Parking sur l'arbre binaire infini

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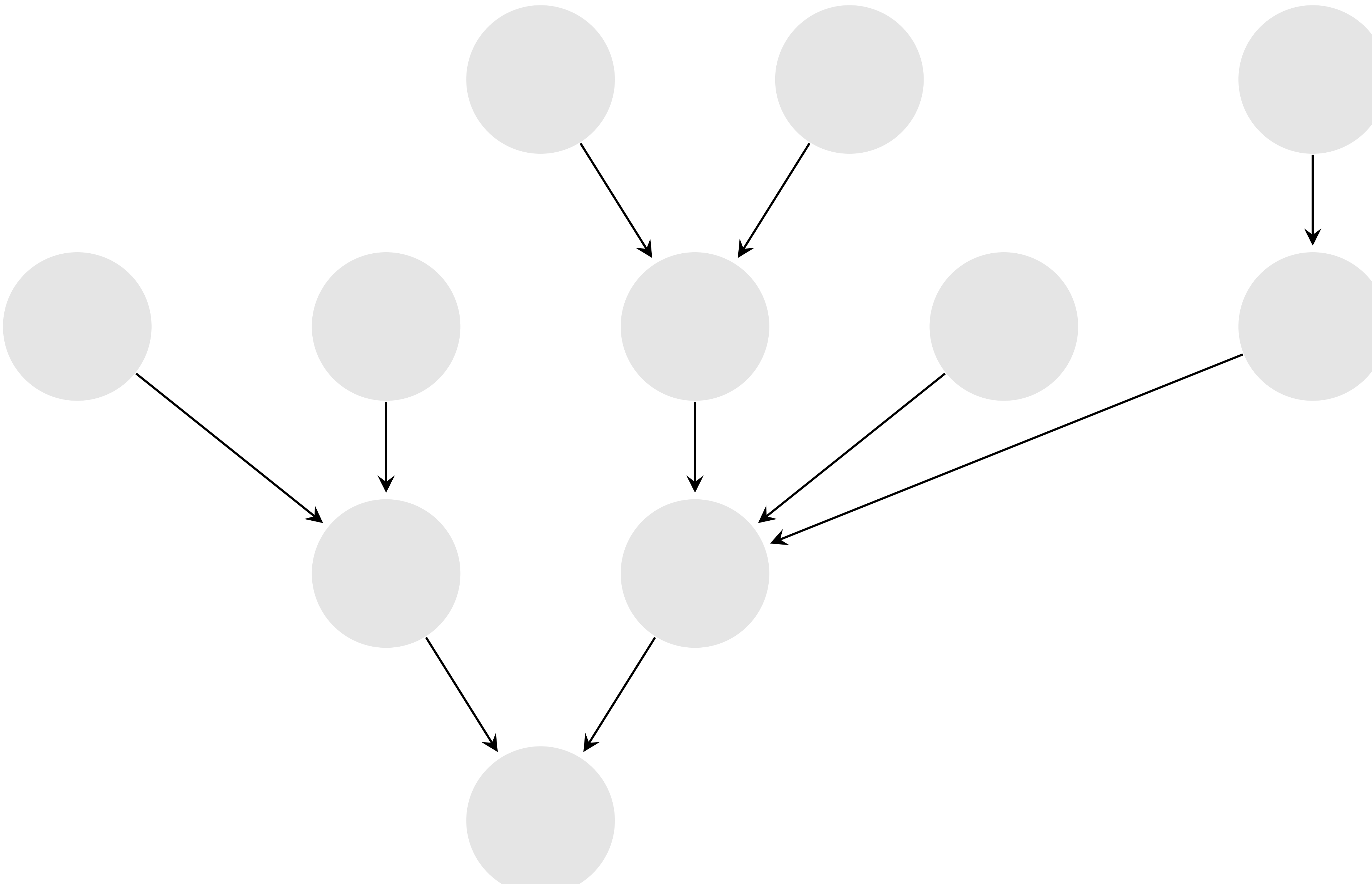
Séminaire de l'équipe CALIN
13 Février 2024

Parking rules on a tree



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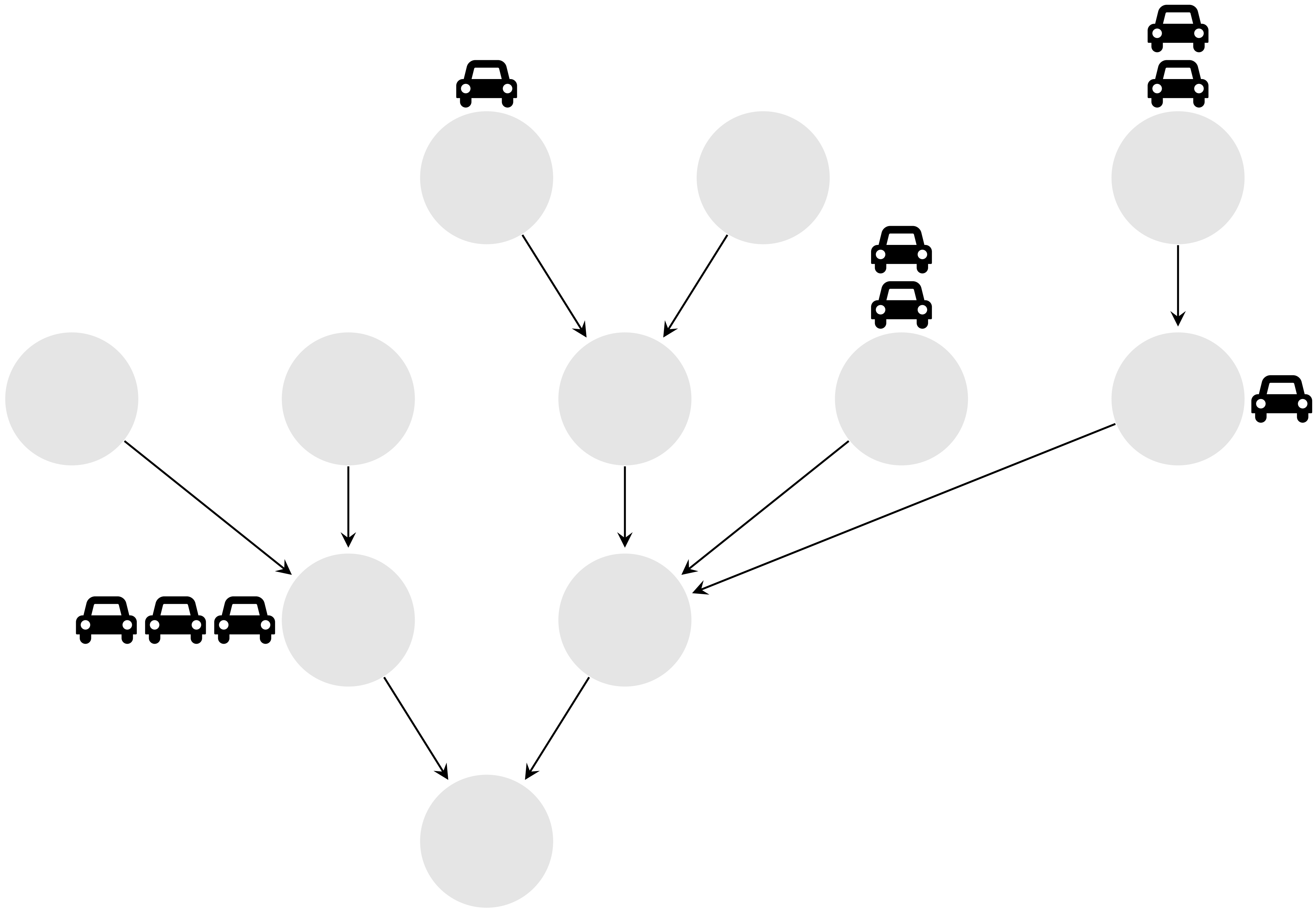

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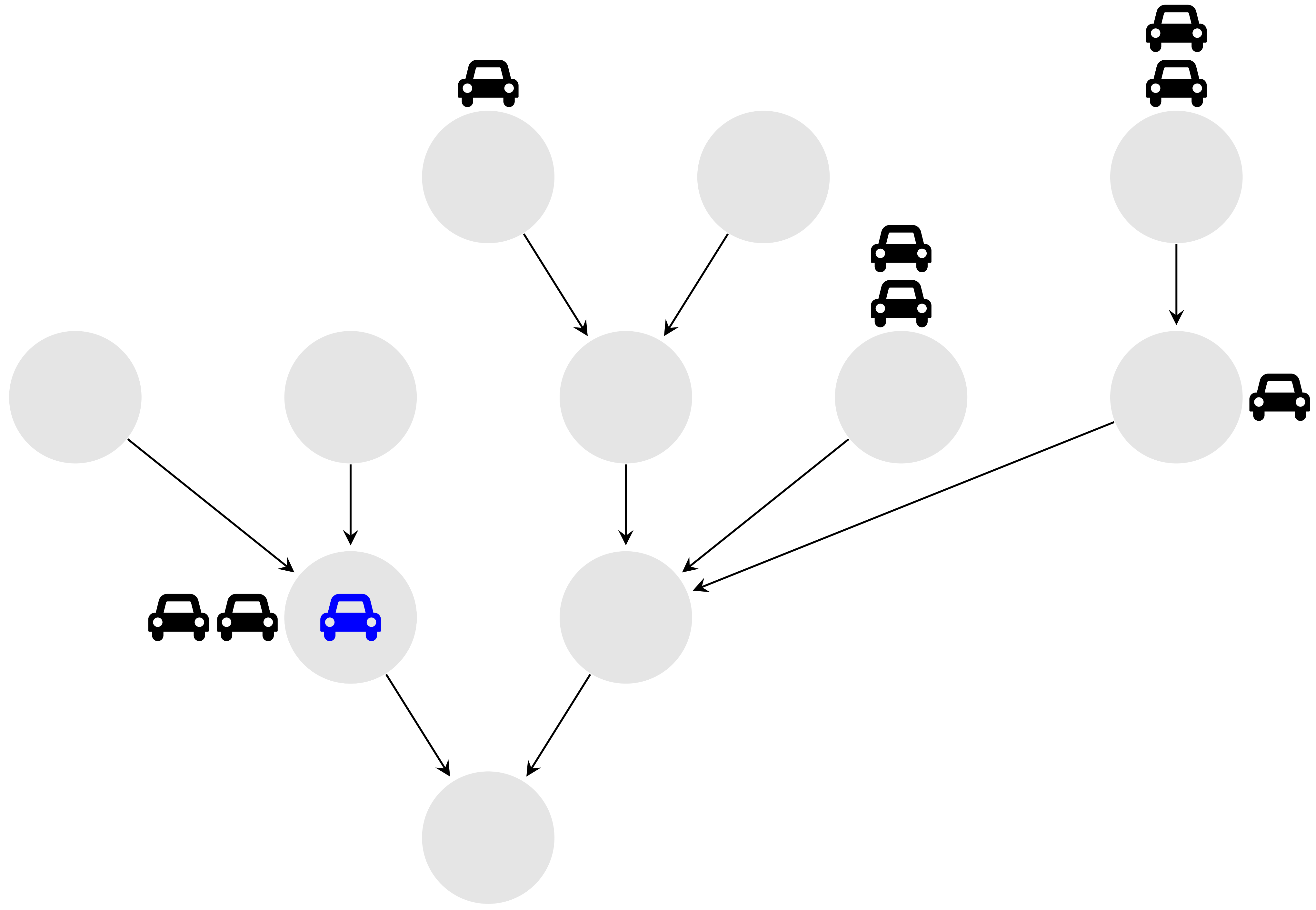

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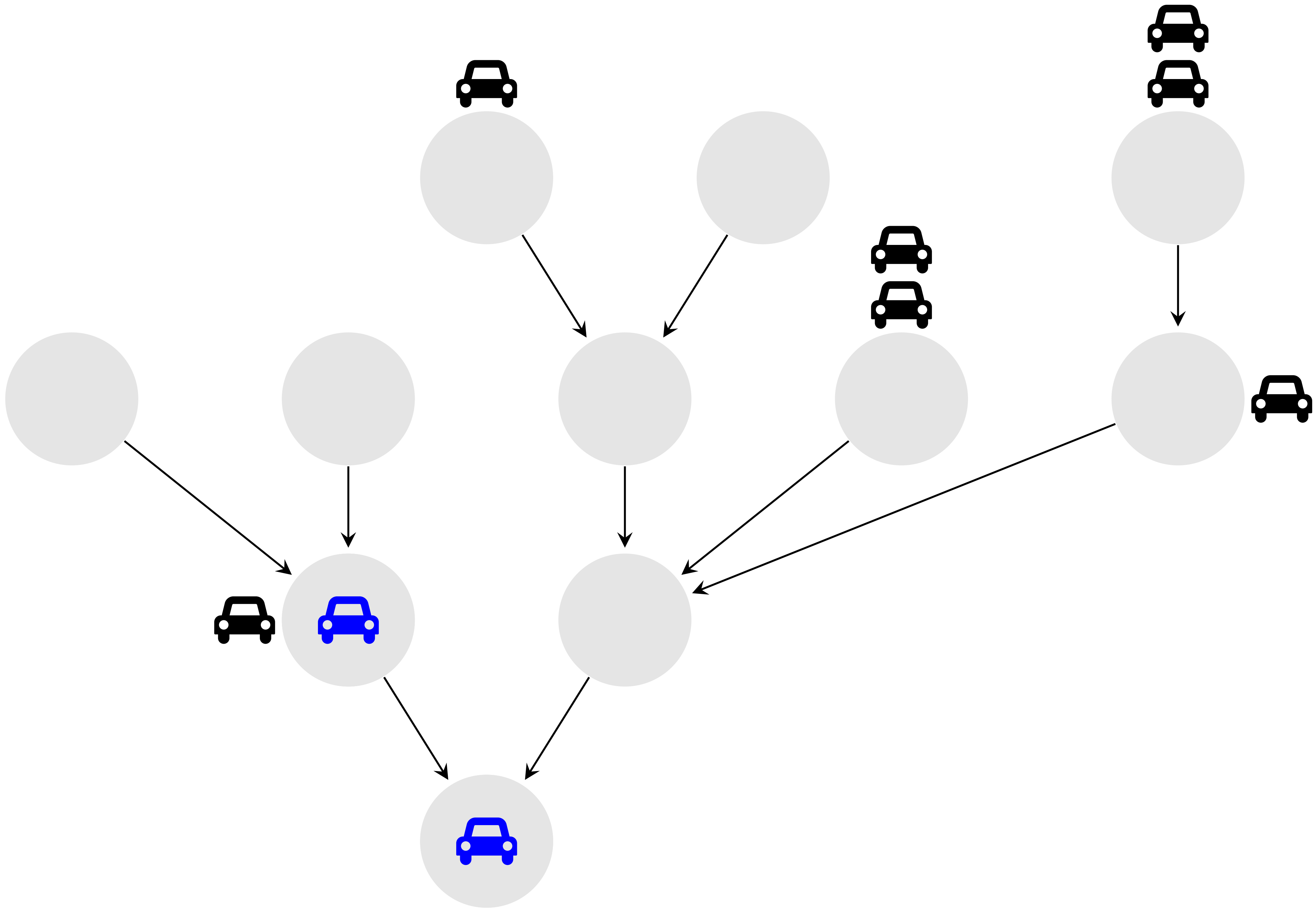

