Differentially algebraic equations in physics

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Based on "*Modular forms, Schwarzian conditions, and symmetries of differential equations in physics*", arXiv 1611.08493

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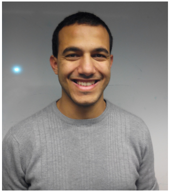
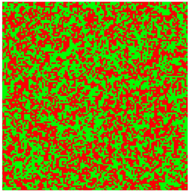
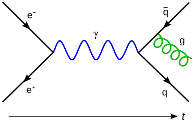
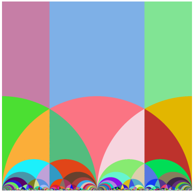
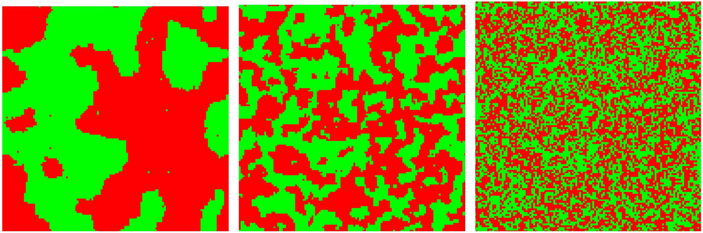







$$H = \sum_{j,k} \{ J_v \sigma_{j,k} \sigma_{j+1,k} + J_h \sigma_{j,k} \sigma_{j,k+1} \}$$



- J_v, J_h: vertical and horizontal coupling constants
- The spins take the values $\sigma_{j,k} = \pm 1$.
- The partition function: $\exp(-\frac{1}{k_b T} H)$

Nature of power series

- Algebraic: $S(x) \in \mathbb{Q}(x)$ root of a polynomial $P(t, S(t)) = 0$
- D-finite: $S(x) \in \mathbb{Q}(x)$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(t)}(t) + s + c_0(t)S(t) = 0$
- Hypergeometric: $S(x) = \sum_{n=0}^{\infty} s_n x^n$ s.t. $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g., the Gauss hypergeometric function:

$$ {}_2F_1([a,b],[c],x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, $$

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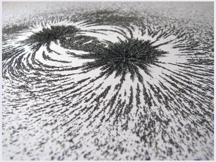
- E.g.: ${}_2F_1(1,1;1;z) = \frac{1}{1-z}, \quad {}_2F_1(1,1;2;z) = -\frac{\ln(1-z)}{z}$
- Partition function 2D square Ising model [Viswanathan, 2014]

$$ {}_4F_3([1,1,\frac{3}{2},\frac{3}{2}],[2,2,2],16k^2]), \quad k = \frac{\tanh(2\beta J)}{2\cosh(2\beta J)} $$

Magnetic susceptibility of 2D Ising model

Magnetic susceptibility $\longrightarrow$ sum of two point correlation functions

$$\chi := \beta \sum_{n=0}^{\infty} \chi^{(2n+1)}$$



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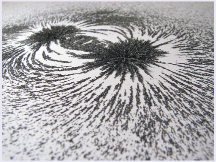


ability of a material to align itself with an external imposed magnetic field

$\chi^{(2n+1)} \longrightarrow 2n$ multiple integrals , e.g. $\chi^{(3)}$ is given by the double integral:

$$\chi^{(3)}(s) = \frac{(1-s)^{1/4}}{s} \frac{1}{4\pi^2} \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \, y_1 y_2 y_3 \frac{1 + x_1 x_2 x_3}{1 - x_1 x_2 x_3} F$$

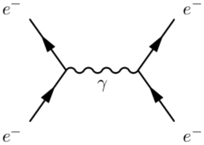
$$x_j = \frac{s}{1 + s^2 - s\cos\phi_j + \sqrt{(1 + s^2 - s\cos\phi_j)^2 - s^2}}$$

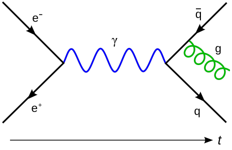
$$y_j = \frac{s}{\sqrt{(1 + s^2 - s\cos\phi_j)^2 - s^2}}, \qquad (j = 1, 2, 3)$$

