# Gaussian Fluctuations for the Elephant Random Walk with Gradually Increasing Memory

(Joint work with Mohamed El Machkouri , Université de Rouen Normandie, France)


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University of Monastir, Tunisa
and
King Saud University, KSA


*University Paris 13*
France

April, 2025

# Plan

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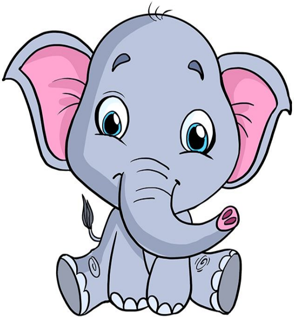

It is a non-markovian random walk $(S_n)_{n\geqslant 0}$ on $\mathbb{Z}$ introduced in 2004 in the paper:

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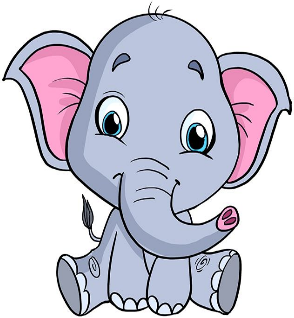

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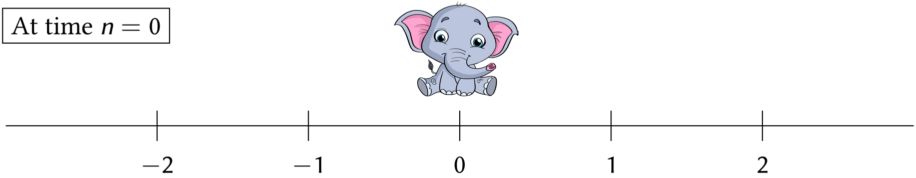
At time $n = 0$



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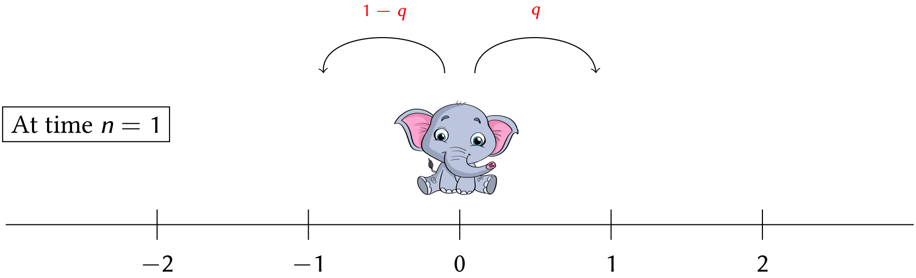


($q \in ]0, 1[$ is a fixed parameter)

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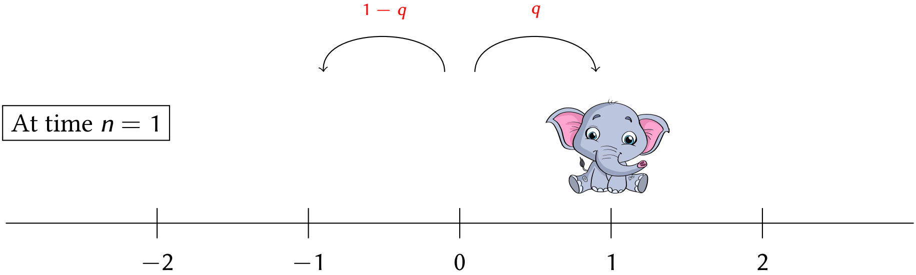

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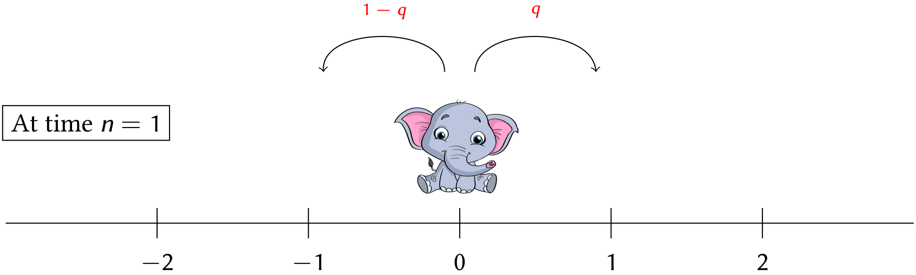
At time $n = 1$



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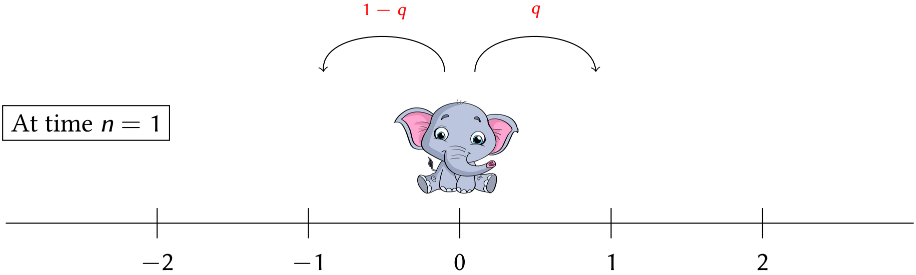

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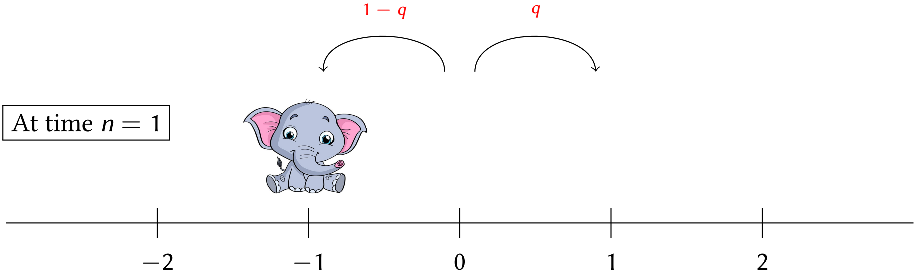

