Hochschild cohomology and combinatorial Dyson-Schwinger equations in noncommutative quantum field theory (NCQFT)

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arXiv:0907.2182, *J. Noncomm. Geom.* (in press)
(in collaboration with Dirk Kreimer)

Villetaneuse, 21st of June 2011

Plan

- Introduction - quantum field theory (QFT) and combinatorics
- NCQFT and ribbon graphs
- Insertions of Feynman ribbon graphs
- The B_+ operator - Hochschild one-cocycle of the Connes-Kreimer Hopf algebra of ribbon graphs
- Combinatorial Dyson-Schwinger equations
- Perspectives

Scalar field theory and Feynman graphs

$\Phi : \mathbb{R}^4 \to \mathbb{K}$ - a scalar field

$\mathbb{R}^4$ - the 4−dimensional space(time), Euclidean metric

the action (functional in the field)

$$S[\Phi(x)] = \int d^4x \left[\frac{1}{2} \sum_{\mu=1}^{4} \left(\frac{\partial}{\partial x_\mu} \Phi(x) \right)^2 + \frac{1}{2} m^2 \Phi^2(x) + \frac{\lambda}{4!} \Phi^4(x) \right]$$

m - the mass of the particle,
λ - the coupling constant

- quadratic part - propagation - edges
- non-quadratic part - interaction potential $V[\Phi(x)] = \frac{\lambda}{4!}\Phi^4(x)$ - vertices:

QFT and combinatorics

partition function of a QFT model:

$$Z := \int \mathcal{D}\phi(x) e^{-S[\phi]}$$

perturbative developement in the *coupling constant* λ
$\hookrightarrow$ Feynman graphs with associated combinatorial weights

QFT - built-in combinatorics

first computations in QFT end in infinite results

a cure for these infinities (such that the theoretical results can be compared with experiments) - renormalization.

huge experimental success

renormalizable theories - building blocks of mathematical physics

Connes-Kreimer Hopf algebra

A. Connes and D. Kreimer, *Commun. Math. Phys.*, '00

$\hookrightarrow$ definition of a coproduct Δ

$\mathcal{H}$ - the algebra generated by Feynman graphs
multiplication: disjoint union of graphs

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \quad \Delta(G) = G \otimes 1 + 1 \otimes G + \sum_{\gamma \in \underline{G}} \gamma \otimes G/\gamma,$$

$\underline{G}$ – primitively divergent subgraphs of G

renormalization as a factorization issue

$$\varepsilon : \mathcal{H} \to \mathbb{K}, \quad \varepsilon(1) = 1, \ \varepsilon(G) = 0, \ \forall G \neq 1,$$

$$S : \mathcal{H} \to \mathcal{H},$$

$$S(1_{\mathcal{H}}) = 1_{\mathcal{H}}, \ G \mapsto -G - \sum_{\gamma \in \underline{G}} S(\gamma) G/\gamma.$$

Theorem: $(\mathcal{H}, \Delta, \varepsilon, S)$ *is a Hopf algebra.*

Algebraic framework for renormalization

R - the map which given a formal integral returns it evaluated at the subtraction point
$R\mathcal{A}(G)$ - the singular part of the Feynman amplitude $\mathcal{A}(G)$

twisted antipode (recursive definition)

$$S_R^{\mathcal{A}}(1_{\mathcal{H}}) = 1,$$
$$S_R^{\mathcal{A}}(G) = -R(\mathcal{A}(G)) - \sum_{\gamma \in \underline{G}} S_R^{\mathcal{A}}(\gamma) R(\mathcal{A}(G/\gamma)).$$

the renormalized amplitude of the graph

$$\mathcal{A}_R = S_R^{\mathcal{A}} \star \mathcal{A}.$$

Connes-Kreimer Hopf algebra structure -
the combinatorial backbone of renormalization

NCQFT - glimpse of the mathematical setup

the Moyal space

The *Moyal algebra* is the linear space of smooth and rapidly decreasing functions $\mathcal{S}(\mathbb{R}^{\mathcal{D}})$ equipped with the *Moyal product*:

$$(f \star g)(x) = \int \frac{d^D k}{(2\pi)^D} d^D y \, f(x + \frac{1}{2}\Theta \cdot k) g(x + y) e^{ik \cdot y}.$$

$\star$ - Moyal product
(non-local, noncommutative, associative product)

$$\Theta = \begin{pmatrix} \Theta_2 & 0 \\ 0 & \Theta_2 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}.$$

Field theory on Moyal space

Φ^4 model:

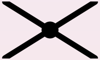
$$\mathcal{S} = \int d^4x \, [\frac{1}{2}\partial_\mu \Phi \star \partial^\mu \Phi + \frac{1}{2} m^2 \Phi \star \Phi + \frac{\lambda}{4!} \Phi \star \Phi \star \Phi \star \Phi],$$

$\int d^4x \, (\Phi \star \Phi)(x) = \int d^4x \, \Phi(x) \, \Phi(x)$
(same propagation as in the commutative theory)

Implications of the use of the Moyal product in QFT

$$\int d^4x\Phi^{\star 4}(x) \propto \int \prod_{i=1}^{4} d^4x_i \Phi(x_i)\delta(x_1 - x_2 + x_3 - x_4)$$
$$e^{2i(x_1-x_2)\Theta^{-1}(x_3-x_4)}$$

oscillation $\propto$ area of parallelogram



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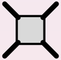
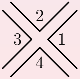


$\hookrightarrow$ non-locality

$\hookrightarrow$ restricted invariance: only under cyclic permutation



$\rightarrow$ ribbon graphs

$\rightarrow$ clear distinction between planar and non-planar graphs

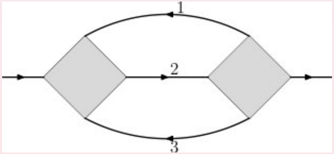
Feynman graphs in NCQFT

n - number of vertices,
L - number of internal lignes,
F - number of faces,

$$2 - 2g = n - L + F$$

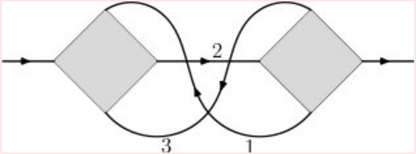
$g \in \mathbb{N}$ - genus

$g = 0$ - planar graph
example:



$n = 2,\ L = 3,\ F = 3,\ g = 0$

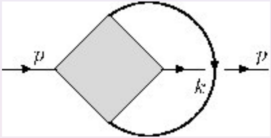
$g \geq 1$ - non-planar graph
example:



$n = 2,\ L = 3,\ F = 1,\ g = 1$

Renormalization on the Moyal space

UV/IR mixing (S. Minwalla *et. al.*, *JHEP*, '00)



$$B = 2$$

B - number of faces broken by external lignes

$B > 1$, *planar irregular graph*

$$\lambda \int d^4k \frac{e^{ik_\mu \Theta^{\mu\nu} p_\nu}}{k^2 + m^2} \to_{|p| \to 0} \frac{1}{\theta^2 p^2}$$

same type of behavior at any order in perturbation theory

J. Magnen, V. Rivasseau and A. T., *Europhys. Lett.* '09

$\to$ non-renormalizability!

A first solution to this problem - the Grosse-Wulkenhaar model

additional harmonic term

(H. Grosse and R. Wulkenhaar, *Comm. Math. Phys.*, '05)

$$s[\phi(x)] = \int d^4x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2}(\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right),$$

$$\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu} x^\nu.$$

modification of the propagator - the model becomes renormalizable several combinatorial developments of these models

analytical combinatorics techniques

the Mellin representation of the Feynman amplitudes

(R. Gurău, A. Malbouisson, V. Rivasseau and A. T., *Lett. Math. Phys.*, 2007)

parametric representation of the Feynman amplitudes

(R. Gurău and V. Rivasseau, *Commun. Math. Phys.*, 2007, A. T. and V. Rivasseau, *Commun. Math. Phys.*, '08,

A. T., *J. Phys. Conf. Series*, 2008, A. T., solicited by de *Modern Encyclopedia Math. Phys.*)

- relation with ribbon graph polynomials (T. Krajewski *et. al.*, *J. Noncomm. Geom.*, 2010)

Translation-invariant renormalizable scalar model

(R. Gurău, J. Magen, V. Rivasseau and A. T., *Commun. Math. Phys.* 2009)

the Grosse-Wulkenhaar model breaks translation-invariance !

the complete propagator:

$$C(p, m, \theta) = \frac{1}{p^2 + a\frac{1}{\theta^2 p^2} + m^2}$$

arbitrary planar irregular 2-point function: same type of $\frac{1}{p^2}$ behavior !

J. Magnen, V. Rivasseau and A. T., *Europhys. Lett.* 2009

$\hookrightarrow$ other modification of the action:

$$S = \int d^4p \, [\frac{1}{2} p_\mu \phi \star p^\mu \phi + \frac{1}{2} a \frac{1}{\theta^2 p^2} \phi \star \phi + \frac{1}{2} m^2 \phi \star \phi + \frac{\lambda}{4!} V^\star[\phi]]. \quad (1)$$

renormalizability at any order in perturbation theory !

