
\gamma_{1,1,1} &= \frac{\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)}{6}, \\
\gamma_{1,1,1,1} &= \frac{80\zeta(3)\gamma - 60\zeta(2)\gamma^2 + 6\zeta(2)^2 + 10\gamma^4}{240}, \\
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - \frac{54}{175}\zeta(2)^4, \\
\gamma_{1,1,6} &= \frac{4}{35}\zeta(2)^3\gamma^2 + [\zeta(2)\zeta(5) + \frac{2}{5}\zeta(3)\zeta(2)^2 - 4\zeta(7)]\gamma \\
&+ \zeta(6,2) + \frac{19}{35}\zeta(2)^4 + \frac{1}{2}\zeta(2)\zeta(3)^2 - 4\zeta(3)\zeta(5), \\
\gamma_{1,1,1,5} &= \frac{3}{4}\zeta(6,2) - \frac{14}{3}\zeta(3)\zeta(5) + \frac{3}{4}\zeta(2)\zeta(3)^2 + \frac{809}{1400}\zeta(2)^4 \\
&- \left(2\zeta(7) - \frac{3}{2}\zeta(2)\zeta(5) + \frac{1}{10}\zeta(3)\zeta(2)^2\right)\gamma \\
&+ \left(\frac{1}{4}\zeta(3)^2 - \frac{1}{5}\zeta(2)^3\right)\gamma^2 + \frac{1}{6}\zeta(5)\gamma^3.
\end{aligned}
$$

NONLINEAR DIFFERENTIAL EQUATIONS

(Sur la trace d'Icare)

Nonlinear differential equation

$y(z) = \sum_{n \geq 0} y_n z^n$ is the output of :

$$(NS) \begin{cases} y(z) &= f(q(z)), \\ \partial_z q(z) &= A_0(q)u_0(z) + A_1(q)u_1(z), \\ q(z_0) &= q_0, \end{cases}$$

$(\rho, \check{\rho}, C_f)$ and $(\rho, \check{\rho}, C_i)$, for $i = 0, .., m$, are convergence modules of f and $\{A_i^j\}_{j=1,..,n}$ respectively at $q \in \mathrm{CV}(f) \Cap_{i=0,..,m, j=1,..,n} \mathrm{CV}(A_i^j)$.

$\sigma f_{|_{q_0}} = \sum_{w \in X^*} \mathcal{A}(w)(f(q_0))\, w$ satisfies the $\chi-$growth condition.

Theorem (extended Fliess' fundamental formula, HNM, 2007)

$$y(z) = \langle \sigma f_{|_{q_0}} \| S_{z_0 \rightsquigarrow z} \rangle = \sum_{w \in X^*} \langle \mathcal{A}(w)(f(q_0)) | w \rangle \langle S_{z_0 \rightsquigarrow z} | w \rangle .$$

Recall that $S_{z_0 \rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}$.

Solution of nonlinear differential equation

Corollary

The output y of the nonlinear dynamical system with singular inputs admits the following functional expansions

$$
\begin{aligned}
y(z) &= \sum_{w\in X^*} g_w(z)\; \mathcal{A}(w)(f(q_0)), \\
&= \sum_{k\geq 0}\sum_{n_1,\ldots,n_k\geq 0} g_{x_0^{n_1}x_1\ldots x_0^{n_k}x_1}(z)\;\; \mathrm{ad}_{A_0}^{n_1} A_1 \ldots \mathrm{ad}_{A_0}^{n_k} A_1 e^{\log zA_0}(f(q_0)), \\
&= \prod_{l\in\mathcal{L}ynX} \exp\Big(g_{S_l}(z)\; \mathcal{A}(\check{S}_l)(f(q_0))\Big) \\
&= \exp\Big(\sum_{w\in X^*} g_w(z)\; \mathcal{A}(\pi_1(w))(f(q_0))\Big),
\end{aligned}
$$

where, for any $w \in X^*$, $g_w \in \mathrm{LI}_{\mathcal{C}}$ and

$$
\pi_1(w) = \sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\cdots,u_k\in X^*+}\langle w|u_1 ⧢ \cdots ⧢ u_k\rangle\; u_1\cdots u_k.
$$

Successive differentiations of L

Let $\partial_z = d/dz$ and $\theta_0 = zd/dz$. For any $n \in \mathbb{N}$, we have

$$\partial_z^n \mathrm{L}(z) = D_n(z)\mathrm{L}(z) \quad \text{and} \quad \theta_0^n \mathrm{L}(z) = E_n(z)\mathrm{L}(z),$$

where

- $D_n(z)$ and $E_n(z)$ in $\mathcal{C}\langle X \rangle$ are defined as follows

$$D_n(z) = \sum_{\mathrm{wgt}(\mathbf{r})=n} \sum_{w \in X^{\deg(\mathbf{r})}} \prod_{i=1}^{\deg(\mathbf{r})} \binom{\sum_{j=1}^{i} r_i + j - 1}{r_i} \tau_{\mathbf{r}}(w),$$

$$E_n(z) = \sum_{\mathrm{wgt}(\mathbf{r})=n} \sum_{w \in X^{\deg(\mathbf{r})}} \prod_{i=1}^{\deg(\mathbf{r})} \binom{\sum_{j=1}^{i} r_i + j - 1}{r_i} \rho_{\mathbf{r}}(w),$$