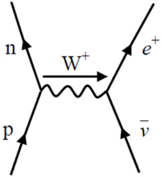
$$\phi_1 + \phi_2 + \phi_3 = 0$$

and $F = f_{23}\left(f_{31} + \frac{f_{23}}{2}\right)$ with $f_{ij} = (\sin\phi_i - \sin\phi_j)\frac{x_i x_j}{1 - x_i x_j}$

Feynman diagrams are D-finite







Feynman diagrams $\longrightarrow$ first order perturbations of n-fold integral of the operator S (scattering operator) giving the probability of such interactions:

$$S = \sum_{n=0}^{\infty} \frac{\iota^n}{n!} \overbrace{\int \cdots \int}^{n \text{ times}} \prod_{j=1}^{n} d^4 x_j \, T \prod_{j=1}^{n} L(x_j)$$

$L_v(x_j) \longrightarrow$ Lagrangian of interaction, T the time ordered product of operators, $d^4 x_j$ four-vectors

Multiple integrals of an algebraic object

Theorem (Kashiwara)

$$\overbrace{\int \cdots \int}^{n\ times} \textit{D-finite function} \quad dx_1 \cdots dx_n \rightarrow \textit{D-finite function}$$

(D-finite = solution of linear ODE with polynomial coefficients)

Diagonal of a rational function

For a formal power series F given by

$$F(z_1, z_2, \cdots, z_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} F_{m_1,\cdots,m_n} z_1^{m_1} \cdots z_n^{m_n},$$

the diagonal of F is defined as the single variable series:

$$Diag(F(z_1, z_2, \cdots, z_n)) := \sum_{m=0}^{\infty} F_{m,\cdots,m} z^m$$

Example. One of the many diagonals leading to Apéry numbers:

$$\mathrm{Diag} \frac{1}{(1-z_1-z_2)(1-z_3-z_4)-z_1z_2z_3z_4} = \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 z^n$$

${}_2F_1$, modular forms, and physics

The Gauss hypergeometric function ${}_2F_1 \rightarrow$ PHYSICS!, e.g. the differential operator of χ^{2n+1} factorizes into operators that annihilate ${}_2F_1$ functions.

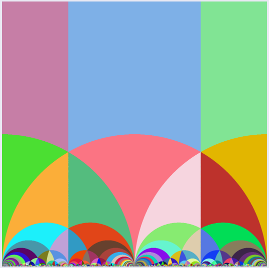
[A. Bostan, S. Boukraa, S. Hassani, M. van Hoeij, J.-M. Maillard, J.-A. Weil, N. Zenine, *The Ising model: from elliptic curves to modular forms and Calabi-Yau equations*, 2011]

[M. Assis, S. Boukraa, S. Hassani, M. van Hoeij, J.-M. Maillard, B. M. McCoy, *Diagonal Ising susceptibility: elliptic integrals, modular forms and Calabi-Yau equations*, 2012]

$$E_4(q) = 1 + 240 \sum_{n=0}^{\infty} n^3 \frac{q^n}{1-q^n}$$

$$= {}_2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728}{j(\tau)}\right)^4$$

$q = \exp(2i\pi\tau)$, $j(\tau) \rightarrow j$-invariant



Modular forms as pullbacked ${}_2F_1$ functions

- Emergence of modular forms in physics through ${}_2F_1$ functions
- Modular forms emerge through covariance properties of ${}_2F_1$:

$$ {}_2F_1\Big([\alpha,\beta],[\gamma],p_1(x)\Big) = \mathcal{A}(x)\,{}_2F_1\Big([\alpha,\beta],[\gamma],p_2(x)\Big) $$