Other developments - relation with combinatorics

- parametric representation (A. T., *J. Phys.* **A** 2009)
- relation with Bollobás-Riordan topologic ribbon graph polynomial

  (T. Krajewski, V. Rivasseau, A. T. and Z. Wang, *J. Noncomm. Geom.* (2010)

  trees $\rightarrow$ $\star-$trees (quasi-trees)

Hopf algebra for renormalizable NCQFTs

A. T. and F. Vignes-Tourneret, *J. Noncomm. Geom.*, 2008

$\hookrightarrow$ definition of a coproduct Δ

$\mathcal{H}$ - the algebra generated by Feynman ribbon graphs

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \quad \Delta(G) = G \otimes 1 + 1 \otimes G + \sum_{\gamma \in \underline{G}} \gamma \otimes G/\gamma,$$

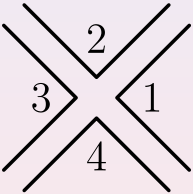
$$\varepsilon : \mathcal{H} \to \mathbb{K}, \quad \varepsilon(1) = 1, \ \varepsilon(G) = 0, \ \forall G \neq 1,$$

$$S : \mathcal{H} \to \mathcal{H}, \quad G \mapsto -G - \sum_{\gamma \in \underline{G}} S(\gamma) G/\gamma.$$

Theorem: $(\mathcal{H}, \Delta, \varepsilon, S)$ *is a Hopf algebra.*

$\hookrightarrow$ 2$-$ and 4$-$point graphs (in commutative ϕ^4)
$\rightarrow$ 2$-$ and 4$-$point planar regular graph

this Hopf algebra structure - the combinatorial backbone of noncommutative renormalization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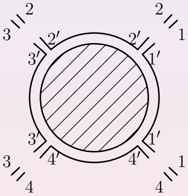
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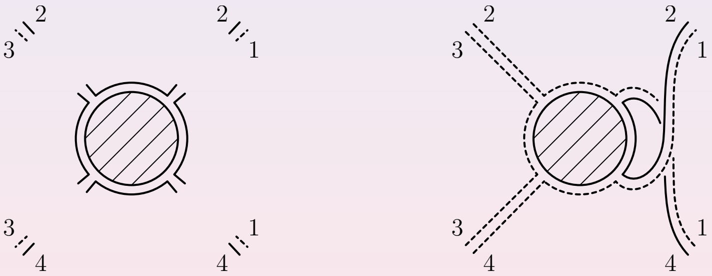
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Quotient combinatorial Hopf algebra

Proposition:

The subspace $\mathcal{H}^{\rm pli}$ generated by the 1PI 2−point planar irregular Feynman graphs is a Hopf coideal in $\mathcal{H}$,

$$\Delta(\mathcal{H}^{\rm pli}) \subseteq \mathcal{H}^{\rm pli} \otimes \mathcal{H} + \mathcal{H} \otimes \mathcal{H}^{\rm pli},\ \varepsilon(\mathcal{H}^{\rm pli}) = 0,\ S(\mathcal{H}^{\rm pli}) \subseteq \mathcal{H}^{\rm pli}.$$

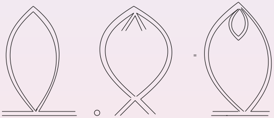
$\Rightarrow$ the quotient Hopf algebra $\mathcal{H}/\mathcal{H}^{\rm pli}$ - the appropriate algebraic structure to work with for describing the renormalization of the non-commutative $1/p^2$ model

discarding the planar irregular graph

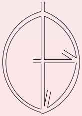
Pre-Lie algebra structures - insertions of ribbon graphs

insertion of a *planar-irregular* $4-$point graph - reduction of the number of broken faces (one can obtain a *planar regular* graph)

example ($B=2$):

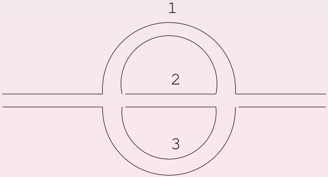


phenomena also possible for $B=3$; impossible for $B=4$
example ($B=4$):



More on ribbon graphs

- symmetry factor
  The symmetry factor of a NCQFT ribbon graph is equal to 1.
- permutation of external edges: $|\Gamma|_V$ - number of graphs equal upon removal of external edges
- number of maximal forests: maxf - number of ways to shrink subdivergencies such that the cograph is primitively divergent.