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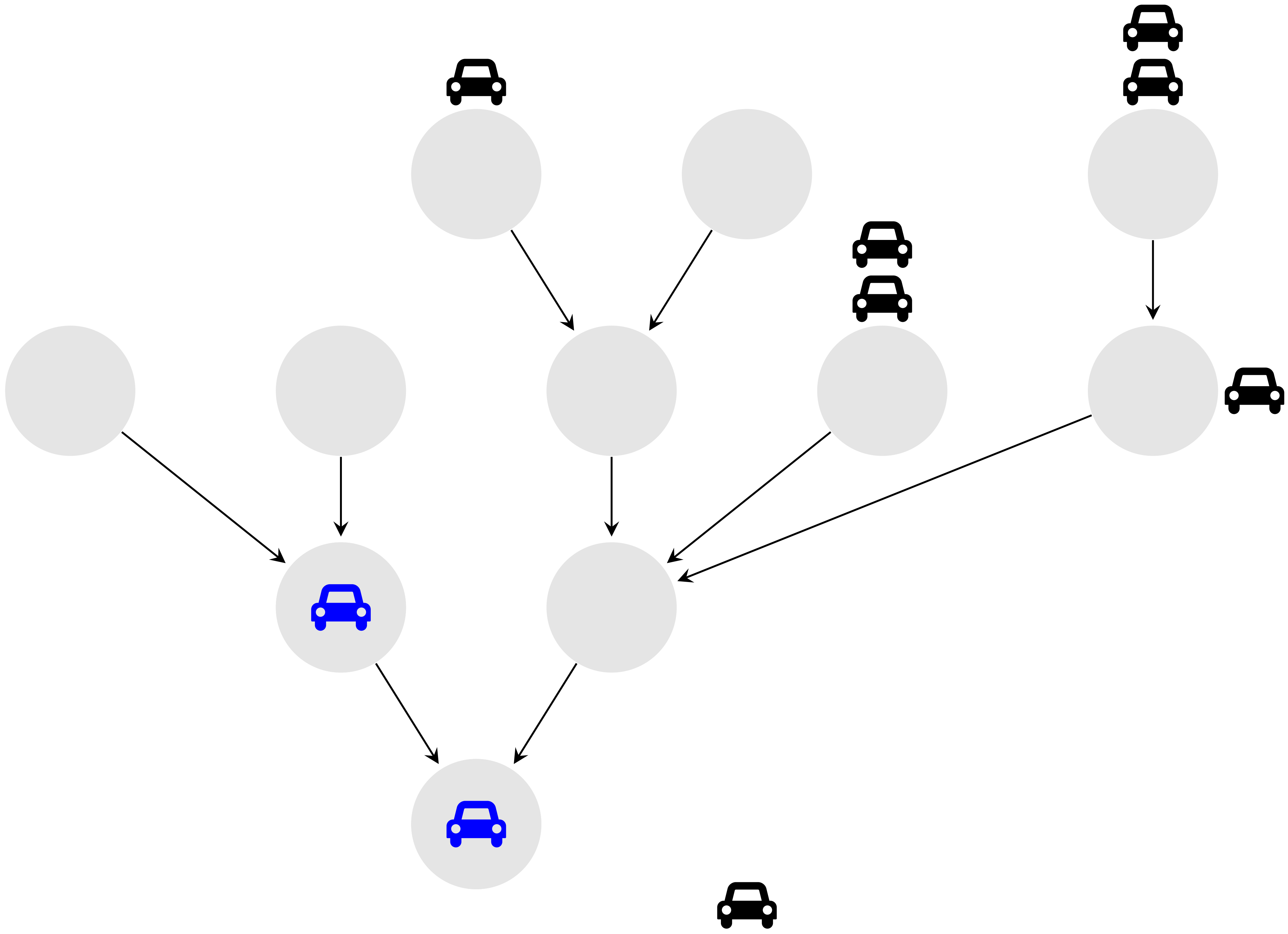

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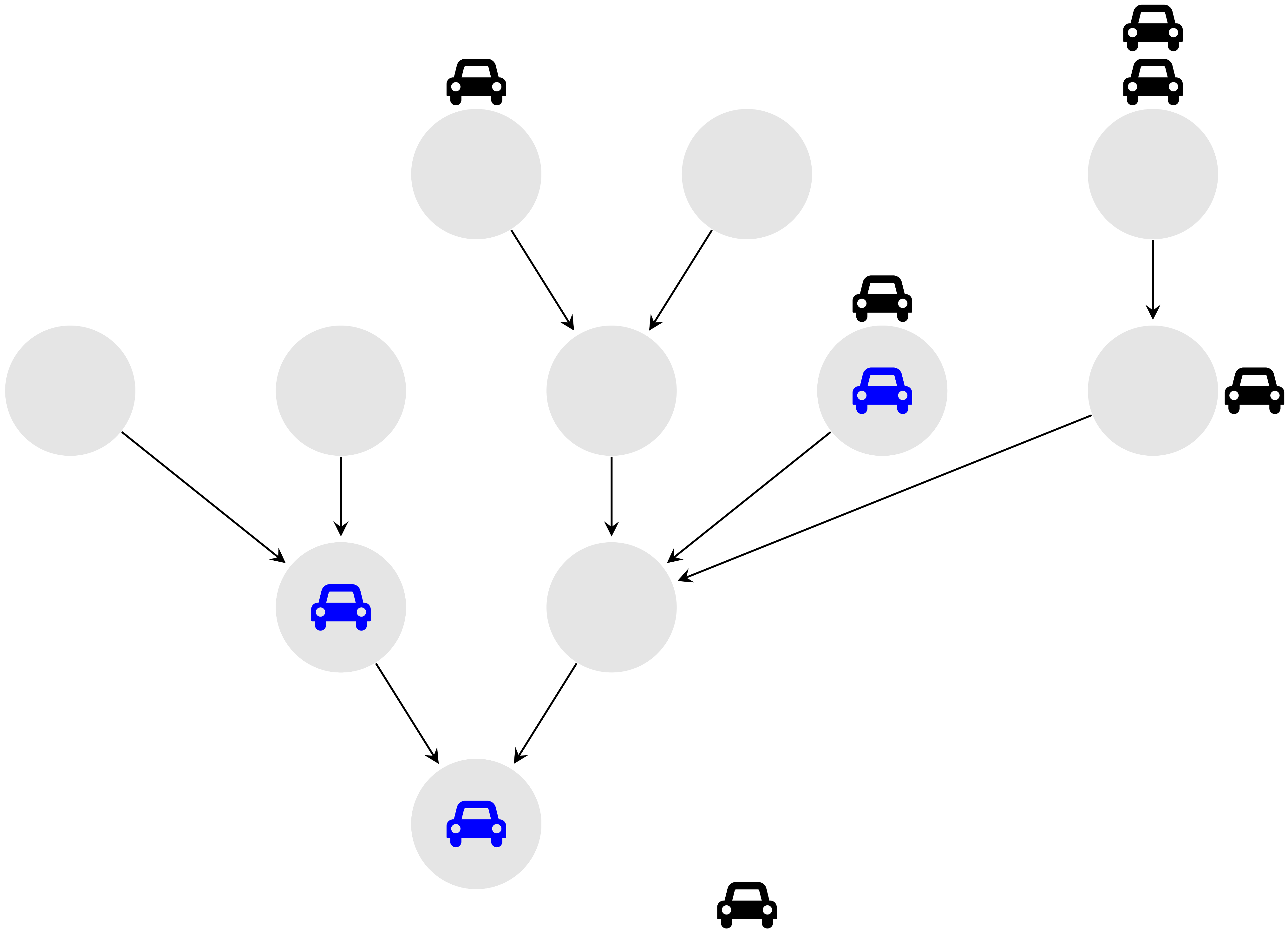

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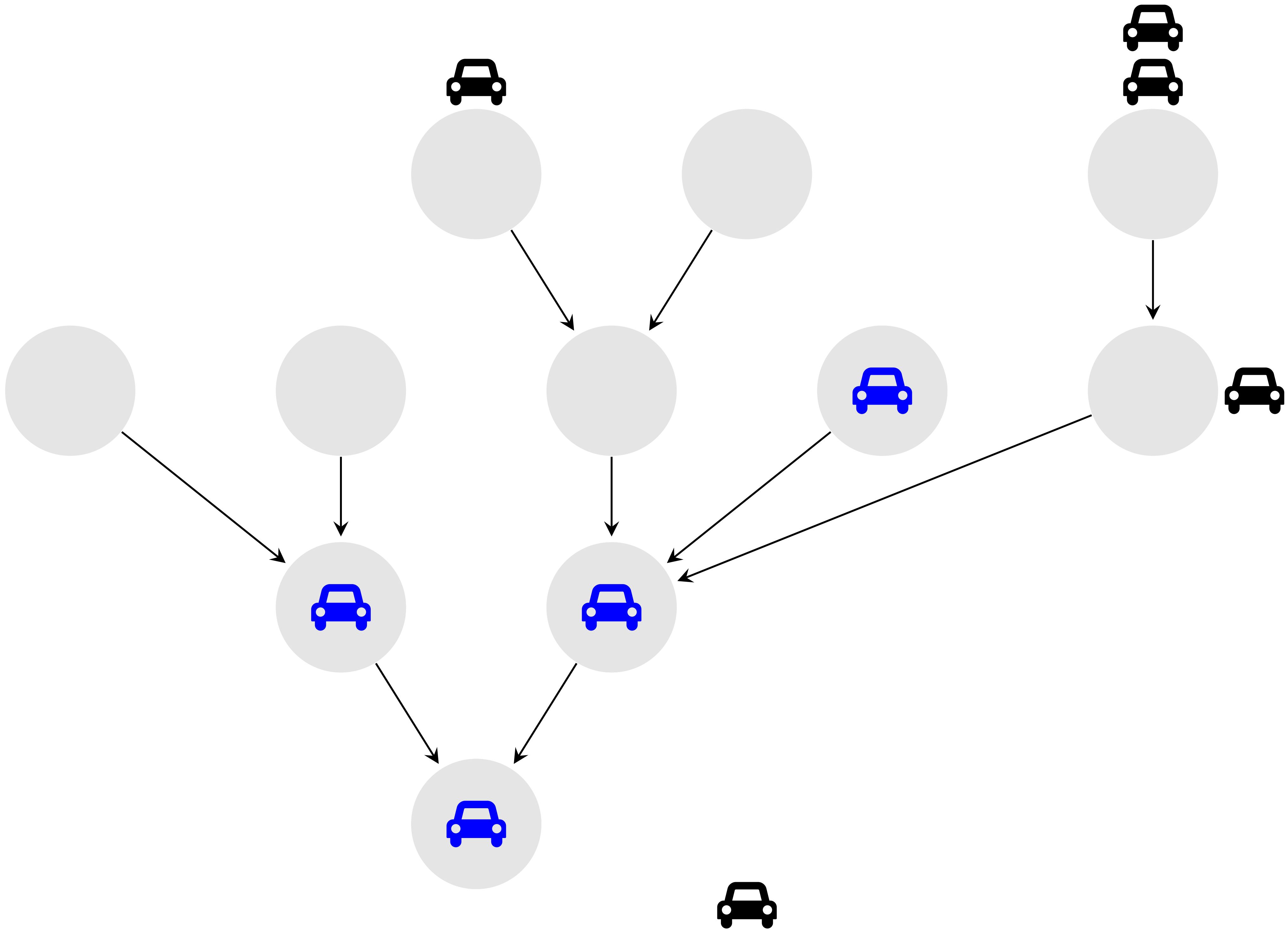

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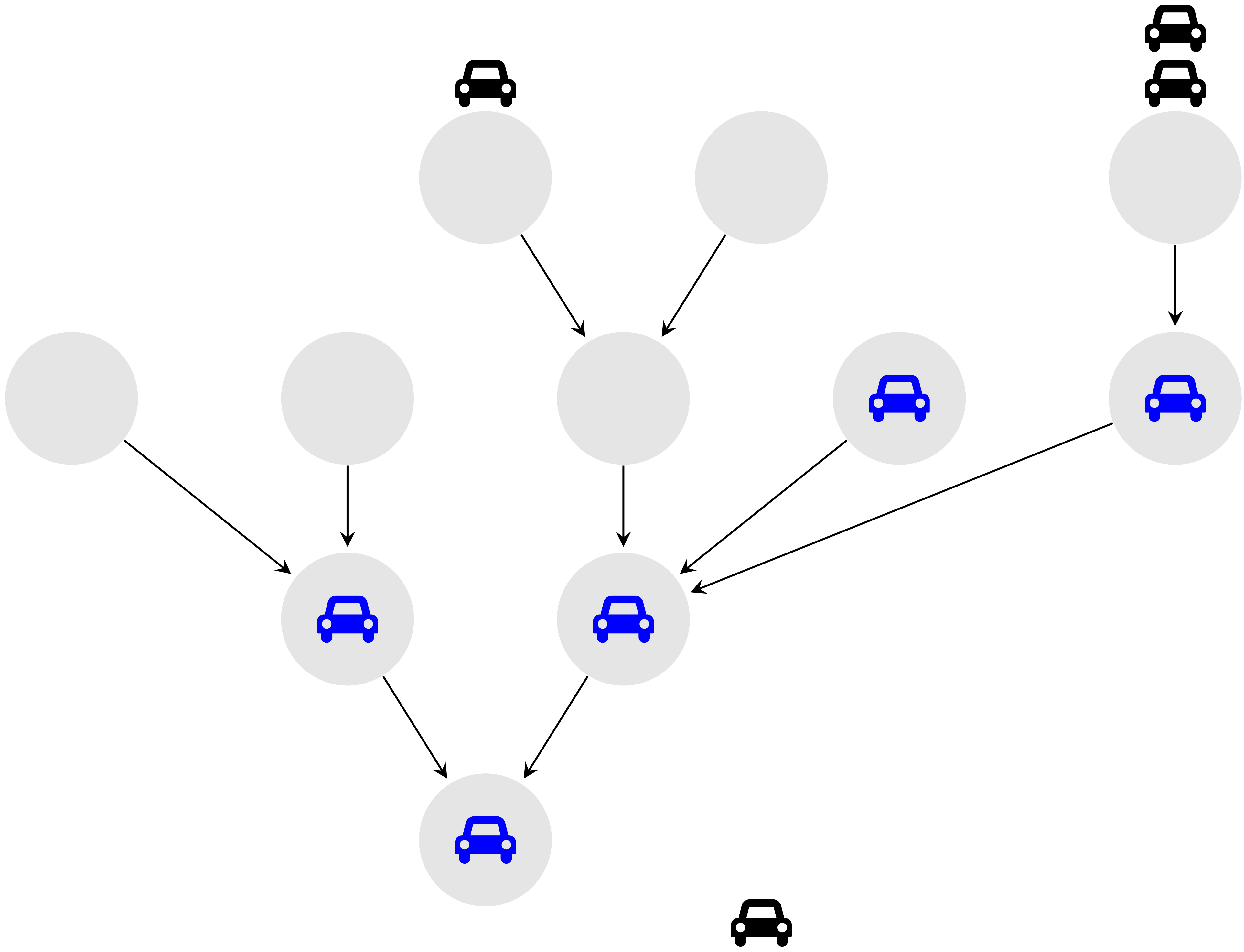