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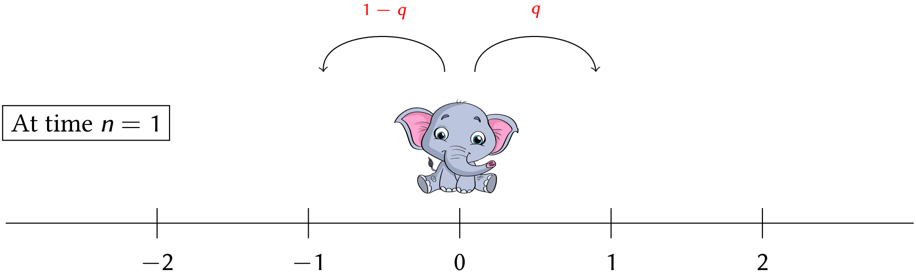

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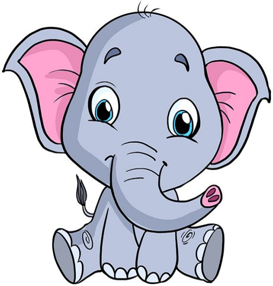

- Let $n \geqslant 1$ be fixed.
- At time $n + 1$, the elephant chooses uniformly at random an instant $\beta_n$ between 1 and $n$.
- According to the memory parameter $p$, the step at time $n + 1$ is given by

$$X_{n+1} = \begin{cases} +X_{\beta_n} & \text{with probability} \quad p \\ -X_{\beta_n} & \text{with probability} \quad 1-p \end{cases}$$

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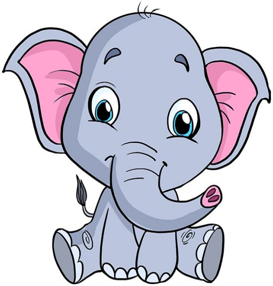

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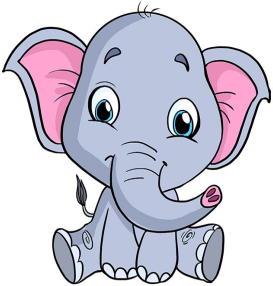

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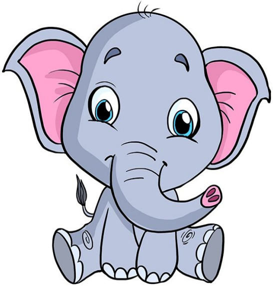

It means that $\boxed{X_{n+1} = \alpha_n X_{\beta_n}}$ with

- $\beta_n \sim \mathcal{U}(\{1, ..., n\})$
- $\alpha_n = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$
- $\alpha_n$, $\beta_n$ and $\mathcal{F}_n := \sigma(X_1, ..., X_n)$ are independent.

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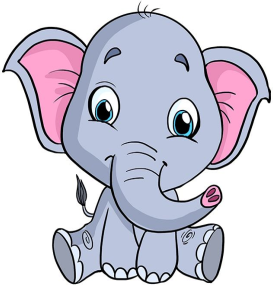

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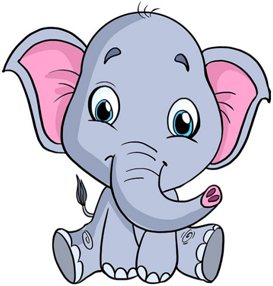

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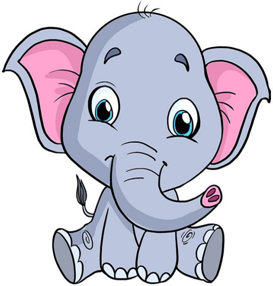

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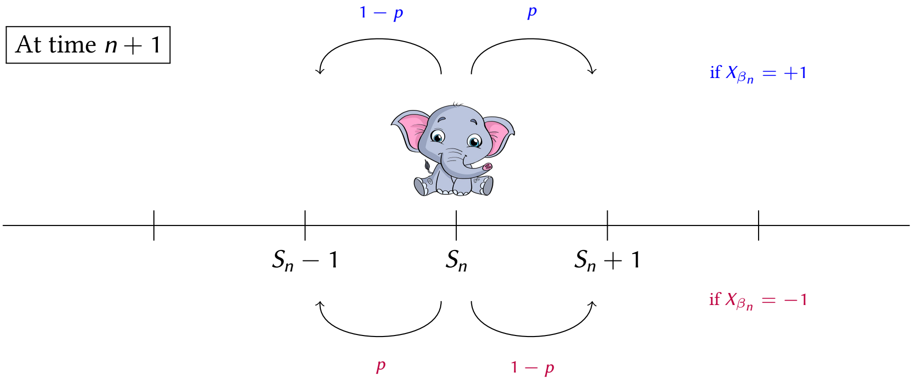
So, the position $S_{n+1}$ of the elephant at time $n+1$ is given by

$$S_{n+1} = S_n + X_{n+1}.$$

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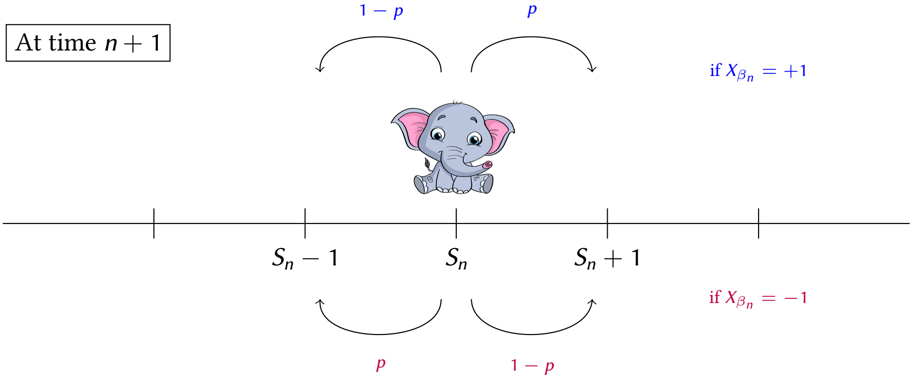


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# When elephants meet martingales...