$\mathcal{A}(x)$, $p_1(x)$ and $p_2(x)$ are rational functions. $p_1(x)$ and $p_2(x)$ are called pullbacks, the ${}_2F_1$ is thus called pullbacked. For instance:

$$ {}_2F_1\Big([\frac{1}{12},\frac{5}{12}],[1],\frac{1728x}{(5+10x+x^2)^3}\Big) = $$

$$ \Big(\frac{5+10x+x^2}{3125+250x+x^2}\Big)^{1/4}{}_2F_1\Big([\frac{1}{12},\frac{5}{12}],[1],\frac{1728x^5}{(3125+250x+x^2)^3}\Big). $$

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- w.l.o.g we have: $\mathcal{A}(x)\,{}_2F_1\Big([\alpha,\beta],[\gamma],y(x)\Big) = \; {}_2F_1\Big([\alpha,\beta],[\gamma],x\Big)$
$\mathcal{A}(x)$ and $y(x)$ algebraic functions.

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$\mathcal{A}(x)$ and $y(x)$ algebraic functions. Modular equation $M(x,y(x)) = 0$:

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Theorem (Abdelaziz–Maillard, 2016)

If we have a pullback given by:

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then we have the following "Schwarzian condition":

$$W(x) - W(y(x))\,y'(x)^2 + \{y(x),x\} = 0$$

$$\textit{where} \qquad W(x) := p'(x) + \frac{p(x)^2}{2} - 2q(x)$$

$$\textit{with} \qquad p(x) = \frac{(\alpha+\beta+1)x - \gamma}{x(x-1)} \qquad q(x) = \frac{\alpha\beta}{x(x-1)}$$

NB: The Schwarzian derivative is defined by

$$\{y(x),x\} := \frac{y'''(x)}{y'(x)} - \frac{3}{2}\left(\frac{y''(x)}{y'(x)}\right)^2$$

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Proof of our theorem of the Schwarzian condition

We introduce

- the operator $L_2 := D_x^2 + p(x) D_x + q(x)$ annihilating $F(x) := {}_2F_1\Big([\alpha, \beta], [\gamma], x\Big)$
- the operator $L_2^{(c)} := \frac{1}{v(x)} L_2 v(x)$, i.e.

$$L_2^{(c)} = D_x^2 + \Big(p(x) + 2\frac{v'(x)}{v(x)}\Big) D_x + q(x) + p(x)\frac{v'(x)}{v(x)} + \frac{v''(x)}{v(x)}$$

NB: $L_2^{(c)}$ annihilates $\mathcal{A}(x)F(x)$ (with $\mathcal{A}(x) = 1/v(x)$):
$L_2^{(c)} \frac{1}{v} F(x) = \frac{1}{v} L_2 \cancel{v} \cancel{\frac{1}{v}} F(x) = 0$

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- So, the operator annihilating $F(y(x))$ is

$$L_2^{(p)} = D_x^2 + \Big(p(y(x))y'(x) - \frac{y''(x)}{y'(x)}\Big)D_x + q(y(x))y'(x)^2$$

- When does the equality $L_2^{(c)} = L_2^{(p)}$ hold?

Proof of our theorem of the Schwarzian condition

Well, identifying $L_2^{(c)} = L_2^{(p)}$ gives us two conditions:

Condition 1: $$p(x) + 2\frac{v'(x)}{v(x)} = p(y(x))y'(x) - \frac{y''(x)}{y'(x)}$$

Condition 2: $$q(x) + p(x)\frac{v'(x)}{v(x)} + \frac{v''(x)}{v(x)} = q(y(x))y''(x)^2$$

Introducing $w(x) := \exp\left(-\int p(x)dx\right)$, i.e. $p(x) = -\frac{w'(x)}{w(x)}$, Condition 1 rewrites

$$-\frac{w'(x)}{w(x)} + 2\frac{v'(x)}{v(x)} = -\frac{y''(x)}{y'(x)} + \frac{w'(y(x))}{w(y(x))}$$