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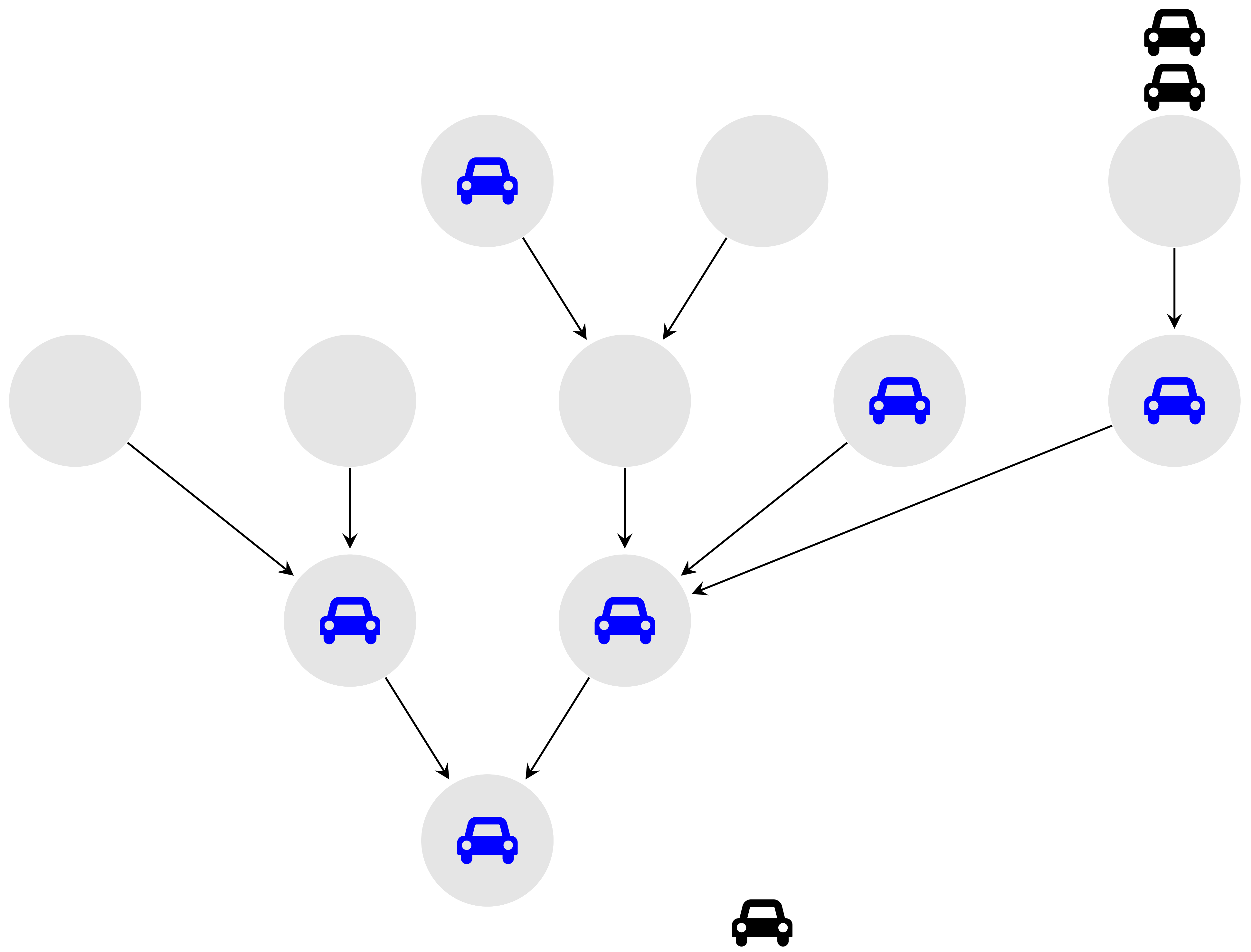

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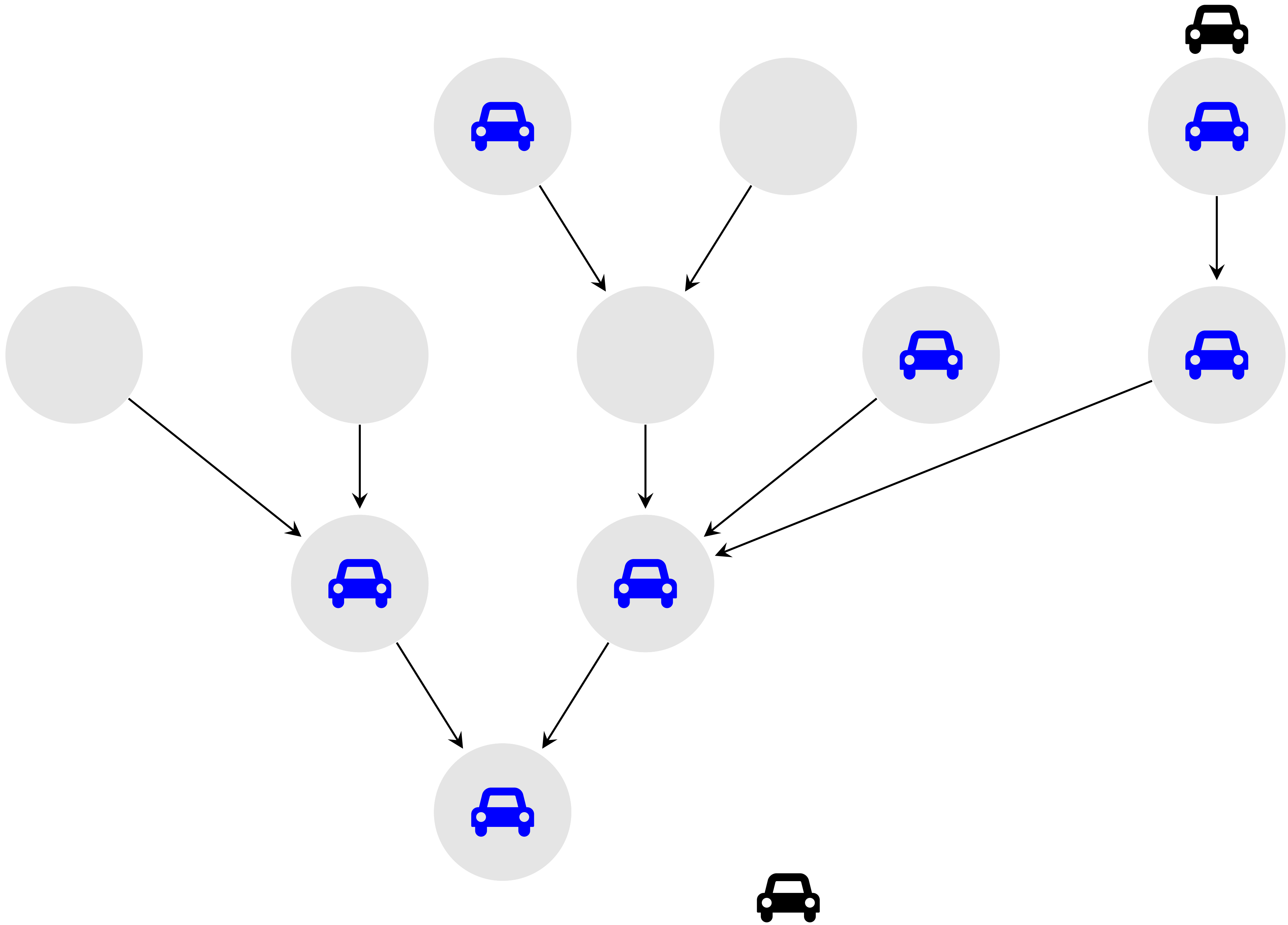

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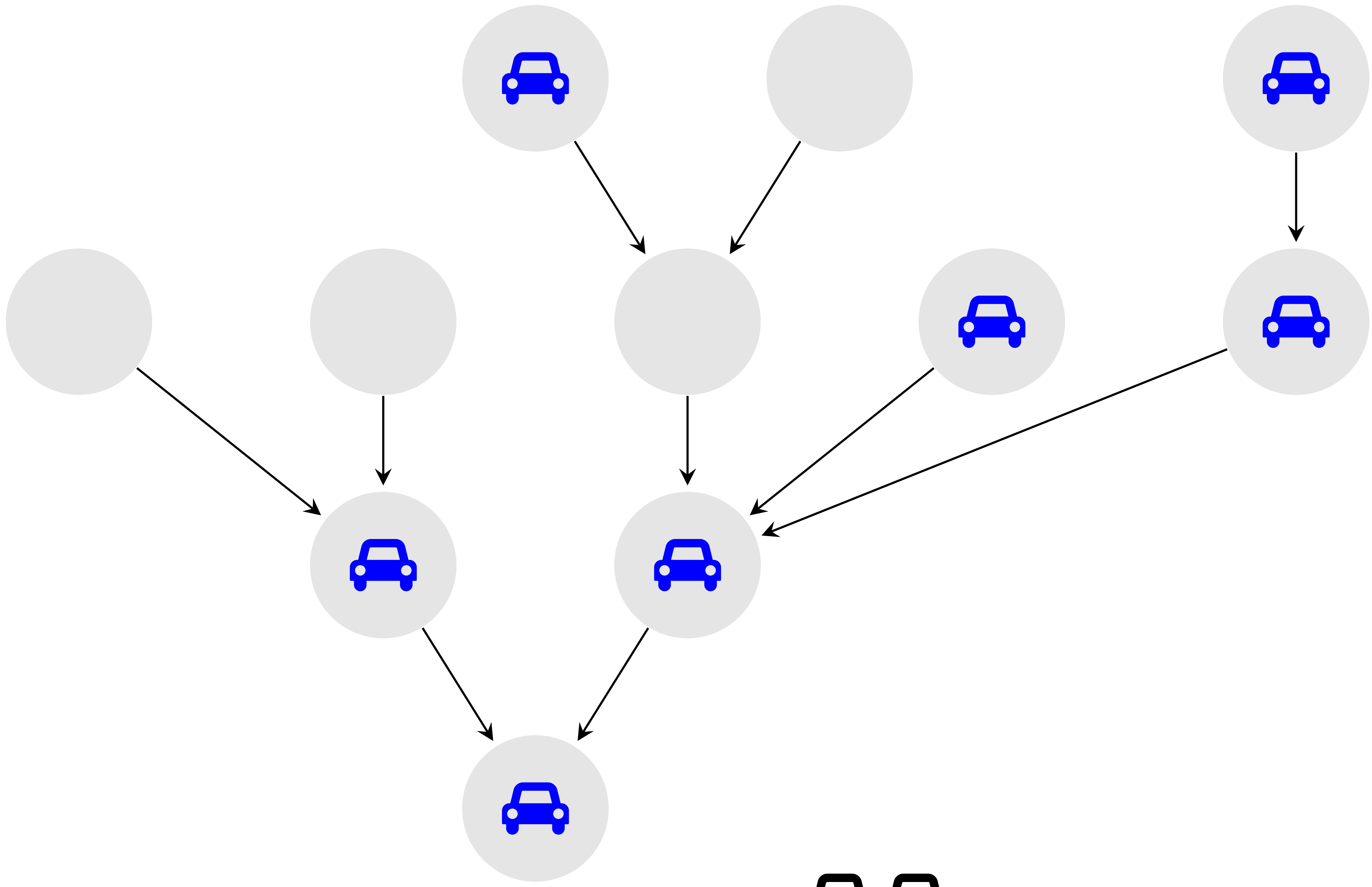

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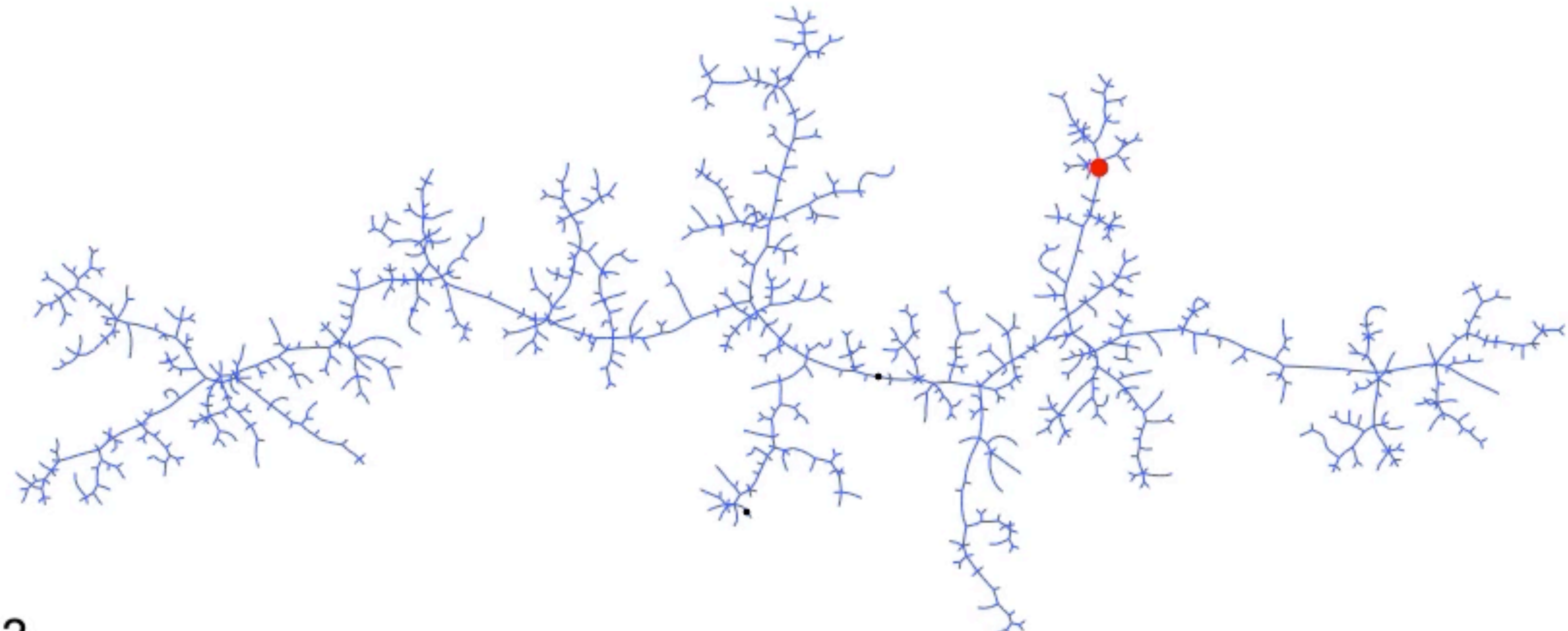
$$\longrightarrow \text{Phase transition}$$