One can notice that

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[\alpha_n]\mathbb{E}[X_{\beta_n}|\mathcal{F}_n] = (2p-1)\,\frac{S_n}{n}.$$

Moreover, since $S_{n+1} = S_n + X_{n+1}$, we obtain

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \gamma_n S_n \quad \text{with} \quad \gamma_n = \frac{n+2p-1}{n}.$$

**Question:** Can we find $(a_n)_{n\geqslant 1} \in \mathbb{R}^{\mathbb{N}^*}$ such that

$$\mathbb{E}[a_{n+1}S_{n+1}|\mathcal{F}_n] = a_n S_n?$$

If it is true then $M := (a_n S_n)_{n\geqslant 1}$ is a martingale !

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That is, for any $n \geqslant 2$, we have

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**Strategy** : Use martingale theory in order to get asymptotic properties for$(S_n)_{n\geqslant 1}$.

*B. Bercu*, A martingale approach for the elephant random walk, *J. Phys. A: Math. Theor., 51 015201, (2018)*

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# Gaussian fluctuations of the ERW

## Theorem (Bercu, 2018)

1. *If* $0 < p < 3/4$ (diffusive regime) *then* $n^{-1/2} S_n \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}\left(0, \frac{1}{3-4p}\right)$.
2. *If* $p = 3/4$ (critical regime) *then* $(n \log n)^{-1/2} S_n \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0, 1)$.
3. *If* $3/4 < p \leqslant 1$ (superdiffusive regime) *then* $n^{-2p+1} S_n \xrightarrow[n\to\infty]{\text{a.s. and } \mathbb{L}^4} L$ *where* $L$ *is a non gaussian random variable.*

## Theorem (Kubota and Takei, 2019)

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3. *If* $3/4 < p \leqslant 1$ (superdiffusive regime) *then* $n^{-2p+1} S_n \xrightarrow[n\to\infty]{a.s. \text{ and } \mathbb{L}^4} L$ *where* $L$ *is a non gaussian random variable.*

## Theorem (Kubota and Takei, 2019)

- *If* $3/4 < p \leqslant 1$ *then* $n^{2p-\frac{3}{2}} \left(n^{-2p+1} S_n - L\right) \xrightarrow[n\to+\infty]{\text{Law}} \mathcal{N}\left(0, \frac{1}{4p-3}\right)$

# Gaussian fluctuations of the ERW

## Theorem (Bercu, 2018)

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## CLT for reversed martingales (Heyde, 1977)

*Let* $(M_n)_{n\geqslant 0}$ *be a square-integrable martingale with* $M_0 = 0$*. Denote* $X_k := M_k - M_{k-1}$*,* $k \geqslant 1$ *and* $s_n^2 := \sum_{k=n}^{+\infty} \mathbb{E}[X_k^2]$*. Assume that* $\sum_{k=1}^{\infty} \mathbb{E}[X_k^2] < +\infty$*. Then,*

$$M_n \xrightarrow[n\to+\infty]{a.s.\ and\ \mathbb{L}^2} M_\infty := \sum_{k=1}^{+\infty} X_k < \infty.$$

*Moreover, if*

1. $s_n^{-2} \sum_{k=n}^{+\infty} X_k^2 \xrightarrow[n\to+\infty]{\mathbb{P}} 1$ *and*
2. $s_n^{-2} \sum_{k=n}^{+\infty} \mathbb{E}[X_k^2 \mathbb{1}_{\{|X_k|>\varepsilon s_n\}}] \xrightarrow[n\to+\infty]{} 0$ *for any* $\varepsilon > 0$*.*

*Then*

$$\frac{M_\infty - M_n}{s_{n+1}} = \frac{\sum_{k=n+1}^{+\infty} X_k}{s_{n+1}} \xrightarrow[n\to+\infty]{Law} \mathcal{N}(0,1).$$

# The Elephant Random Walk with Gradually Increasing Memory

# ERW with gradually increasing memory

Gut and Stadtmüller (2022) introduced a variation of the ERW.

*A. Gut and U. Stadtmüller,* The elephant random walk with gradually increasing memory, *Statist. Proab. Lett., 189:Paper No 109598, 10 (2022)*

Let $(m_n)_{n\geqslant 1}$ be positive integers such that for any $k$, $\ell$ and $n$,

- $m_n \leqslant n$,
- $k < \ell \Rightarrow m_k \leqslant m_\ell$.

At time $n+1$, the elephant remembers the steps $\{1, 2, ..., m_n\}$ instead of $\{1, 2, ..., n\}$.