- for any $w = x_{i_1} \cdots x_{i_k}$ and $\mathbf{r} = (r_1, \ldots, r_k)$ of degree $\deg(\mathbf{r}) = k$ and of weight $\mathrm{wgt}(\mathbf{r}) = k + r_1 + \cdots + r_k$, $\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1}) \cdots \tau_{r_k}(x_{i_k})$ and $\rho_{\mathbf{r}}(w) = \rho_{r_1}(x_{i_1}) \cdots \rho_{r_k}(x_{i_k})$ are defined by

$$\tau_r(x_0) = \partial^r \frac{x_0}{z} = \frac{-r!x_0}{(-z)^{r+1}} \quad \text{and} \quad \tau_r(x_1) = \partial^r \frac{x_1}{1-z} = \frac{r!x_1}{(1-z)^{r+1}},$$

$$\rho_r(x_0) = \theta_0^r \frac{x_0}{z} = 0 \quad \text{and} \quad \rho_r(x_1) = \theta_0^r \frac{zx_1}{1-z} = \mathrm{Li}_{-r}(z)x_1.$$

Examples of the coefficients of $\theta_0^n \mathrm{L}$

$$
\begin{aligned}
\theta_0 \,\mathrm{Li}_{x_0}(z) &= 1,\\
\theta_0 \,\mathrm{Li}_{x_1}(z) &= z(1-z)^{-1} =: \mathrm{Li}_{0}(z),\\
\theta_0^2 \,\mathrm{Li}_{x_1}(z) &= \sum_{n\geq 1} n z^n =: \mathrm{Li}_{-1}(z),\\
\theta_0^3 \,\mathrm{Li}_{x_1}(z) &= \sum_{n\geq 1} n^2 z^n =: \mathrm{Li}_{-2}(z),\\
\theta_0^4 \,\mathrm{Li}_{x_1}(z) &= \sum_{n\geq 1} n^3 z^n =: \mathrm{Li}_{-3}(z),\\
&\ldots\\
\theta_0 \,\mathrm{Li}_{x_1^2}(z) &= \mathrm{Li}_0(z)\,\mathrm{Li}_1(z),\\
\theta_0^2 \,\mathrm{Li}_{x_1^2}(z) &= \mathrm{Li}_{-1}(z)\,\mathrm{Li}_1(z) + \mathrm{Li}_0^2(z),\\
\theta_0^3 \,\mathrm{Li}_{x_1^2}(z) &= \mathrm{Li}_{-2}(z)\,\mathrm{Li}_1(z) + 3\,\mathrm{Li}_{-1}(z)\,\mathrm{Li}_0(z),\\
\theta_0^4 \,\mathrm{Li}_{x_1^2}(z) &= \mathrm{Li}_{-3}(z)\,\mathrm{Li}_1(z) + \mathrm{Li}_{-2}(z)\,\mathrm{Li}_0(z)\\
&+ 3\,\mathrm{Li}_{-2}(z)\,\mathrm{Li}_0(z) + 3\,\mathrm{Li}_{-1}^2(z),\\
&\ldots\\
\theta_0 \,\mathrm{Li}_{x_0 x_1}(z) &= \mathrm{Li}_1(z).
\end{aligned}
$$

Corollary

Let $\partial_z = d/dz$ and $\theta_0 = zd/dz$. For any $n \in \mathbb{N}$, we have

$$\partial_z^n y(1) \underset{\varepsilon \to 0^+}{\widetilde{\quad}} \sum_{w \in X^*} \langle \mathcal{A}(w) \circ f_{|_{q_0}} | w \rangle \langle D_n(1-\varepsilon) e^{-x_1 \log \varepsilon} Z_{\sqcup\!\sqcup} e^{-x_0 \log \varepsilon} | w \rangle$$

and

$$\theta_0^n y(1) \underset{\varepsilon \to 0^+}{\widetilde{\quad}} \sum_{w \in X^*} \langle \mathcal{A}(w) \circ f_{|_{q_0}} | w \rangle \langle E_n(1-\varepsilon) e^{-x_1 \log \varepsilon} Z_{\sqcup\!\sqcup} e^{-x_0 \log \varepsilon} | w \rangle.$$

Asymptotic of the Taylor coefficients of the output

Corollary

The n-order differentiation of the output y of the system (NDE) is a $\mathcal{C}$-combination of the elements belonging to the polylogarithm algebra.
Moreover, if the ordinary Taylor expansions of $\partial^n y$ and $\theta_0^n y$ exist :

$$\partial^n y(z) = \sum_{k\geq 0} d_k z^n \quad \text{and} \quad \theta_0^n y(z) = \sum_{k\geq 0} t_k z^k$$

then the coefficients of these expansions belong to the algebra of harmonic sums and there exist algorithmically computable coefficients $a_i, a_i' \in \mathbb{Z}, b_i, b_i' \in \mathbb{N}$ and $c_i, c_i' \in \mathcal{Z}[\gamma]$ such that

$$d_k \underset{k\to\infty}{\widetilde{}} \sum_{i\geq 0} c_i k^{a_i} \log^{b_i} k \quad \text{and} \quad t_k \underset{k\to\infty}{\widetilde{}} \sum_{i\geq 0} c_i' k^{a_i'} \log^{b_i'} k.$$

THANK YOU FOR YOUR ATTENTION

(Mars brûle-t-il ?)