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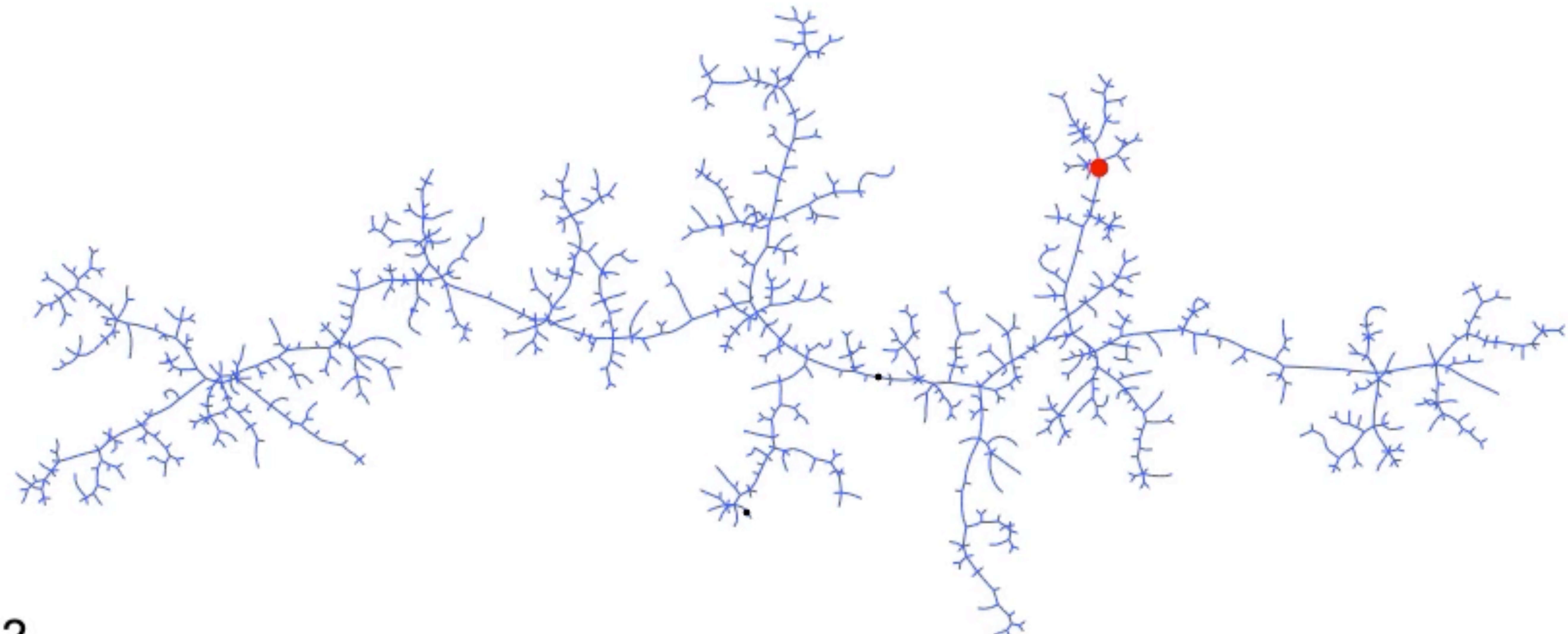

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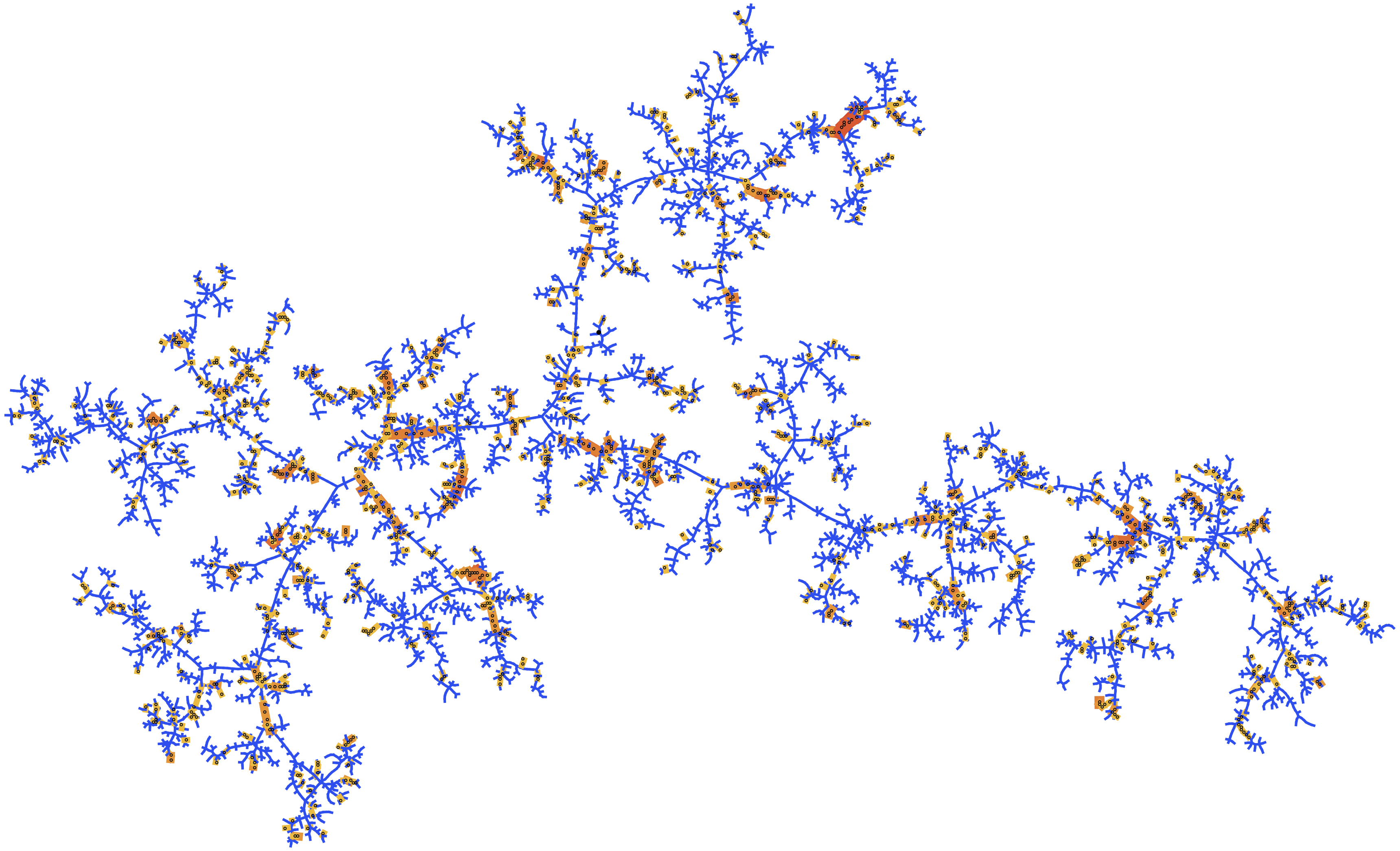
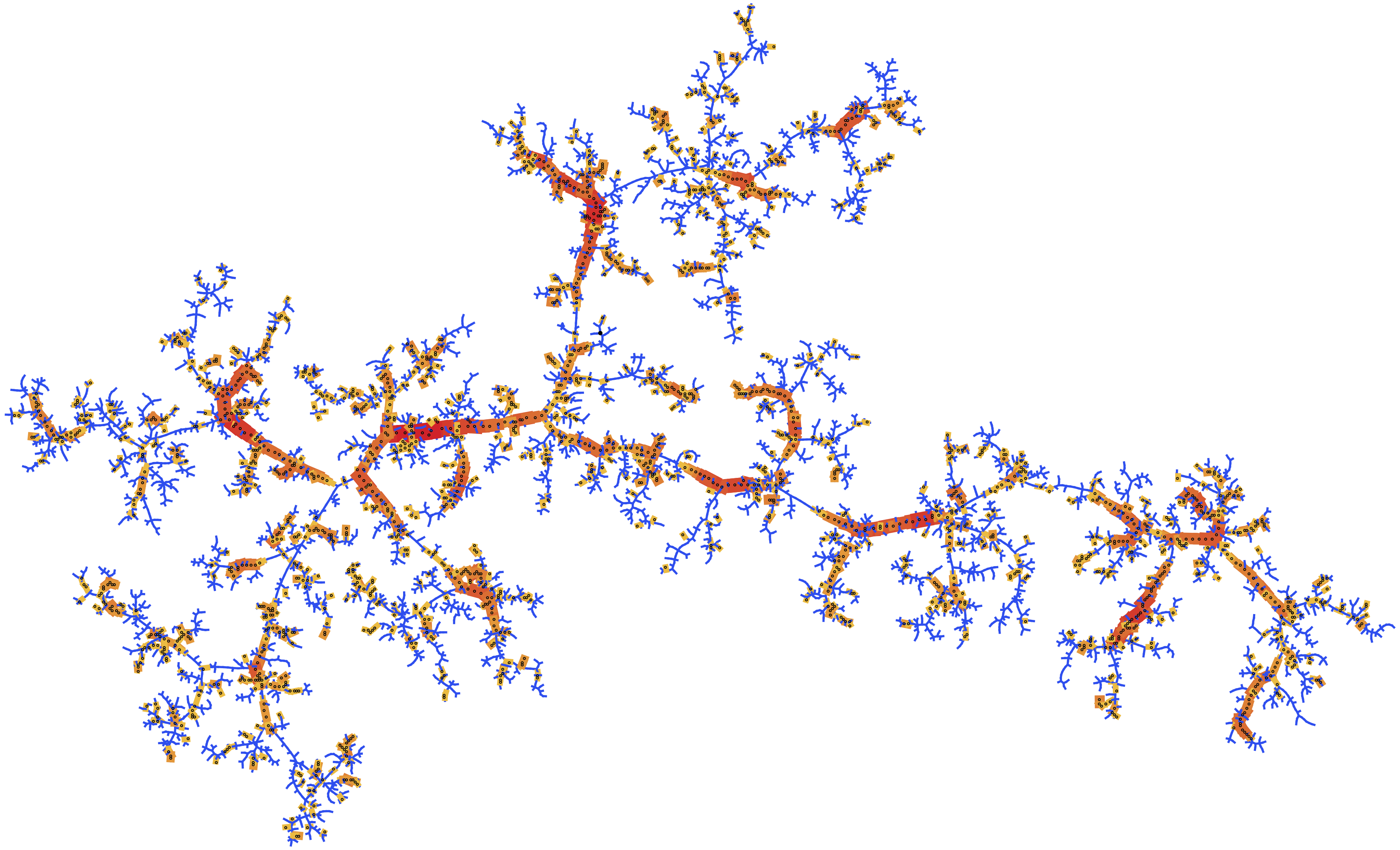
Régime sous-critique



Flux de voitures sortantes $= o_{\mathbb{P}}(n)$

Régime critique



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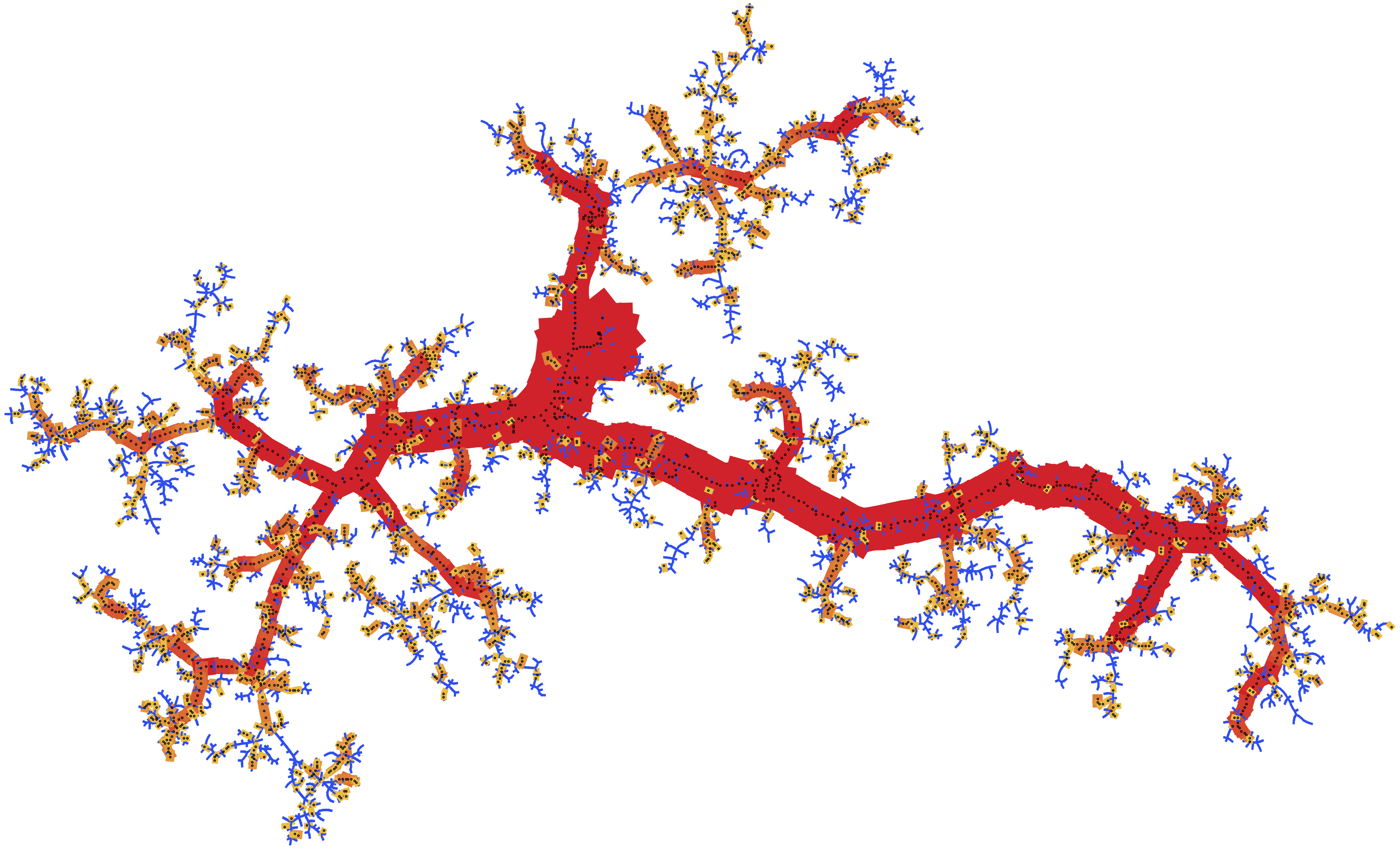

Flux de voitures sortantes $= (c + o_{\mathbb{P}}(1))n$

Phase transition

If t is a deterministic infinite tree (e.g. line or binary tree)

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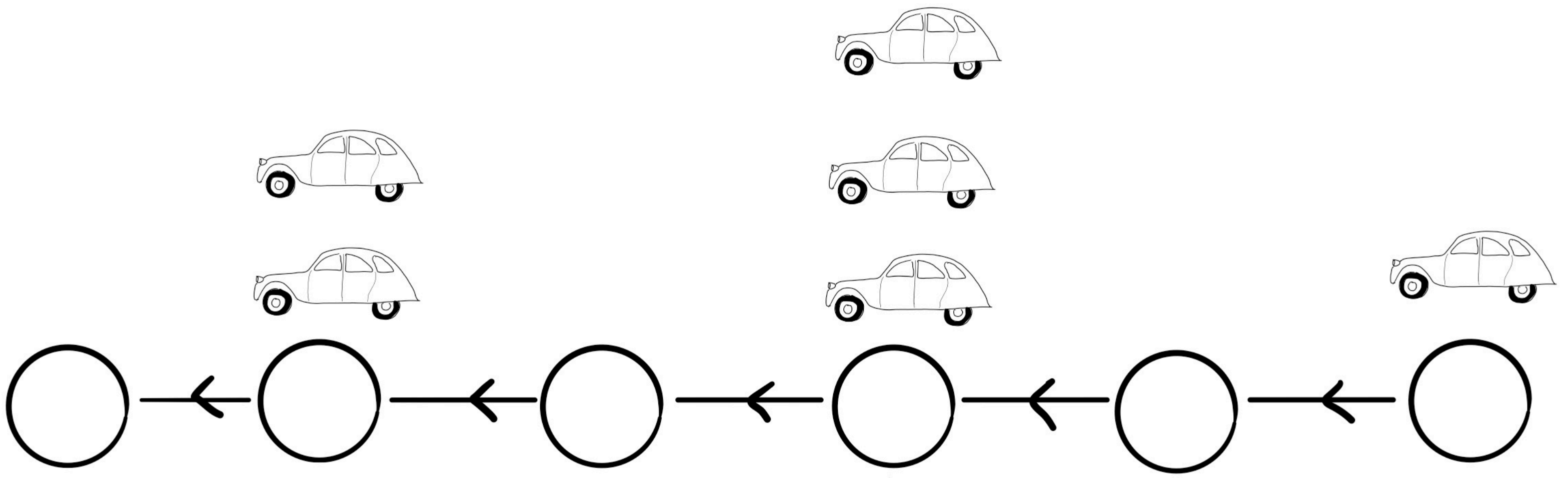
We can define a "phase transition" for finite but large trees (see later).