It means that $\boxed{X_{n+1} = \alpha_n X_{\beta_n}}$ with

$$\mathbb{P}(\alpha_n = 1) = \mathbb{P}(\alpha_n = -1) = 1/2 \quad \text{and} \quad \beta_n \hookrightarrow \mathcal{U}(\{1, 2, ..., m_n\}).$$

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# ERW with gradually increasing memory

$$n = 6 \text{ and } m_n = [n/2] = 3$$

$X_1$ $X_2$ $X_3$ $X_4$ $X_5$ $X_6$

$$\forall k \in \{2, 3\} \quad X_k = \alpha_{k-1} X_{\beta_{k-1}} \quad \text{with} \quad \beta_{k-1} \hookrightarrow \mathcal{U}(\{1, .., k-1\})$$

$$\forall k \in \{4, 5, 6\} \quad X_k = \alpha_{k-1} X_{\beta_{k-1}} \quad \text{with} \quad \beta_{k-1} \hookrightarrow \mathcal{U}(\{1, 2, 3\})$$

# ERW with gradually increasing memory

$$n = 7 \text{ and } m_n = [n/2] = 3$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ |

$$\forall k \in \{2,3\} \quad X_k = \alpha_{k-1} X_{\beta_{k-1}} \quad \text{with} \quad \beta_{k-1} \hookrightarrow \mathcal{U}\left(\{1,..,k-1\}\right)$$

$$\forall k \in \{4,5,6,7\} \quad X_k = \alpha_{k-1} X_{\beta_{k-1}} \quad \text{with} \quad \beta_{k-1} \hookrightarrow \mathcal{U}\left(\{1,2,3\}\right)$$

# ERW with gradually increasing memory

$$n = 8 \text{ and } m_n = [n/2] = 4$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |

$$\forall k \in \{2,3,4\} \quad X_k = \alpha_{k-1} X_{\beta_{k-1}} \quad \text{with} \quad \beta_{k-1} \hookrightarrow \mathcal{U}\left(\{1,..,k-1\}\right)$$

$$\forall k \in \{5,6,7,8\} \quad X_k = \alpha_{k-1} X_{\beta_{k-1}} \quad \text{with} \quad \beta_{k-1} \hookrightarrow \mathcal{U}\left(\{1,2,3,4\}\right)$$

# ERW with gradually increasing memory

For any integer $1 \leqslant \ell \leqslant n$,

$$S_\ell = \sum_{k=1}^{\ell} X_k \quad \text{and} \quad \mathcal{F}_\ell = \sigma(X_k\,;\, 1 \leqslant k \leqslant \ell).$$

It is important to note that the elephant can not choose among its steps $m_n + 1, m_n + 2, \ldots, k$ to determine its $(k+1)$th step for any $m_n \leqslant k < n$.

So, we have

$$\mathbb{E}[X_{k+1}|\mathcal{F}_k] = \begin{cases} (2p-1)m_n^{-1}S_{m_n} & \text{if} \quad m_n < k \leqslant n, \\ (2p-1)k^{-1}S_k & \text{if} \quad 1 \leqslant k \leqslant m_n. \end{cases}$$

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# ERW with gradually increasing memory

Consequently, for $1 \leqslant k < m_n$,

$$\mathbb{E}[S_{k+1}|\mathcal{F}_k] = S_k + \mathbb{E}[X_{k+1}|\mathcal{F}_k] = \gamma_k S_k \quad \text{with} \quad \gamma_k = 1 + \frac{2p-1}{k}.$$

If we denote $a_k = \prod_{\ell=1}^{k-1} \gamma_\ell^{-1}$ and $\overline{M}_k := a_k S_k$ then $(\overline{M}_k)_{1 \leqslant k \leqslant m_n}$ is a martingale with respect to the filtration $(\mathcal{F}_k)_{1 \leqslant k \leqslant m_n}$ which can be written

$$\overline{M}_k = \sum_{\ell=1}^{k} a_\ell \varepsilon_\ell \quad \text{with} \quad \varepsilon_\ell = S_\ell - \gamma_{\ell-1} S_{\ell-1}$$

satisfying $\mathbb{E}[\varepsilon_\ell|\mathcal{F}_{\ell-1}] = 0$ for any $1 \leqslant \ell \leqslant m_n$.

# ERW with gradually increasing memory

Consequently, for $1 \leqslant k < m_n$,

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# ERW with gradually increasing memory

Consider also the martingale $(M_k)_{1\leqslant k\leqslant n}$ defined by

$$M_k := \sum_{\ell=1}^{k} a_\ell \left(S_\ell - \mathbb{E}[S_\ell|\mathcal{F}_{\ell-1}]\right)$$
$$= \begin{cases} \sum_{\ell=1}^{k} a_\ell \varepsilon_\ell & \text{if} \quad k \leqslant m_n \\ \sum_{\ell=1}^{m_n} a_\ell \varepsilon_\ell + \sum_{\ell=m_n+1}^{k} a_\ell \left(X_\ell - \mathbb{E}[X_\ell|\mathcal{F}_{\ell-1}]\right) & \text{if} \quad k > m_n \end{cases}$$

# Gaussian fluctuations (previous result)

## Theorem (Gut and Stadtmüller, 2022)

*Let* $(m_n)_{n\geqslant 1}$ *be a nondecreasing sequence of positive integers satisfying* $m_n \leqslant n$ *and* $\lim_{n\to+\infty} n^{-1}m_n = 0$.

1) *If* $0 < p < 3/4$ *then* $n^{-1}\sqrt{m_n}S_n \xrightarrow[n\to+\infty]{Law} \mathcal{N}\left(0, \frac{(2p-1)^2}{3-4p}\right)$.
2) *If* $p = 3/4$ *then* $n^{-1}S_n\sqrt{m_n/\log m_n} \xrightarrow[n\to+\infty]{Law} \mathcal{N}\left(0, \frac{1}{4}\right)$.
3) *If* $3/4 < p < 1$ *then* $n^{-1}S_n m_n^{2(1-p)} \xrightarrow[n\to+\infty]{a.s.} (2p-1)L$

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**Question**: what happen if $\lim_{n\to+\infty} n^{-1}m_n = \theta$ for some $\theta \in ]0,1]$ ?