Integrating the log-derivative terms we get:

$$-\ln w(x) + 2\ln v(x) = -\ln y'(x) - \ln w(y(x))$$

Taking exponential gives

$$v(x) = \sqrt{\frac{w(x)}{w(y(x))y'(x)}}$$

inserting it in Condition 2 gives the Schwarzian condition in the theorem. □

Global nilpotence

Assuming that the operator is globally nilpotent is equivalent to:

$$p(x) = -\frac{w'(x)}{w(x)}$$

The following statements are a consequence of global nilpotence:

- The Wronskian is the n-th root of a rational function
- The solutions of the differential equation have rational coefficients
- The p-curvature is a nilpotent matrix mod prime
- Global nilpotence $\rightarrow$ rational coefficients of solutions $\rightarrow$ $p(x) = \frac{d}{dx} \ln w(x)$

Modular equation M(x,y(x))=0 and modular invariant

The j-invariant of the elliptic curve:

$$j(k) = 256 \frac{(1 - k^2 + k^4)^3}{k^4(1 - k^2)^2}$$

The Landen transformation:

$$k_L = \frac{2\sqrt{k}}{1 + k}$$

The transform of the elliptic invariant through k_L:

$$j(k_L) = 16 \frac{(1 + 14k^2 + k^4)^3}{k^2(1 - k^2)^4}$$

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The two corresponding Hauptmoduls (similar to a group generator):

$$x = \frac{1728}{j(k)} \qquad y = \frac{1728}{j(k_L)}$$

are related through the modular equation $\tau \rightarrow 2\tau$:

$$M(x,y) = 1953125x^3y^3 - 187500x^2y^2(x+y) + 375xy(16x^2 - 4027xy + 16y^2)$$
$$-64(x+y)(x^2 + 1487xy + y^2) + 110592xy = 0$$

Isogeny structure, commutation

For one ${}_2F_1\Big([a,b],[1],x\Big)$ with two different pullbacks

$$\alpha x + \cdots$$
$$\alpha x^2 + \cdots$$
$$\alpha x^3 + \cdots$$

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This set of solutions is either:

- Algebraic: e.g. ${}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], x\Big)$, we recover "some" commutation like in the case of isogenies (as we will see below)
- Transcendent

Schwarzian condition and modular forms: $\tau \rightarrow 2\tau$ and beyond

The modular form:

$$\mathcal{A}(x)\, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], x\Big) = {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], y(x)\Big) \qquad (1)$$

- $\mathcal{A}(x)$ is an algebraic function
- $y(x)$ is an algebraic function corresponding to the modular equation corresponding to $\tau \rightarrow 2\tau$

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$$y(x) = \frac{1}{1728}x^2 + \frac{31}{62208}x^3 + \frac{1337}{3359232}x^4 + \frac{349115}{1088391168}x^5 + \cdots$$

The Schwarzian condition is verified in this case with:

$$W(x) = -\frac{32x^2 - 41x + 36}{72x^2(x-1)^2}, p(x) = \frac{3x-2}{2x(x-1)}, q(x) = \frac{5}{144x(x-1)}$$

Schwarzian condition and modular forms: $\tau \rightarrow 2\tau$ and beyond

The modular form:

$$\mathcal{A}(x)_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], x\Big) =_2 F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], y(x)\Big) \qquad (1)$$

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It turns out that one can write, for the modular equations corresponding to $\tau \rightarrow N\tau$, the function in the form of (1) above. Thus the equation (1) above encapsulates all the modular equations corresponding to $\tau \rightarrow N\tau$.

Modular equations of higher order

The modular equation of order three $\tau \to 3\tau$:

$$262144000000000x^3y^3(x+y) + 4096000000x^2y^2(27x^2 - 45946xy + 27y^2)$$
$$+15552000xy(x+y)(x^2 + 241433xy + y^2)$$
$$+729x^4 - 779997924x^3y + 1886592284694x^2y^2 - 779997924xy^3 + 729y^4$$
$$+2811677184xy(x+y) - 2176782336xy = 0$$

has the series expansion starting in x^3 and given by:

$$y(x) = \frac{x^3}{2985984} + \frac{31x^4}{71663616} + \frac{36221x^5}{82556485632} + \frac{29537101x^6}{71328803586048} + \dots$$