Case of the line



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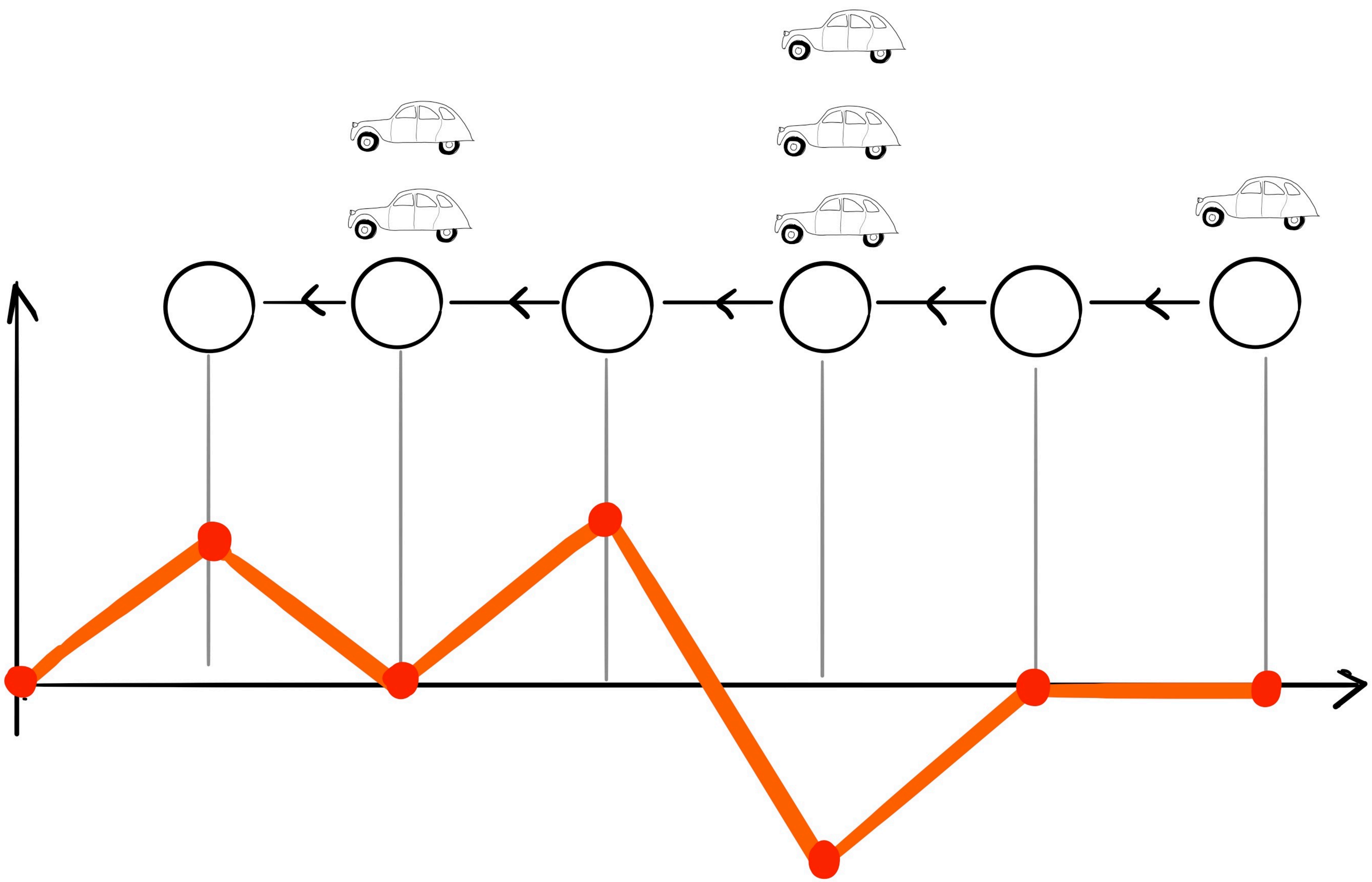

"Trivial" phase transition: always at $\alpha = 1$ whatever the distribution.

And on random trees ?

- Fix $\mathrm{t} = \mathcal{T}_n$ a Bienaymé–Galton–Watson tree conditioned to have n vertices with offspring distribution

$$\nu = \sum_{k=0}^{\infty} \nu_k \delta_k \text{ aperiodic with mean 1 and finite variance } \Sigma^2.$$

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- The car arrivals on each vertex are independent.
- The law of the car arrivals only depends on the degree of the vertex.

Phase transition on critical random trees

Building on [Curien, Hénard 2019]

Theorem (C. 2020)

We observe a phase transition which depends only on

$$\Theta = (1-\alpha)^2 - \Sigma^2(\alpha + \alpha^2 - \sigma^2) \qquad \textit{or} \qquad \Theta = \Theta(\Sigma^2, \dots)$$

	subcritical $\Theta > 0$	*critical* $\Theta = 0$	*supercritical* $\Theta < 0$
$\varphi(\mathcal{T}_n)$ *when* $n \to \infty$	*finite*	$o(n)$	$\sim cn$ *with* $c > 0$
$\mathbb{E}[\varphi(\mathcal{T})]$			
$\lvert C_{\max}(n) \rvert$ *when* $n \to \infty$			

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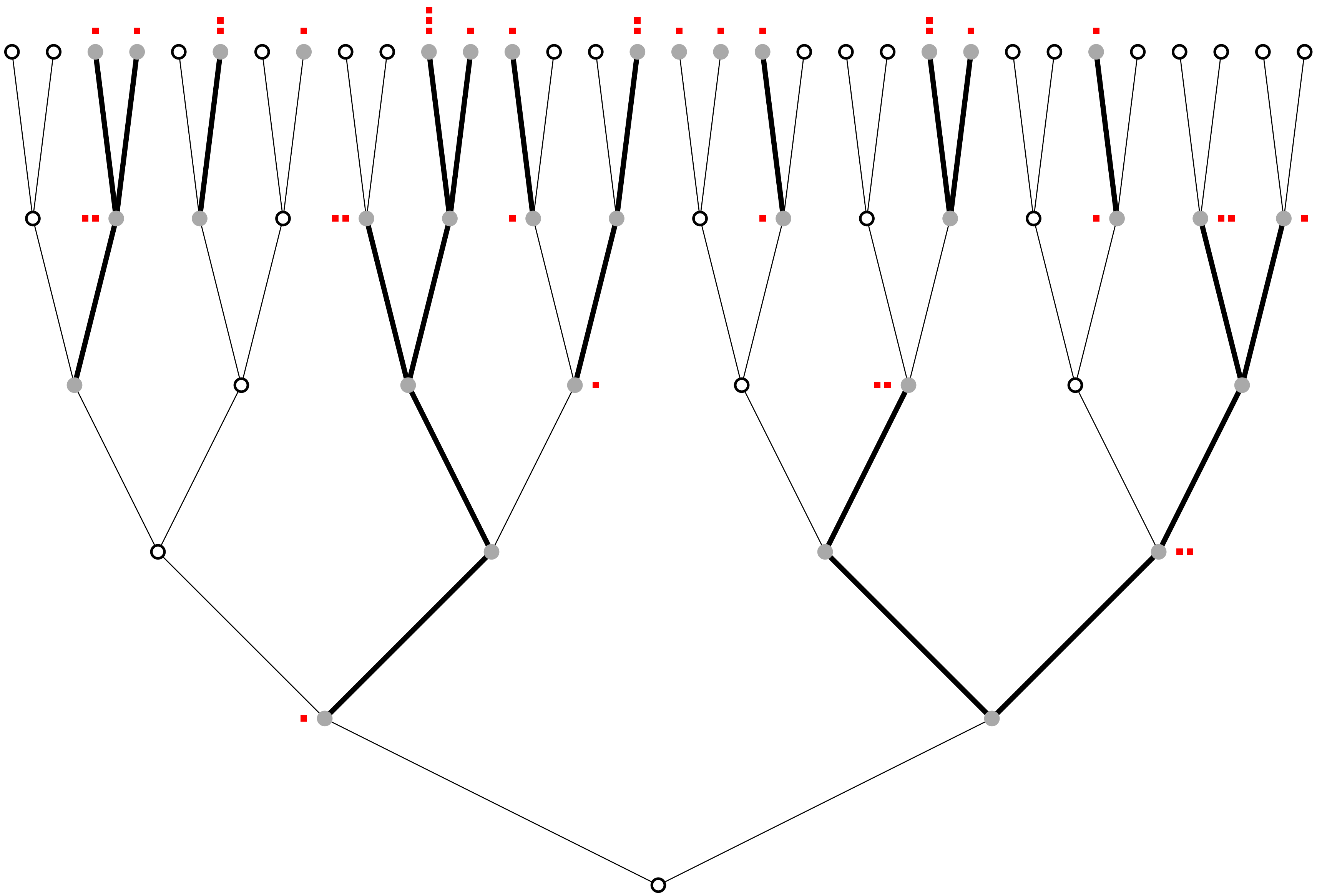
$\lvert C_{\max}(n)\rvert$ *when* $n \to \infty$	$\leq A\log(n)$	?	$\sim Cn$ *avec* $C > 0$

Location of the transition in the binary case



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Take t the infinite binary tree. Let G be the generating function of the law μ of the car arrivals.

Theorem (Aldous, C., Curien, Hénard, 2022)

Suppose there exists

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In the generic situation, the time t_c exists.

Examples

Car arrivals	Critical value α_c
Binary $0/2$ $\mu^\alpha = (1 - \frac{\alpha}{2})\delta_0 + \frac{\alpha}{2}\delta_2$	
Binary $0/k$ $\mu^\alpha = (1 - \frac{\alpha}{k})\delta_0 + \frac{\alpha}{k}\delta_k$	
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Probabilistic consequences

- In the subcritical regime, we need $\mathbb{E}[2^{\#\text{cars}}] < \infty$.
- $\mathbb{E}[X] < \infty$ in the subcritical and critical regime.
- Size of the cluster of parked cars/empty spots ?

Sketch of proof: combinatorial decomposition

- Decomposition of the final configuration into clusters of parked cars

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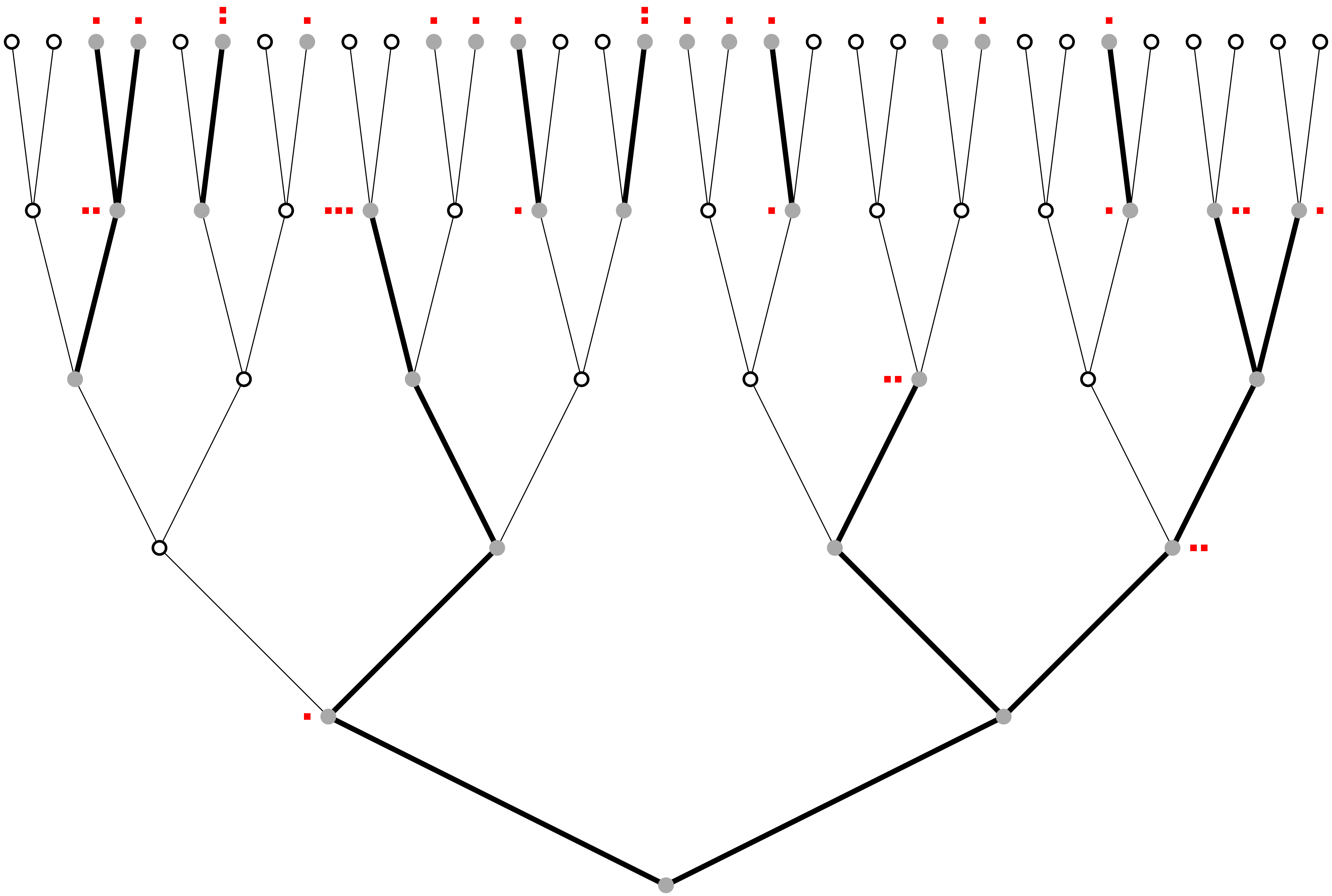
We denote by