# Gaussian fluctuations (previous result)

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# Gaussian fluctuations (new result)

## Theorem (Aguech, E.M., 2023)

*Let $(m_n)_{n\geqslant 1}$ be a non-decreasing sequence of positive integers such that $\lim_{n\to+\infty} n^{-1}m_n = \theta$ for some $\theta \in [0,1]$ and denote $\tau := \theta + (1-\theta)(2p-1)$.*

1) *if* $0 < p < 3/4$ *then* $n^{-1}\sqrt{m_n}\, S_n \xrightarrow[n\to+\infty]{Law} \mathcal{N}\left(0, \frac{\tau^2}{3-4p} + \theta(1-\theta)\right)$.

2) *if* $p = 3/4$ *then* $n^{-1}\sqrt{m_n/\log m_n}\, S_n \xrightarrow[n\to+\infty]{Law} \mathcal{N}\left(0, \frac{(1+\theta)^2}{4}\right)$.

3) *if* $3/4 < p < 1$ *then* $n^{-1}m_n^{2(1-p)}\, S_n \xrightarrow[n\to+\infty]{a.s.\ and\ \mathbb{L}^4} \tau L$.

   *In addition, if* $\lim_{n\to+\infty} m_n^{2p-\frac{3}{2}}|n^{-1}m_n - \theta| = 0$ *then*

$$m_n^{2p-\frac{3}{2}}\left(n^{-1}S_n m_n^{2(1-p)} - \tau L\right) \xrightarrow[n\to+\infty]{Law} \mathcal{N}\left(0, \frac{\tau^2}{4p-3} + \theta(1-\theta)\right).$$

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# Idea of proof

Let $\theta \in [0,\ 1]$ such that $n^{-1}m_n \to \theta$ and $\tau := \theta + (1-\theta)(2p-1)$.

First, we write

$$\frac{S_n\sqrt{m_n}}{n} = \frac{\sqrt{m_n}}{n}\sum_{k=1}^{m_n} X_k + \frac{\sqrt{m_n}}{n}\sum_{k=m_n+1}^{n} X_k.$$

Since $\mathbb{E}[X_k|\mathcal{F}_{k-1}] = (2p-1)m_n^{-1}S_{m_n}$ for $k > m_n$, we get

$$\frac{S_n\sqrt{m_n}}{n} = \frac{\tau_n S_{m_n}}{\sqrt{m_n}} + \frac{\sqrt{m_n}}{n}\sum_{k=m_n+1}^{n}\left(X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]\right)$$

where $\tau_n := \frac{m_n}{n} + \left(1 - \frac{m_n}{n}\right)(2p-1) \xrightarrow[n\to+\infty]{} \tau := \theta + (1-\theta)(2p-1)$.

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where $\tau_n := \frac{m_n}{n} + \left(1 - \frac{m_n}{n}\right)(2p-1) \xrightarrow[n\to+\infty]{} \tau := \theta + (1-\theta)(2p-1)$.

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Since $\mathbb{E}[X_k|\mathcal{F}_{k-1}] = (2p-1)m_n^{-1}S_{m_n}$ for $k > m_n$, we get

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# Idea of proof

Since $a_{m_n} S_{m_n} = \sum_{\ell=1}^{m_n} a_\ell \varepsilon_\ell$ with $\varepsilon_\ell = S_\ell - \gamma_{\ell-1} S_{\ell-1}$, we obtain

$$\frac{S_n\sqrt{m_n}}{n} = \frac{\tau_n \sum_{\ell=1}^{m_n} a_\ell \varepsilon_\ell}{a_{m_n}\sqrt{m_n}} + \frac{\sqrt{m_n}}{n}\sum_{k=m_n+1}^{n}\left(X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]\right) = \sum_{k=1}^{n}\Delta_k$$

where

$$\Delta_k = \begin{cases} \tau_n \left(\sqrt{m_n}\, a_{m_n}\right)^{-1} a_k \varepsilon_k & \text{if} \quad k \leqslant m_n \\ n^{-1}\sqrt{m_n}\left(X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]\right) & \text{if} \quad k > m_n. \end{cases}$$

# Idea of proof

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function with compact support and two bounded and continuous derivatives.

Let $(Z_k)_{1 \leqslant k \leqslant n}$ be independent $\mathcal{N}(0, \mathbb{E}[\Delta_k^2])$-random variables which are assumed to be independent of the sequence $(X_k)_{1 \leqslant k \leqslant n}$.

Applying Lindeberg's method, it suffices to show that the term

$$I_n(\varphi) := \mathbb{E}\left[\varphi\left(\sum_{k=1}^{n} \Delta_k\right)\right] - \mathbb{E}\left[\varphi\left(\sum_{k=1}^{n} Z_k\right)\right]$$