- number of bijections: bij$(\gamma_1, \gamma_2, \gamma)$ - number of bijections between the external edges of γ_2 and adjacent edges of places in γ_1 such that γ is obtained.
- number of insertion places of X in γ - $[\gamma|X]$

The B_+ operator - Hochschild 1−cocycle

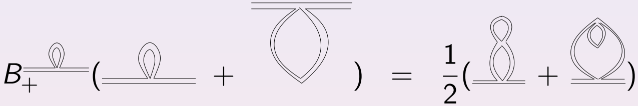
$$B_+^{k;r} := \sum_{\substack{|\gamma|=k \\ \mathrm{res}(\gamma)=r}} B_+^{\gamma},$$

$$B_+^{\gamma}(X) := \sum_{\Gamma} \frac{bij(\gamma, X, \Gamma)}{|X|_{\vee}} \frac{1}{\mathrm{maxf}(\Gamma)} \frac{1}{[\gamma|X]} \Gamma$$

$$B_+^{\gamma}(1_{\mathcal{H}}) := \gamma$$

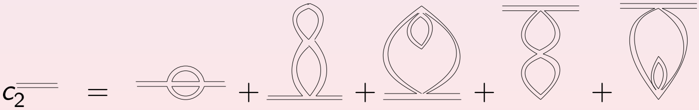
$\hookrightarrow$ generalization of the pre-Lie insertion into γ

a 2−*loop example:*



$c_k^r := \sum_{\substack{|\Gamma|=k \\ \mathrm{res}(\Gamma)=r}} \Gamma$ sum of graphs of a given loop number and residue

a 2−*loop example:*



Dyson-Schwinger equations (DSE)

1. combinatorial DSE: power series written in terms of the insertion operator B_+
2. analytical DSE: equations obtained by applying the Feynman rules to the combinatorial DSE
(equations a physicist would recognize)

solving the analytical DSE $\Leftrightarrow$ *non-perturbative solution!*

every divergent graph γ determines a Hochschild $1-$cocyle; any relevant graph in the perturbative expansion (with the right combinatorial weight!) is in the range of such a cocyle

DSE - formal construction based on the Hochschild cohomology of the renormalization Hopf algebra

Theorem:

$$i)\ \sum_{\Gamma\in M_r}\Gamma=\sum_{k=1}^{\infty}B_+^{k;r}(X_{k,r})$$

M_r - set of graphs with residue r

$$ii)\ \Delta(B_+^{k;r}(X_{k,r}))=B_+^{k;r}(X_{k,r})\otimes 1_{\mathcal{H}}+(\mathrm{id}\otimes B_+^{k;r})\Delta(X_{k,r})$$

$$iii)\ \Delta(c_k^r)=\sum_{j=0}^{k}\mathrm{Pol}_j^r(c)\otimes c_{k-j}^r$$

$\mathrm{Pol}_j^r(c)$ - polynomial in the variables c_m^r (total degree j)

Hopf subalgebras

2-loops examples (Hopf subalgebras)

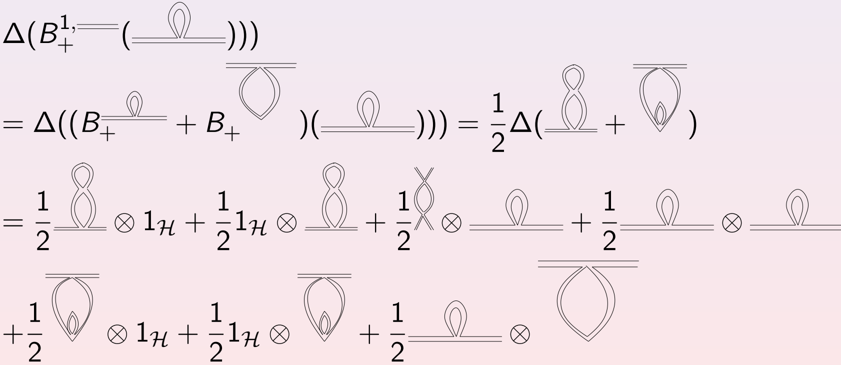
$k = 2$ (2 loops)

$$\Delta'(c_2^{=}) = (c_1^{\times} + c_1^{=}) \otimes c_1^{=}$$

$$\Delta'(c_2^{\times}) = (2c_1^{\times} + 2c_1^{=}) \otimes c_1^{\times}$$

2-loops examples (Hochschild cohomology)

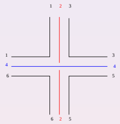
$k = 2$ (2 loops)



etc.

Perspectives - can things be (even) more complicated?