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$$+15552000xy(x+y)(x^2 + 241433xy + y^2)$$
$$+729x^4 - 779997924x^3y + 1886592284694x^2y^2 - 779997924xy^3 + 729y^4$$
$$+2811677184xy(x+y) - 2176782336xy = 0$$

has the series expansion starting in x^3 and given by:

$$y(x) = \frac{x^3}{2985984} + \frac{31x^4}{71663616} + \frac{36221x^5}{82556485632} + \frac{29537101x^6}{71328803586048} + \dots$$

Similarly for $\tau \to 4\tau$, we get a series starting in x^4:

$$y(x) = \frac{x^4}{5159780352} + \frac{31x^5}{92876046336} + \frac{43909x^6}{106993205379072} + \cdots$$

Modular equations of higher order

- Except for this last series solution, the solution series corresponding to the isogenies $\tau \rightarrow N\tau$ have the form $ax^N + \cdots$
- The series solution corresponding to $\tau \rightarrow 3\tau$ and $\tau \rightarrow 4\tau$ are solution of the Schwarzian condition

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Generalizing the solution series corresponding to $\tau \to 2\tau$ we seek solution series of the Schwarzian condition of the form $ax^2 + \cdots$:

$$y_2 = ax^2 + \frac{31ax^3}{36} - \frac{a(5952a - 9511)}{13824}x^4 + \cdots$$

reducing to the solution of $\tau \to 2\tau$ when $a = 1/1728$

Modular equations of higher order

A one-parameter family of solution-series $bx^3 + \cdots$ for the modular equation corresponding to $\tau \to 3\tau$:

$$y_3 = bx^3 + \frac{31b}{24}x^4 + \frac{36221b}{27648}x^5 + \cdots$$

reduces to a previous series having the form $x^3 + \cdots$ when $b = 1/1728^2$.

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Finally the one-parameter series

$$y_4 = cx^4 + \frac{31c}{18}x^5 + \frac{43909c}{20736}x^6 + \cdots$$

reduces to a previous series of the form $x^4 + \cdots$ for
$c = 1/5159780352 = 1/1728^3$

Commuting series

- These series do not commute: $y_i(y_j(x)) \neq y_j(y_i(x))$.
- Composing the solution series y_3 and y_2 with $d = ab^2$:

$$y_2(y_3(x)) = dx^6 + \frac{31dx^7}{12} + \frac{59285d}{13824}x^8 + \cdots$$

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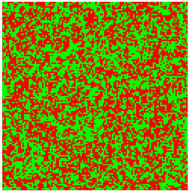
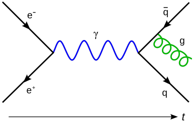
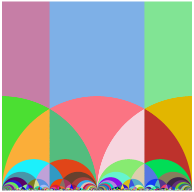
Conclusion

- The Schwarzian condition encapsulates the infinite number of modular equations $\tau \to N\tau$.
- Strong incentive to develop more differentially algebraic tools from an algorithmic perspective : to test the non-D-finiteness of the Ising susceptibility for example!
- Strong incentive to examine further the occurence of non-linear symmetries (like the Landen transformation) in physics.







Questions: non-linear differential Galois group

- Built to generalize the differential Galois group to non-linear ODE's and non linear functional equations having the form $f(x+1) = y(f(x))$.
- Having a finite non-linear differential Galois group guarantees "some integrability" and this is guaranteed by Casale's condition:

$$\nu(y) y''(x)^2 - \nu(x) + \frac{y'''(x)}{y'(x)} - \frac{3}{2}\Big(\frac{y''(x)}{y'(x)}\Big)^2 = 0$$