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# Idea of proof

For any $1 \leqslant k \leqslant n$, we consider the notation

$$U_k = \Delta_1 + \cdots + \Delta_k + Z_{k+1} + \cdots + Z_n.$$

Then, using the Lindeberg's trick, we have

$$I_n(\varphi) = \sum_{k=1}^{n} \mathbb{E}[\varphi(U_k) - \varphi(U_{k-1})] = \sum_{k=1}^{n} b_k(\varphi)$$

where

$$b_k(\varphi) = \mathbb{E}\left[(\varphi(W_k + \Delta_k) - \varphi(W_k)) - (\varphi(W_k + Z_k) - \varphi(W_k))\right]$$

and

$$W_k = \Delta_1 + \cdots + \Delta_{k-1} + Z_{k+1} + \cdots + Z_n.$$

# Idea of proof

Using Taylor's formula, we have

$$b_k(\varphi) = \mathbb{E}\left[\Delta_k \varphi^{'}(W_k)\right] + \frac{1}{2}\mathbb{E}\left[\varphi''(W_k)\Delta_k^2\right] + \mathbb{E}[R_k] - \left(\mathbb{E}\left[Z_k \varphi^{'}(W_k)\right] + \frac{1}{2}\mathbb{E}\left[\varphi''(W_k)Z_k^2\right] + \mathbb{E}[r_k]\right)$$

with $|R_k| \leqslant |\Delta_k|^3$ and $|r_k| \leqslant |Z_k|^3$.

Since $\mathbb{E}[\Delta_k|\mathcal{F}_{k-1}] = 0$, we have $\mathbb{E}[\Delta_k \varphi^{'}(W_k)] = 0$.

Moreover, $\mathbb{E}[Z_k \varphi^{'}(W_k)] = \mathbb{E}[Z_k]\mathbb{E}[\varphi^{'}(W_k)] = 0$. Consequently,

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So, it suffices to control the partial sums of each term above...

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## How to estimate the memory parameter $p$ ?

# How to estimate the memory $p$ ?

We follow the approach by Bercu and Laulin (2022).

If $m_n \leqslant k \leqslant n$ then $\beta_k \hookrightarrow \mathcal{U}(\{1, 2, ..., m_n\})$ and

$$\begin{aligned}
\mathbb{P}(X_{k+1} = 1|\mathcal{F}_k) &= \mathbb{P}(X_{k+1} = 1|\mathcal{F}_{m_n}) \\
&= \mathbb{P}(X_{k+1} = 1, \alpha_k = 1|\mathcal{F}_{m_n}) + \mathbb{P}(X_{k+1} = -1, \alpha_k = -1|\mathcal{F}_{m_n}) \\
&= p\mathbb{P}(X_{\beta_k} = 1|\mathcal{F}_{m_n}) + (1-p)\mathbb{P}(X_{\beta_k} = -1|\mathcal{F}_{m_n}) \\
&= \frac{p}{m_n}\sum_{k=1}^{m_n} \mathbb{1}_{\{X_k=1\}} + \frac{1-p}{m_n}\sum_{k=1}^{m_n} \mathbb{1}_{\{X_k=-1\}} \\
&= \frac{p\,(m_n + S_{m_n})}{2m_n} + \frac{(1-p)\,(m_n - S_{m_n})}{2m_n} \\
&= \frac{1}{2}\left(1 + (2p-1)\frac{S_{m_n}}{m_n}\right).
\end{aligned}$$

B. Bercu and L. Laulin, *How to estimate the memory of the elephant random walk*, Communications in Statistics - Theory and Methods (2022).

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If $1 \leqslant k < m_n$ then $\beta_k \hookrightarrow \mathcal{U}(\{1, 2, ..., k\})$ and

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&= p\mathbb{P}(X_{\beta_k} = 1 | \mathcal{F}_k) + (1-p)\mathbb{P}(X_{\beta_k} = -1 | \mathcal{F}_k) \\
&= \frac{p}{k} \sum_{j=1}^{k} \mathbb{1}_{\{X_j = 1\}} + \frac{1-p}{k} \sum_{j=1}^{k} \mathbb{1}_{\{X_j = -1\}} \\
&= \frac{p(k + S_k)}{2k} + \frac{(1-p)(k - S_k)}{2k} \\
&= \frac{1}{2}\left(1 + (2p-1)\frac{S_k}{k}\right).
\end{aligned}$$

# How to estimate the memory $p$ ?

It means that $\mathbb{P}\left(X_{k+1} = 1 | \mathcal{F}_k\right) = p_k$ where

$$p_\ell = \begin{cases} \frac{1}{2}\left(1 + a\frac{S_\ell}{\ell}\right) & \text{if} \quad \ell < m_n \\ \frac{1}{2}\left(1 + a\frac{S_{m_n}}{m_n}\right) & \text{if} \quad \ell \geqslant m_n \end{cases}$$

and $a = 2p - 1$. Therefore, for any $x_{k+1} \in \{-1, 1\}$,

$$\mathbb{P}\left(X_{k+1} = x_{k+1} | \mathcal{F}_k\right) = p_k^{\frac{1+x_{k+1}}{2}} \left(1 - p_k\right)^{\frac{1-x_{k+1}}{2}}.$$

So, for any $n \geqslant 1$ and any $x = (x_1, ..., x_n) \in \{-1, 1\}^n$, we have

$$\begin{aligned} &\mathbb{P}\left(X_1 = x_1, X_2 = x_2, ..., X_n = x_n\right) \\ &= \prod_{k=1}^{n-1} \mathbb{P}\left(X_{k+1} = x_{k+1} | X_1 = x_1, ..., X_k = x_k\right) \mathbb{P}\left(X_1 = x_1\right) \\ &= \prod_{k=1}^{n-1} p_k^{(1+x_{k+1})/2} (1 - p_k)^{(1-x_{k+1})/2} q^{(1+x_1)/2} (1 - q)^{(1-x_1)/2} \end{aligned}$$

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# How to estimate the memory $p$ ?