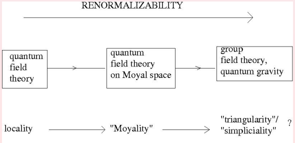
- generalization to *tensor models* (appearing in *group field theory*, recent approach for a theory of quantum gravity)



various non-trivial combinatorial structures:

  - topological graph polynomial (A. T., arXiv:1102.4231 [math.CO], *J. Math. Phys.* (in press))
  - combinatorial Hopf algebras

- renormalizability study of quantum gravity models (A. T., *Class. Quant. Grav.* 2010, T. Krajewski *et. al.*, *Phys. Rev.* **D** - saddle point techniques for computing Feynman amplitudes)



Thank you for your attention!

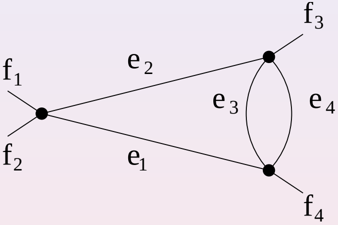
Cohomology

A - bialgebra
linear maps $L : A \to A^{\otimes n}$ - $n-$cochain

coboundary map b ($b^2 = 0$)

$$bL := (id \otimes L) \circ \Delta + \sum_{i=1}^{n} (-1)^i \Delta_i \circ L + (-1)^{n+1} L \otimes 1_{\mathcal{H}}, \qquad (2)$$

example of a Feynman graph:



Fourier transform: position space (x) $\rightarrow$ momentum space (p)

$$S[\Phi] = \int d^4 p \left[\frac{1}{2} \sum_{\mu=1}^{4} (p_\mu \Phi)^2 + \frac{1}{2} m^2 \Phi^2 + \tilde{V}_{\rm int} \right]$$

- propagation - $\frac{1}{p^2+m^2}$
- interaction - $\lambda\,\delta$(sum of incoming/outgoing momentae)

$\Longrightarrow$ Feynman amplitude (in momentum space)

Scales - noncommutative renormalization group

definition of the RG scales:

- locus where $C^{-1}(p)$ is big
- locus where $C^{-1}(p)$ is low

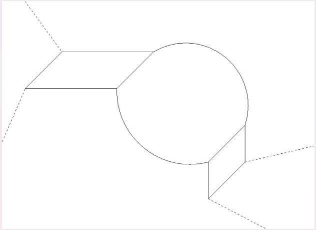
$$C^{-1}_{\mathrm{comm}}(p) = p^2$$

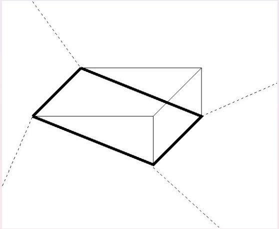
$$C^{-1}_{GW} = p^2 + \Omega^2 x^2$$

$$C^{-1}(p) = p^2 + \frac{a}{\theta^2 p^2}$$

mixing of the UV and IR scales - key of the renormalization

The principle of “Moyality” - ribbon Feynman graph level





valid iff the graph is planar

renormalization necessary only for the planar sector!

"The amount of theoretical work one has to cover before being able to solve
problems of real practical value is rather large, but this circumstance is [...] likely to become more pronounced in the theoretical physics of the future."
P.A.M. Dirac, *"The principles of Quantum Mechanics"*, 1930

renormalization conditions

$$\Gamma^4(0,0,0,0) = -\lambda_r,\ G^2(0,0) = \frac{1}{m^2},\ \frac{\partial}{\partial p^2} G^2(p,-p)|_{p=0} = -\frac{1}{m^4}. \quad (3)$$

where Γ^4 and G^2 are the connected functions and

$0 \to p_m$ (the minimum of $p^2 + \frac{a}{\theta^2 p^2}$)

Main ingredients of renormalizability

1. power counting theorem: indicates which Feynman graphs are primitively divergent
   superficial degree of divergence ω -
   should not depend on the internal structure
   exemple: the ϕ^4 model

$$\omega = N - 4.$$

   N - number of external legs of the graph
   primitively divergent graphs: $2-$ and $4-$point graphs

2. locality

$\hookrightarrow$ Bogoliubov subtraction operator R
(defined as a *sum over forests*)

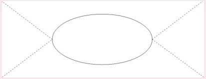
subtraction of divergences

The physical principle of locality (Feynman graph level)

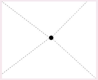
connected graphs can be reduced to points

graph made of internal propagators of
high energy (or short distance) (ultraviolet (UV) regime) - local

example:



$\rightarrow$



subtraction of local counterterms -
i. e. counterterms have the same form as the terms of the action

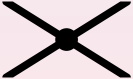
via Taylor expansion

$\Longrightarrow$ renormalized amplitude $\mathcal{A}_R$: finite!

Renormalizability of NCQFT: locality $\to$ "Moyality"

QFT $\to$ NCQFT

locality $\to$ "Moyality"

 $\to$