$$p_\circ = \mathbb{P}(\text{the root is empty}), \text{ and}$$

$$p_\bullet = \mathbb{P}(X = 0 \text{ and the root is parked}).$$

Decomposition into clusters



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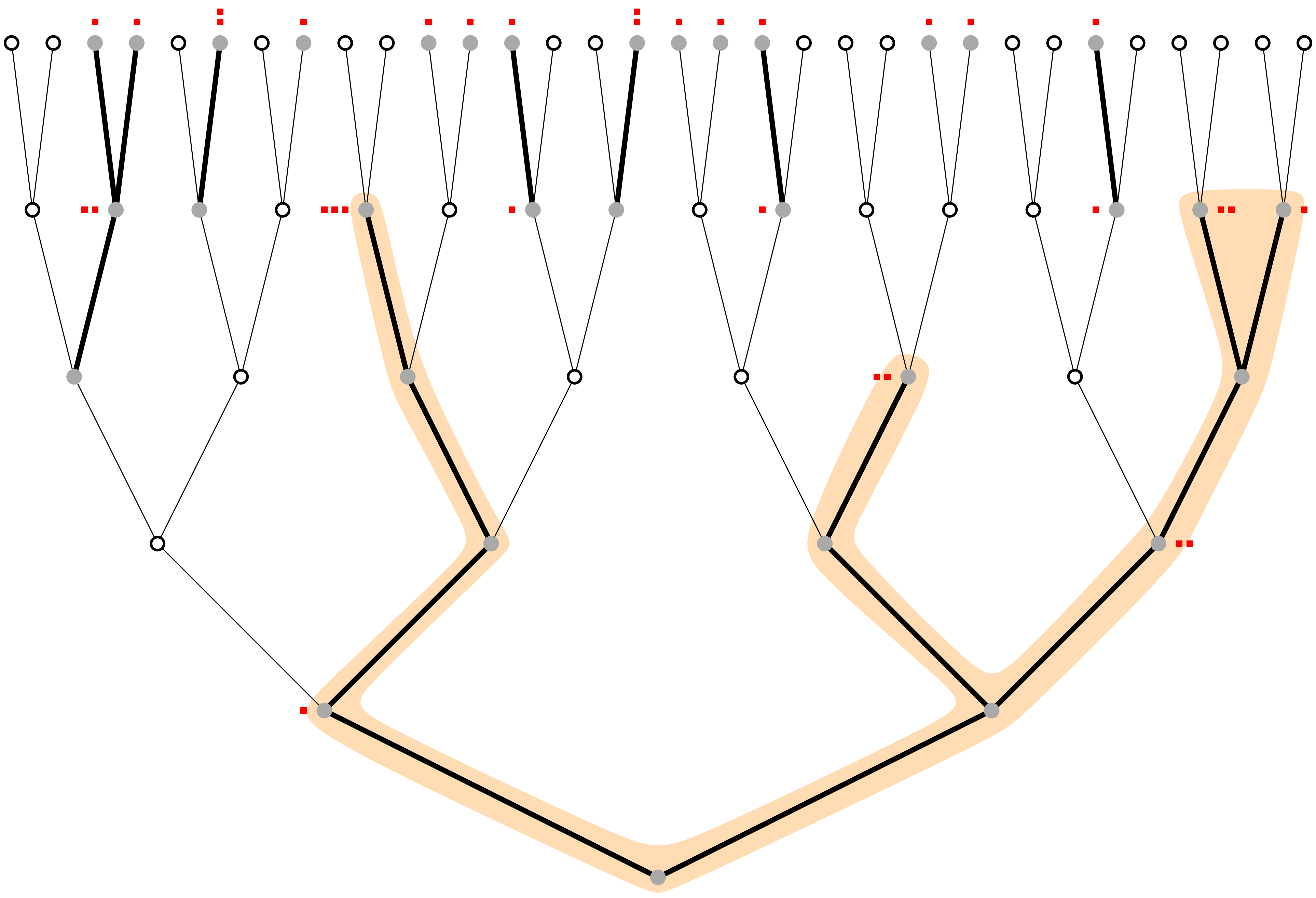

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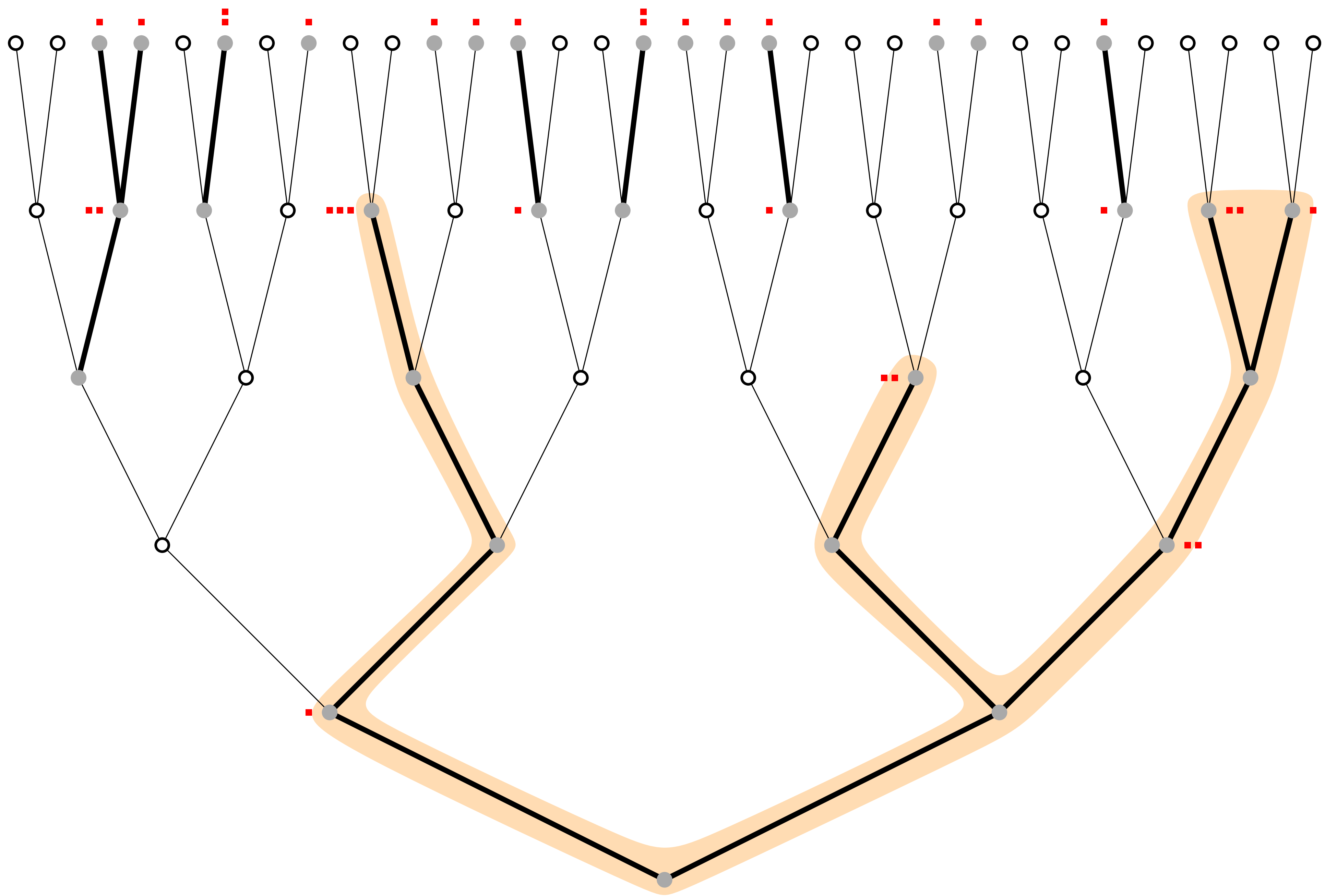

$$\mathbb{P}(X = k) = \sum_{n \geq 1} \sum_{\mathbf{t} \in \mathbb{T}_n^k} w(\mathbf{t}) p_\circ^{n+1}$$

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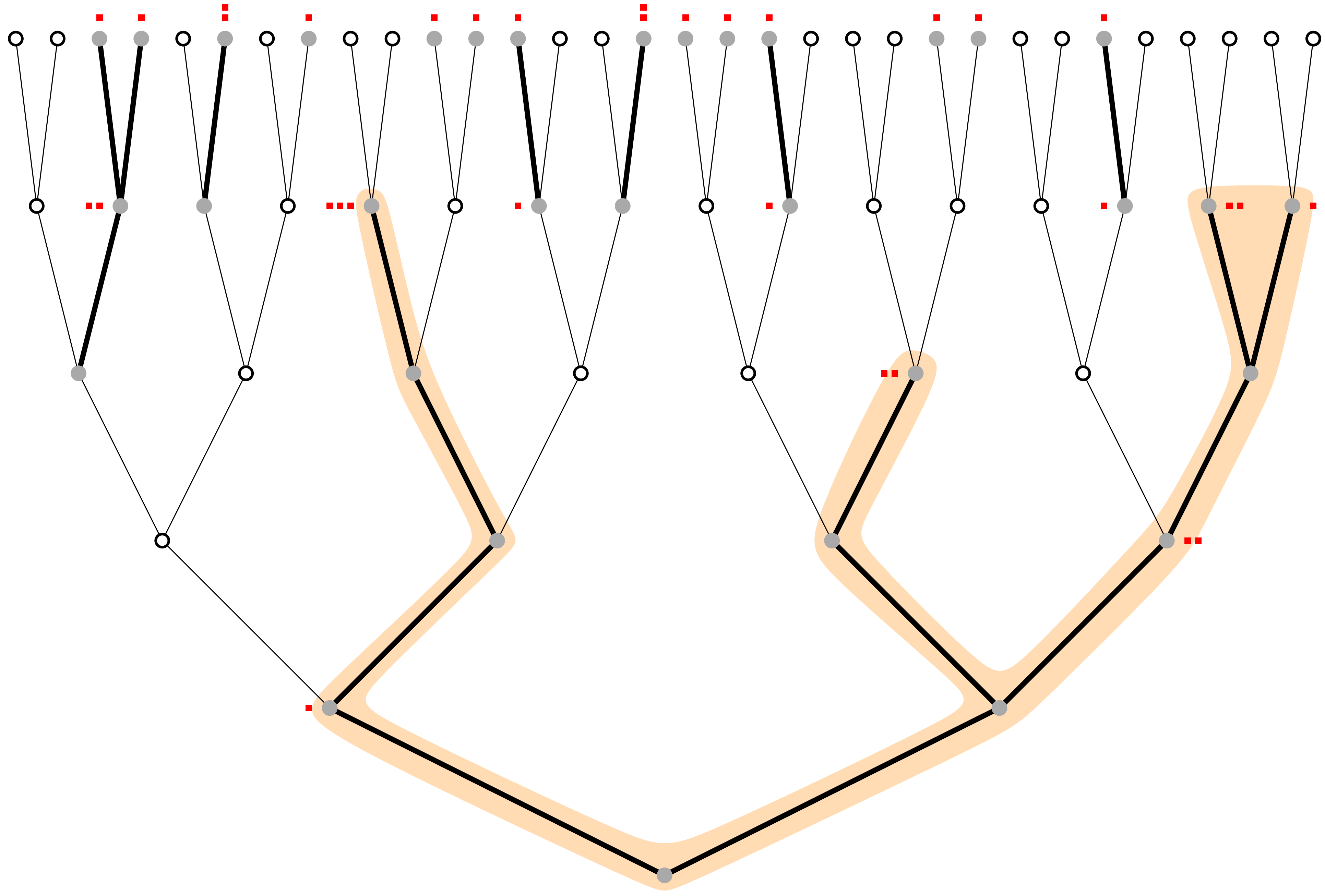

$$p_\bullet = \sum_{n\geq 1} \sum_{\mathbf{t}\in\mathbb{T}_n^1} w(\mathbf{t}) p_\circ^{n+1}$$

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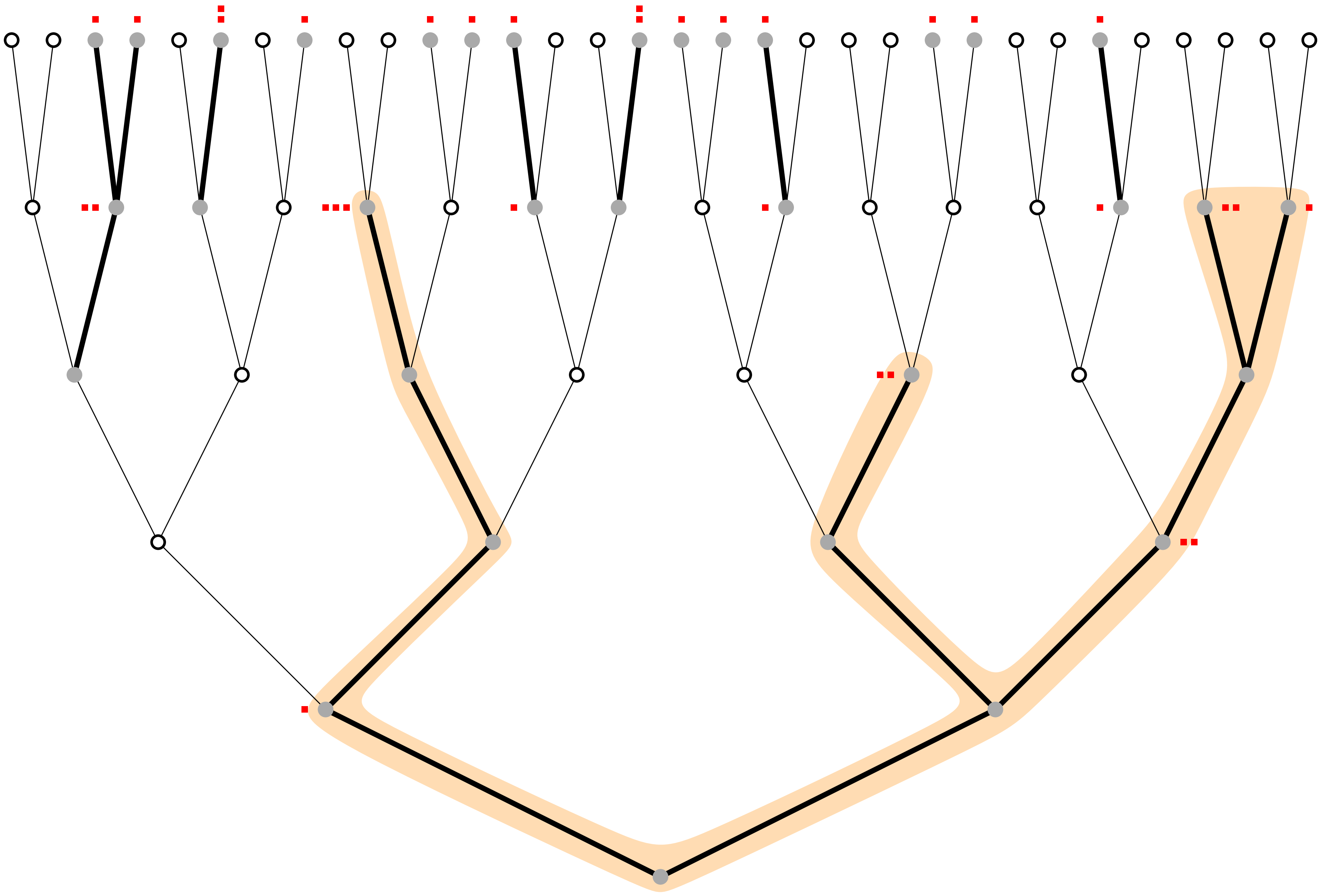

$$p_\circ = \mu_0(p_\circ + p_\bullet)^2$$

Characterization of the subcritical regime

$$F(x, y) = \sum_{n \geq 1} \sum_{p \geq 0} \sum_{\mathbf{t} \in \mathbb{T}_n^p} w(\mathbf{t}) x^n y^p$$