Consequently, the log-likelihood function associated with $(X_1, \dots, X_n)$ is given by

$$\ell_n(a) = c(q, X_1) + \sum_{k=1}^{n-1} \left(\frac{1 + X_{k+1}}{2}\right) \log p_k + \left(\frac{1 - X_{k+1}}{2}\right) \log(1 - p_k)$$

with $c(q, X_1) := \left(\frac{1+X_1}{2}\right) \log q + \left(\frac{1-X_1}{2}\right) \log(1 - q)$ and

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The second order Taylor approximation of $\ell_n(a)$ is given by

$$\lambda_n(a) = -(n-1)\log 2 + c(q, X_1) + a\sum_{k=1}^{n-1} X_{k+1} T_k - \frac{a^2}{2} \sum_{k=1}^{n-1} T_k^2$$

where

$$T_k = \begin{cases} k^{-1} S_k & \text{if} \quad k < m_n \\ m_n^{-1} S_{m_n} & \text{if} \quad k \geqslant m_n \end{cases}$$

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Since

$$\lambda_n'(a) = \sum_{k=1}^{n-1} X_{k+1} T_k - a \sum_{k=1}^{n-1} T_k^2,$$

we get

$$\lambda_n'(a) = 0 \Leftrightarrow \hat{a} = \frac{\sum_{k=1}^{n-1} X_{k+1} T_k}{\sum_{k=1}^{n-1} T_k^2}.$$

Since $a = 2p - 1$, we obtain

$$\boxed{\hat{p}_n = \frac{\sum_{k=1}^{n-1} T_k \left(X_{k+1} + T_k\right)}{2\sum_{k=1}^{n-1} T_k^2} \quad \text{where} \quad T_k = \begin{cases} k^{-1} S_k & \text{if} \quad k < m_n \\ m_n^{-1} S_{m_n} & \text{if} \quad k \geqslant m_n \end{cases}}$$

# Consistency of $\hat{p}_n$

Consequently, we get

$$\hat{p}_n - p = \frac{\sum_{k=1}^{n-1} T_k \left(X_{k+1} - (2p-1)T_k\right)}{2\sum_{k=1}^{n-1} T_k^2}.$$

Since $(2p-1)T_k = \mathbb{E}[X_{k+1}|\mathcal{F}_k]$ for any $1 \leqslant k \leqslant n$, we derive

$$\hat{p}_n - p = \frac{M_n}{V_n}$$

where $M_n := \sum_{k=1}^{n-1} T_k \left(X_{k+1} - \mathbb{E}[X_{k+1}|\mathcal{F}_k]\right)$ and $V_n := 2\sum_{k=1}^{n-1} T_k^2$.

**Remark:** $(M_n)_{n\geqslant 0}$ is a martingale !

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Moreover, its quadratic variation is given by

$$\langle \mathrm{M} \rangle_n = \sum_{k=1}^{n-1} \mathbb{E}\left[ \left( M_k - M_{k-1} \right)^2 | \mathcal{F}_k \right] = \sum_{k=1}^{n-1} T_k^2 (1 - a^2 T_k^2).$$

So,

$$\frac{\langle \mathrm{M} \rangle_n}{V_n} = \frac{\sum_{k=1}^{n-1} T_k^2 (1 - a^2 T_k^2)}{2 \sum_{k=1}^{n-1} T_k^2} \xrightarrow[n \to +\infty]{\text{a.s.}} \frac{1}{2}.$$

From Bercu (2018), we know that

$$\frac{1}{\log n} \sum_{k=1}^{m_n} \left( \frac{S_k}{k} \right)^2 \xrightarrow[n \to +\infty]{\text{a.s.}} \frac{1}{3 - 4p}.$$

So, we get

$$\langle \mathrm{M} \rangle_n \geqslant (1 - a^2) \sum_{k=1}^{m_n - 1} \left( \frac{S_k}{k} \right)^2 \xrightarrow[n \to +\infty]{\text{a.s.}} +\infty$$

# Consistency of $\hat{p}_n$

## LLN for martingales (Neveu, 1968)

*Let $(M_n)_{n\geqslant 0}$ be a square-integrable martingale. If $\lim_{n\to+\infty}\langle M\rangle_n = +\infty$ a.s. then, for any $\gamma > 0$,*

$$\frac{M_n}{\langle M\rangle_n} = o_{\text{a.s.}}\left(\frac{\log^{1+\gamma}\langle M\rangle_n}{\langle M\rangle_n}\right).$$

Consequently, we obtain

$$\hat{p}_n - p = \frac{M_n}{V_n} = \frac{\langle M\rangle_n}{V_n} \times \frac{M_n}{\langle M\rangle_n} \xrightarrow[n\to+\infty]{\text{a.s.}} \frac{1}{2} \times 0 = 0.$$

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# Gaussian fluctuations of $\hat{p}_n$

One can notice that

$$\frac{1}{\log n}\sum_{k=1}^{n} T_k^2 = \frac{1}{\log n}\sum_{k=1}^{m_n}\left(\frac{S_k}{k}\right)^2 + \frac{n-m_n}{\log n}\left(\frac{S_{m_n}}{m_n}\right)^2.$$

From Bercu (2018), we know that

$$\frac{1}{\log n}\sum_{k=1}^{m_n}\left(\frac{S_k}{k}\right)^2 \xrightarrow[n\to+\infty]{\text{a.s.}} \frac{1}{3-4p}.$$