Modular equations: definition through θ functions

With $q = \exp(i\pi\tau)$, $\tau = iK'/K$ the θ_3 and θ_4 functions are defined as follows:

$$\theta_2 = 2q^{1/4} \prod_{n\geq 1} \left(\frac{1-q^{4n}}{1-q^{4n-2}} \right), \quad \theta_3 = \sum_{-\infty}^{\infty} q^{n^2}, \quad \theta_4 = \sum_{-\infty}^{\infty} (-1)^n q^{n^2}$$

where $K = (\pi/2)\theta_3^2(\tau)$ and $K'(\tau) = K(-1\tau)$. We can write the identity:

$$\theta_3(\tau)^2 + \theta_4(\tau)^2 = 2\theta_3(2\tau)^2 = \frac{2}{1+k'}$$

with $\sqrt{k(\tau)} = \frac{\theta_2(\tau)}{\theta_3(\tau)}$, $\sqrt{k'(\tau)} = \frac{\theta_4(\tau)}{\theta_3(\tau)}$ and $l'(\tau) = k'(p\tau)$ where p is given by a positive integer, we have:

$$\frac{1}{l'} = \frac{1}{2}(\sqrt{k'} + \frac{1}{\sqrt{k'}})$$

giving in the case $p = 2$ the modular equation that sends τ to 2τ.

Painlevé equations

- The hypergeometric function,the Bessel function, the Airy function, the Hermite polynomials, are all "special" (appearing in problems elated to physics) functions solution of linear differential equations.
- Elliptic functions are also "special" functions: they appear in physics as we shall see here, yet they are solution of simple, yet non-linear differential equations.
- Painlevé was set out to find special functions satisfying non-linear differential equations, yet have nice properties (all their singularities are poles).

Painlevé wanted to classify all differential equations of order two having the form:

$$u_{xx} = R(x, u, u_x)$$

with R being a rational function. Painlevé found 50 equations having this form, six of these were irreducible to known functions; they are known today as the six Painlevé equations.

Magnetic susceptibility = ratio of D-finite functions?

The hypergeometric function:

$$ {}_2F_1([1/3, 1/3], [1], 27x) $$

is D-finite and verifies the following **linear** differential equation

$$ (27x^2 - x)\left(\frac{d^2}{dx^2}F(x)\right) + (45x - 1)\left(\frac{d}{dx}F(x)\right) + 3F(x)\,. $$

Similarly the hypergeometric function given by

$$ {}_2F_1([1/2, 1/2], [1], 16x) $$

verifies the **D-finite** equation

$$ (16x^2 - x)\left(\frac{d^2}{dx^2}F(x)\right) + (32x - 1)\left(\frac{d}{dx}F(x) + 4F(x)\right)\,. $$

Reminder: A function is D-finite when it is solution of a *linear* differential equation and with *rational* coefficients in x.

Magnetic susceptibility = ratio of D-finite functions?

The ratio of these two D-finite functions is given by:

$$\frac{{}_2F_1([1/3, 1/3], [1], 27x)}{{}_2F_1([1/2, 1/2], [1], 16x)}$$

- While the product of two D-finite functions is always D-finite, the ratio of two D-finite functions is generally **not so** (except if the D-finite function at the denominator is an algebraic function)!
- In fact the differential equation that this ratio verifies is non-linear as we can see in the next slide

$$
\begin{aligned}
-2x^2(27x-1)(-1+16x)((27x-1)(-1+16x)\frac{d}{dx}F(x) \\
-72xF(x)-F(x))\frac{d^3}{dx^3}F(x) \\
+3x^2(27x-1)^2(-1+16x)^2\left(\frac{d^2}{dx^2}F(x)\right)^2 \\
-2x(93312\frac{d}{dx}F(x)x^4-7992\frac{d}{dx}F(x)x^3 \\
-93312x^3F(x)+87\frac{d}{dx}F(x)x^2+168x^2F(x) \\
+3\frac{d}{dx}F(x)x+297xF(x)-4F(x))\frac{d^2}{dx^2}F(x) \\
+(-1+16x)(1944x^3-1569x^2+58x-1)\left(\frac{d}{dx}F(x)\right)^2 \\
+2F(x)(29376x^3+5580x^2-221x+1)\frac{d}{dx}F(x) \\
+(144x^2-432x+1)F(x)^2=0
\end{aligned}
$$