The parking process is subcritical iff there exists a positive solution to

$$1 = \mu_0 x (1 + F(x, 0))^2$$

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The parking process is subcritical iff at x_c radius of convergence of F

$$1 \leq \mu_0 x_c (1 + F(x_c, 0))^2$$

Enumeration of *FPT* : decomposition "à la Tutte"

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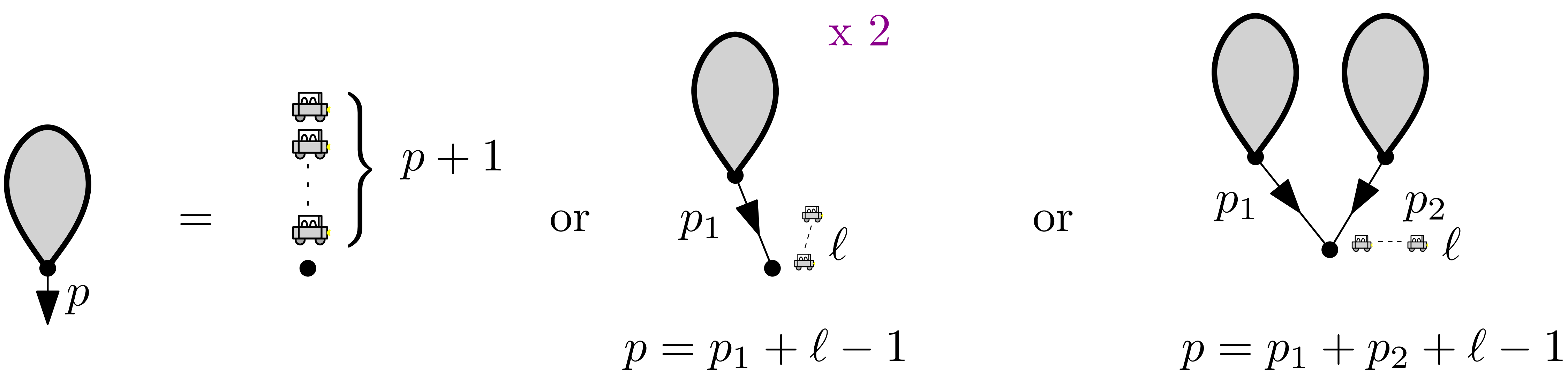

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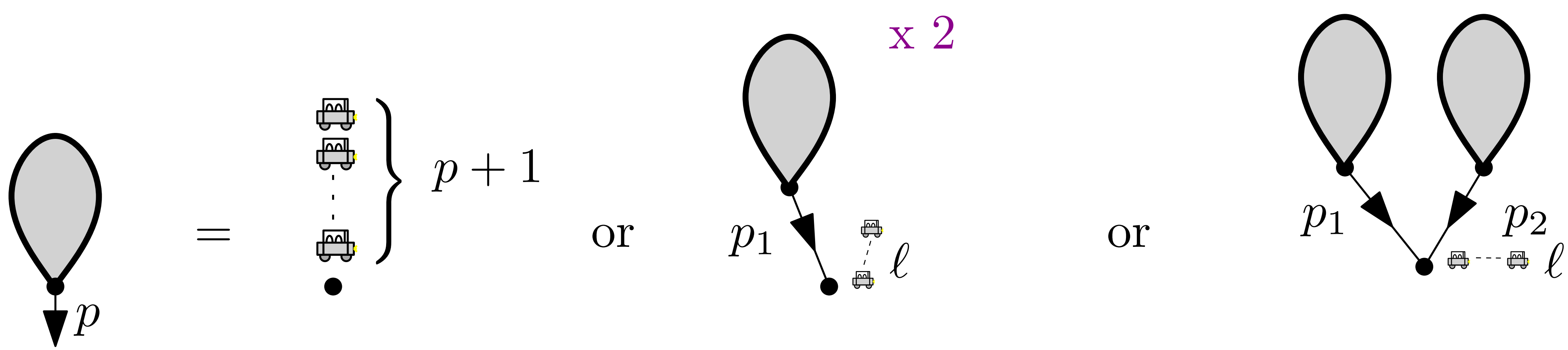

$p = p_1 + \ell - 1$ $\qquad$ $p = p_1 + p_2 + \ell - 1$

$$F(x,y) = \frac{x}{y}G(y) + 2\frac{x}{y}F(x,y)G(y) + \frac{x}{y}F(x,y)^2G(y)$$
$$- \frac{x}{y}G(0) - 2\frac{x}{y}F(x,0)G(0) - \frac{x}{y}F(x,0)^2G(0)$$

Solving the equation

Tutte's equation can be written in the form

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Key: Find $Y = Y(x)$ such that

$$\partial_f P(\mathsf{F}(x, Y(x)), \mathsf{F}_0(x), x, Y(x)) = 0,$$

since we will also get

$$\partial_y P(\mathsf{F}(x, Y(x)), \mathsf{F}_0(x), x, Y(x)) = 0.$$

We get 3 equations :

$$\begin{cases} Y - 2x\mathsf{F}G(Y) = 0, \\ 1 + xG'(Y)\mathsf{F}^2 = \mathsf{F}, \\ Y + xG(Y)\mathsf{F}^2 = Y\mathsf{F} + xG(0)\mathsf{F}_0^2 \end{cases}$$

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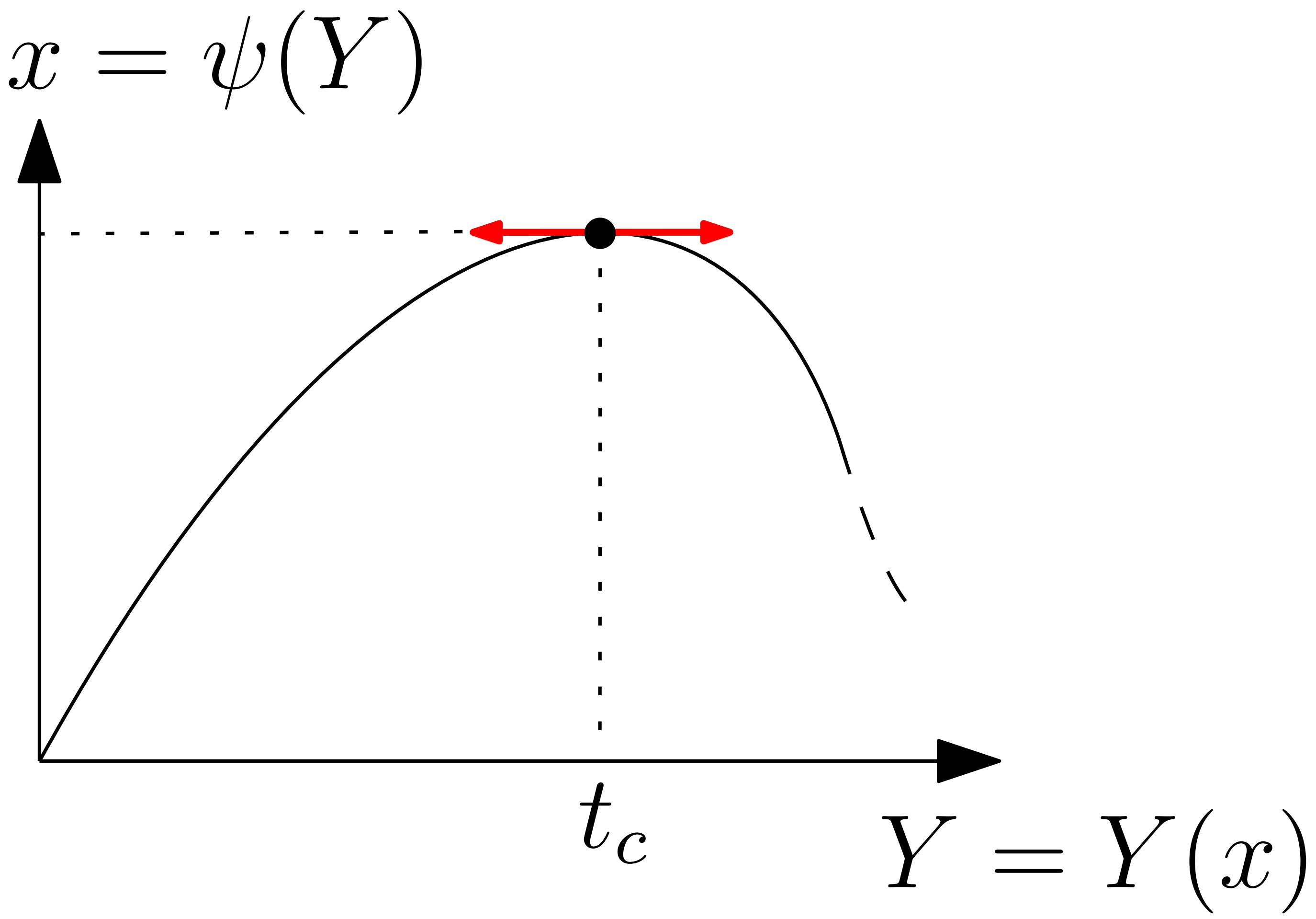
We obtain

$$x = \frac{Y(2G(Y) - YG'(Y))}{4G(Y)^2} \text{ and } \mathsf{F}_0(x) = \frac{2G(Y)\sqrt{G(Y) - YG'(Y)}}{(2G(Y) - YG'(Y))\sqrt{G(0)}}$$



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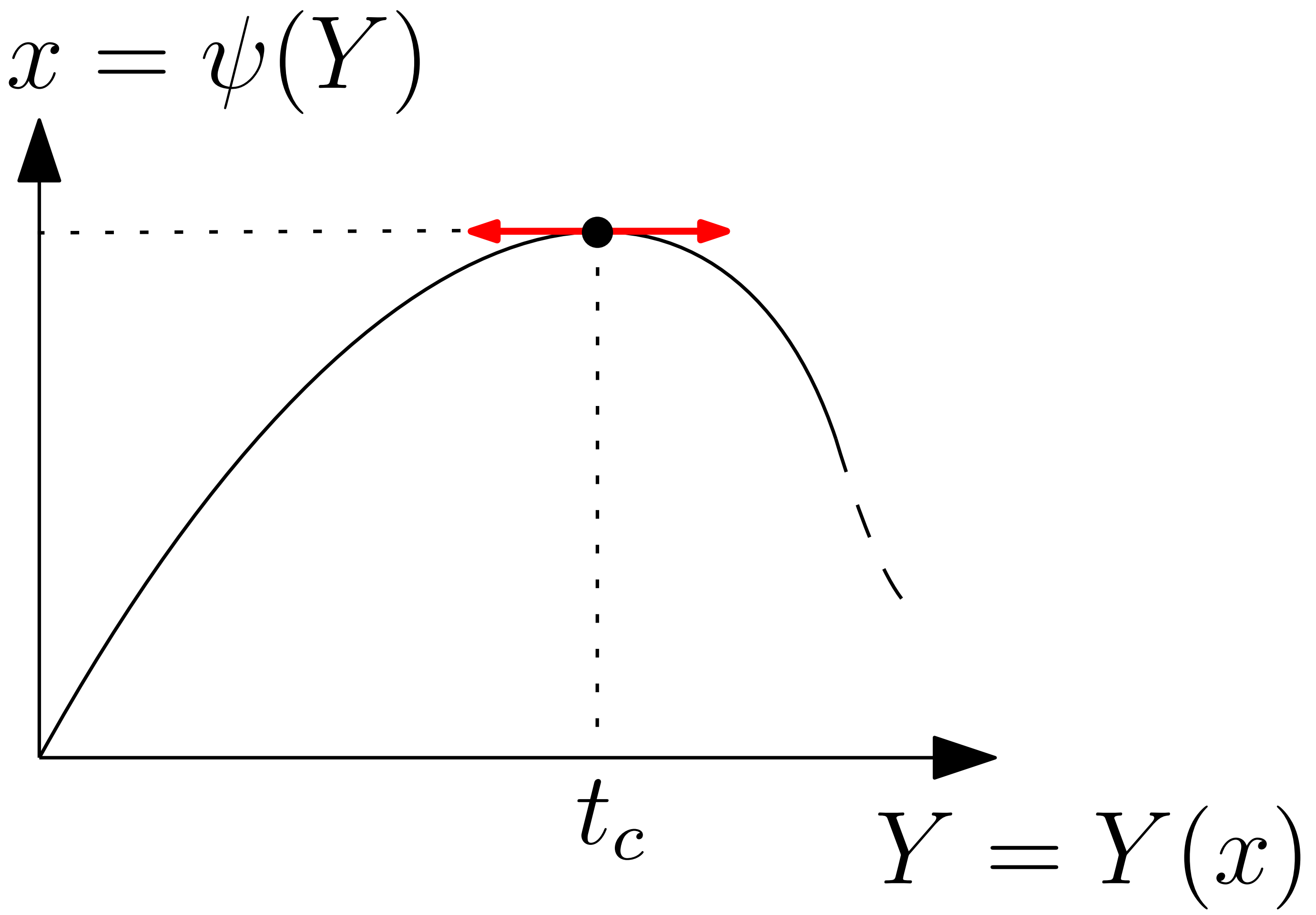