# Gaussian fluctuations of $\hat{p}_n$

Moreover, for any $\lambda > 0$,

$$\mathbb{P}\left(\left|\frac{n-m_n}{\log n}\left(\frac{S_{m_n}}{m_n}\right)^2\right| > \lambda^2\right) = \mathbb{P}\left(|S_{m_n}| > \frac{\lambda m_n\sqrt{\log n}}{\sqrt{n-m_n}}\right)$$
$$= \mathbb{P}\left(|a_{m_n}S_{m_n}| > \frac{\lambda m_n a_{m_n}\sqrt{\log n}}{\sqrt{n-m_n}}\right)$$
$$= \mathbb{P}\left(\left|\sum_{k=1}^{m_n} a_k\varepsilon_k\right| > \frac{\lambda m_n a_{m_n}\sqrt{\log n}}{\sqrt{n-m_n}}\right)$$
$$\leqslant \exp\left(\frac{-\lambda^2 a_{m_n}^2 m_n^2 \log n}{(n-m_n)\nu_{m_n}}\right)$$

where $\nu_{m_n} = \sum_{k=1}^{m_n} a_k^2$ and $\varepsilon_k = S_k - \gamma_{k-1}S_{k-1}$

# Gaussian fluctuations of $\hat{p}_n$

Since $0 \leqslant p < 3/4$, we have

$$\lim_{n \to +\infty} n^{2p-1} a_n = \Gamma(2p) \quad \text{and} \quad \lim_{n \to +\infty} n^{4p-3} \nu_n = \frac{\Gamma^2(2p)}{3-4p}.$$

So, we get

$$\mathbb{P}\left( \left| \frac{n - m_n}{\log n} \left( \frac{S_{m_n}}{m_n} \right)^2 > \lambda^2 \right) \lesssim \exp\left( \frac{-\lambda^2 (3-4p) m_n \log n}{n - m_n} \right)$$

$$\lesssim n^{\frac{-\lambda^2(3-4p)\theta}{1-\theta}} \xrightarrow[n \to +\infty]{} 0.$$

That is,

$$\frac{n - m_n}{\log n} \left( \frac{S_{m_n}}{m_n} \right)^2 \xrightarrow[n \to +\infty]{\mathbb{P}} 0.$$

# Gaussian fluctuations of $\hat{p}_n$

So, we get the following QWLLN:

$$\frac{1}{\log n}\sum_{k=1}^{n} T_k^2 = \frac{1}{\log n}\sum_{k=1}^{m_n}\left(\frac{S_k}{k}\right)^2 + \frac{n-m_n}{\log n}\left(\frac{S_{m_n}}{m_n}\right)^2 \xrightarrow[n\to+\infty]{\mathbb{P}} \frac{1}{3-4p}$$

and $(\log n)^{-1}V_n \xrightarrow[n\to+\infty]{\mathbb{P}} 2/(3-4p)$. Moreover, for any $\varepsilon > 0$,

$$L_n(\varepsilon) := \frac{1}{\log n}\sum_{k=1}^{n-1}\mathbb{E}[(M_{k+1}-M_k)^2\ \mathbb{1}_{\{|M_{k+1}-M_k|>\varepsilon\sqrt{\log n}\}}|\mathcal{F}_k] \leqslant \frac{2V_n}{\varepsilon^2(\log n)^2}$$

So, we get $L_n(\varepsilon) \xrightarrow[n\to+\infty]{\mathbb{P}} 0$. Moreover, from the QWLLN, we have

$$\sum_{k=1}^{n-1}\left(\frac{M_{k+1}-M_k}{\sqrt{\log n}}\right)^2 \xrightarrow[n\to+\infty]{\mathbb{P}} \frac{1}{3-4p}.$$

## Theorem (McLeish, 1974)

*Let $(M_{k,n}, \mathcal{F}_{k,n})_{1\leqslant k\leqslant n}$ be a zero-mean, square-integrable martingale array with differences $(X_{k,n})_{1\leqslant k\leqslant n}$ and let $\eta^2$ be an a.s. finite random variable. Assume that*

- $\sum_{k=1}^{n} X_{k,n}^2 \xrightarrow[n\to+\infty]{\mathbb{P}} \eta^2$
- *For any* $\varepsilon > 0$, $\sum_{k=1}^{n} \mathbb{E}[X_{k,n}^2 | \mathcal{F}_{k-1,n}] \xrightarrow[n\to+\infty]{\mathbb{P}} 0$
- $\mathcal{F}_{k,n} \subset \mathcal{F}_{k,n+1}$ *for any* $1 \leqslant k \leqslant n$

*Then,*

$$\sum_{k=1}^{n} X_{k,n} \xrightarrow[n\to+\infty]{Law} Z$$

*where $Z$ has characteristic function* $\mathbb{E}[\exp(-\eta^2 t^2/2)]$.

# Gaussian fluctuations of $\hat{p}_n$

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# Gaussian fluctuations of $\hat{p}_n$

Applying McLeish's CLT, we obtain

$$\frac{M_n}{\sqrt{\log n}} \xrightarrow[n\to+\infty]{\text{Law}} \mathcal{N}\left(0, \frac{1}{3-4p}\right).$$

Finally, by Slutsky's lemma, we get

$$\boxed{\sqrt{\log n}\,(\hat{p}_n - p) = \left(\frac{V_n}{\log n}\right)^{-1} \frac{M_n}{\sqrt{\log n}} \xrightarrow[n\to+\infty]{\text{Law}} \mathcal{N}\left(0, \frac{3-4p}{4}\right)}.$$

# Gaussian fluctuations of $\hat{p}_n$

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# Conclusion

# Open questions

- How to obtain the rate of convergence in the CLT for the ERW with gradually increasing memory ?
- How to obtain the rate of convergence for the estimator of $p$ ?
- How to extend the results for an ERW with memory $\{n - m_n, ..., n\}$ intead of $\{1, ..., m_n\}$ with $\lim_{n\to+\infty} n^{-1} m_n = \theta$ ?

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**Thank you !**

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