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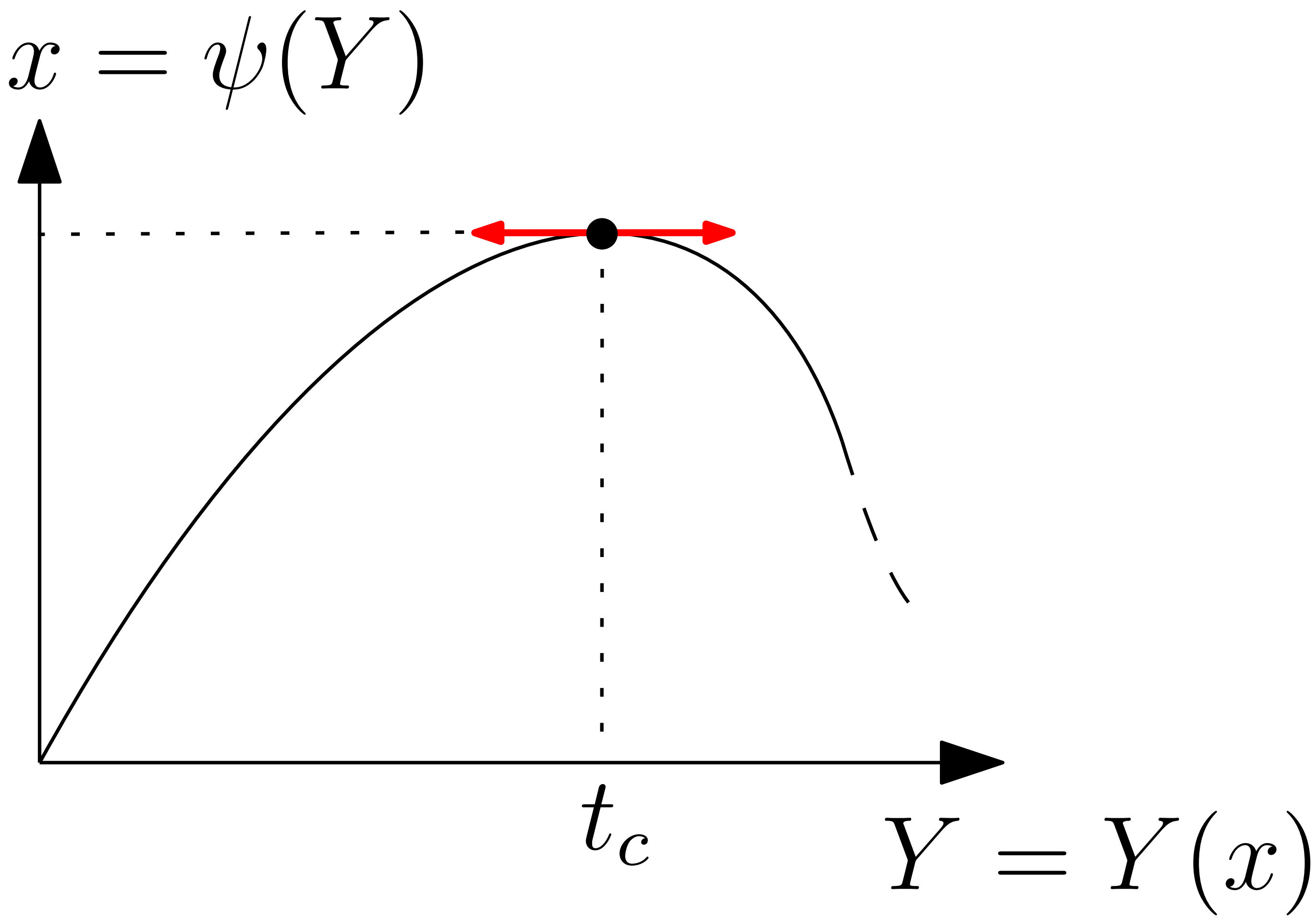

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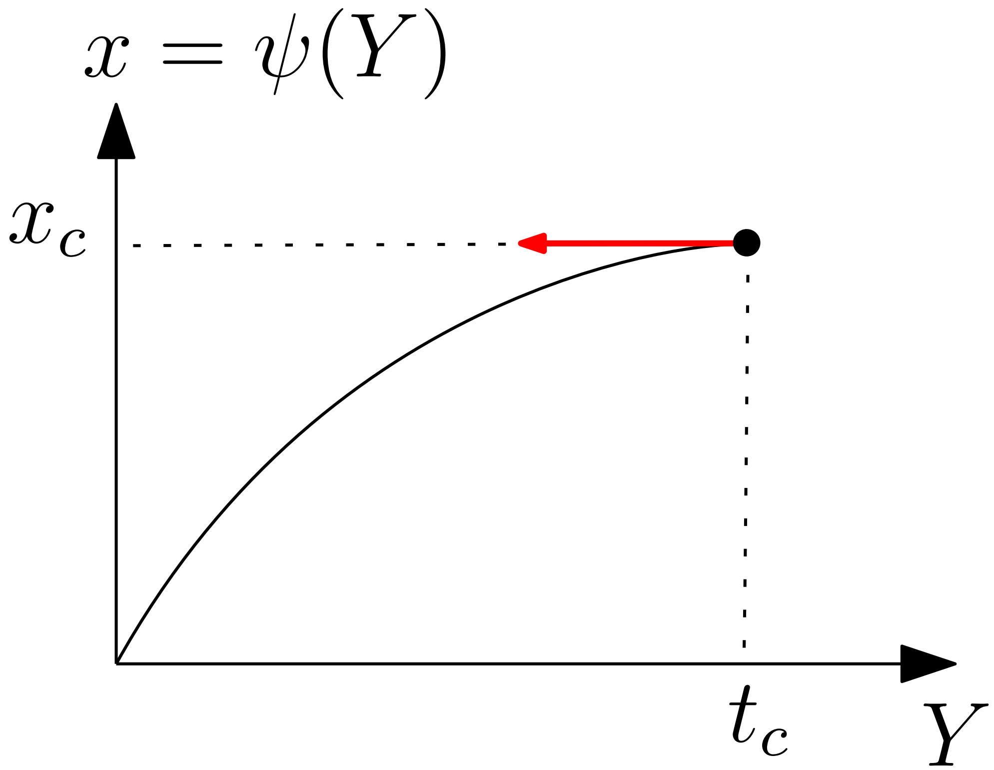
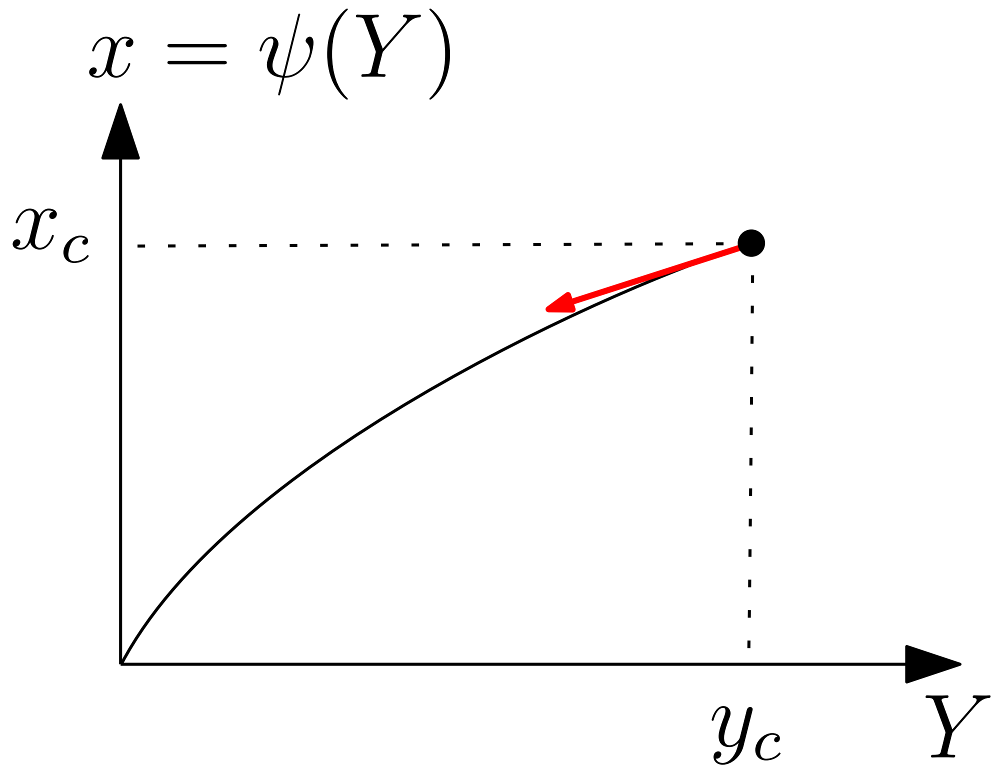
$$1 \leq \mu_0 x_c (1 + F(x_c, 0))^2 \Leftrightarrow (t_c - 2) G(t_c) \geq t_c (t_c - 1) G'(t_c).$$

Non generic case?

dilute non-generic



dense



Bonus:

Supercritical Bienaymé–Galton–Watson trees with geometric offspring distribution

- Consider a Bienaymé—Galton—Watson tree $\mathscr{T}$ with geometric offspring distribution

$$\nu_q = \sum_{k=0}^{+\infty} q^k(1-q)\delta_k$$

with $q > 1/2$.

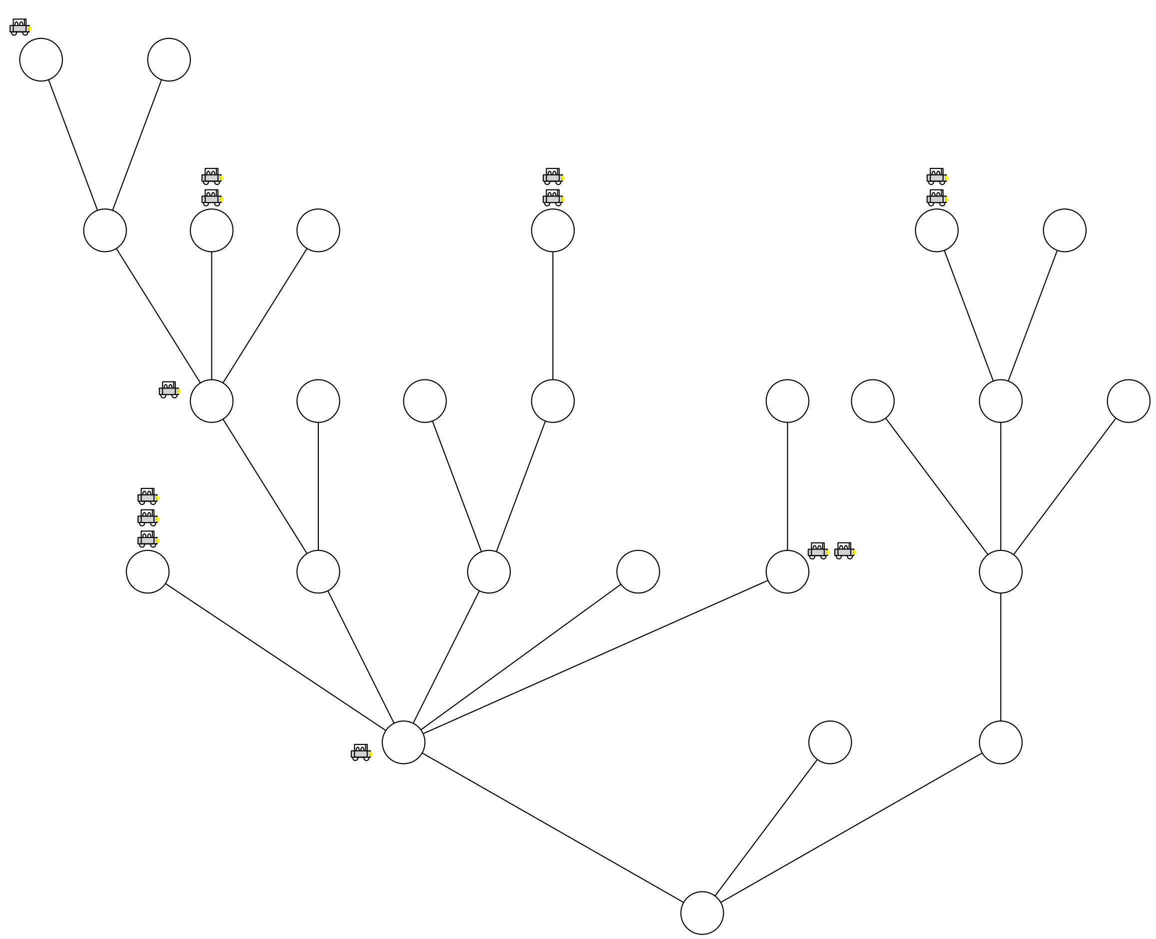
- Again, we denote by X the number of outgoing cars.

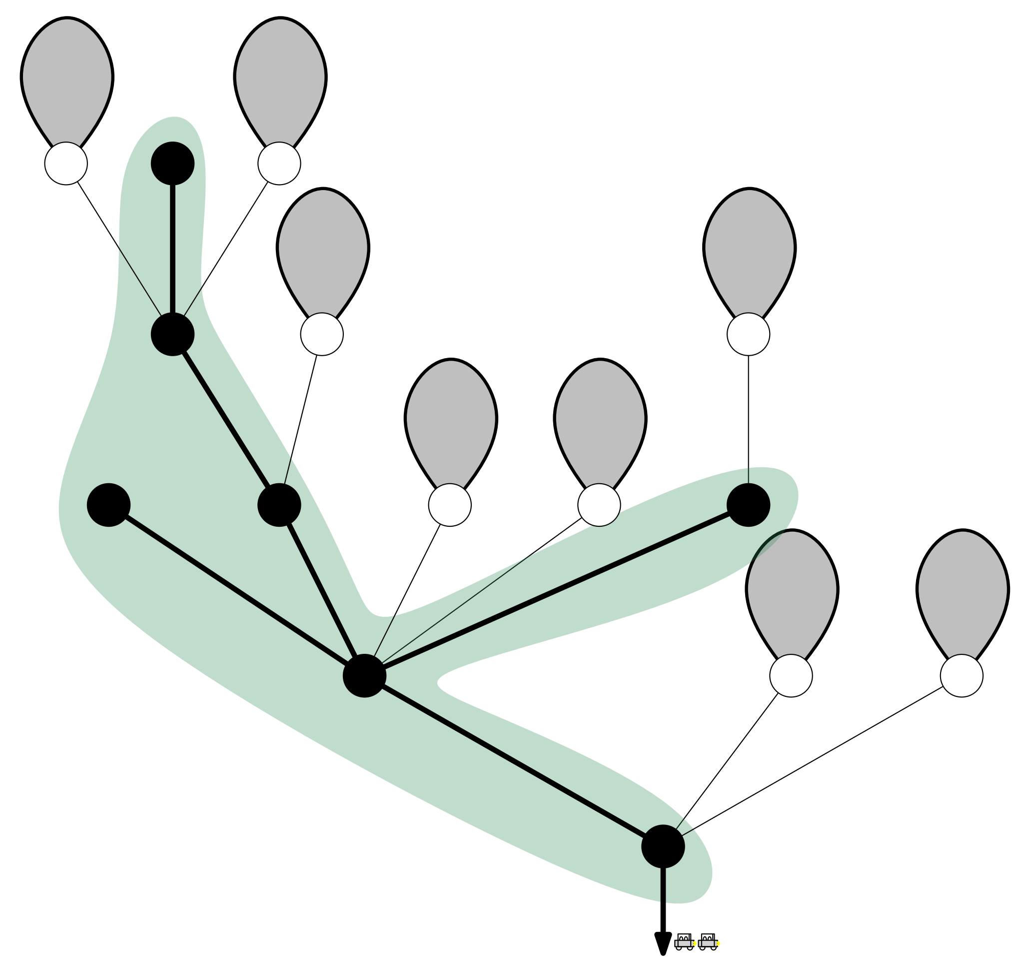
  - **Subcritical :** X is almost surely finite.
  - **Supercritical :** X is infinite as soon an $\mathscr{T}$ is infinite.

- Similarly, we obtain a collection of equations.

$$p_\circ = \frac{(1-q)G(0)}{1-q(p_\circ + p_\bullet)}$$

$$\forall k \geq 0, \quad \mathbb{P}\left(X = k+1\right) = \frac{1-qp_\circ}{q}[y^k]F\left(\frac{q(1-q)}{(1-qp_\circ)^2}, y\right)$$

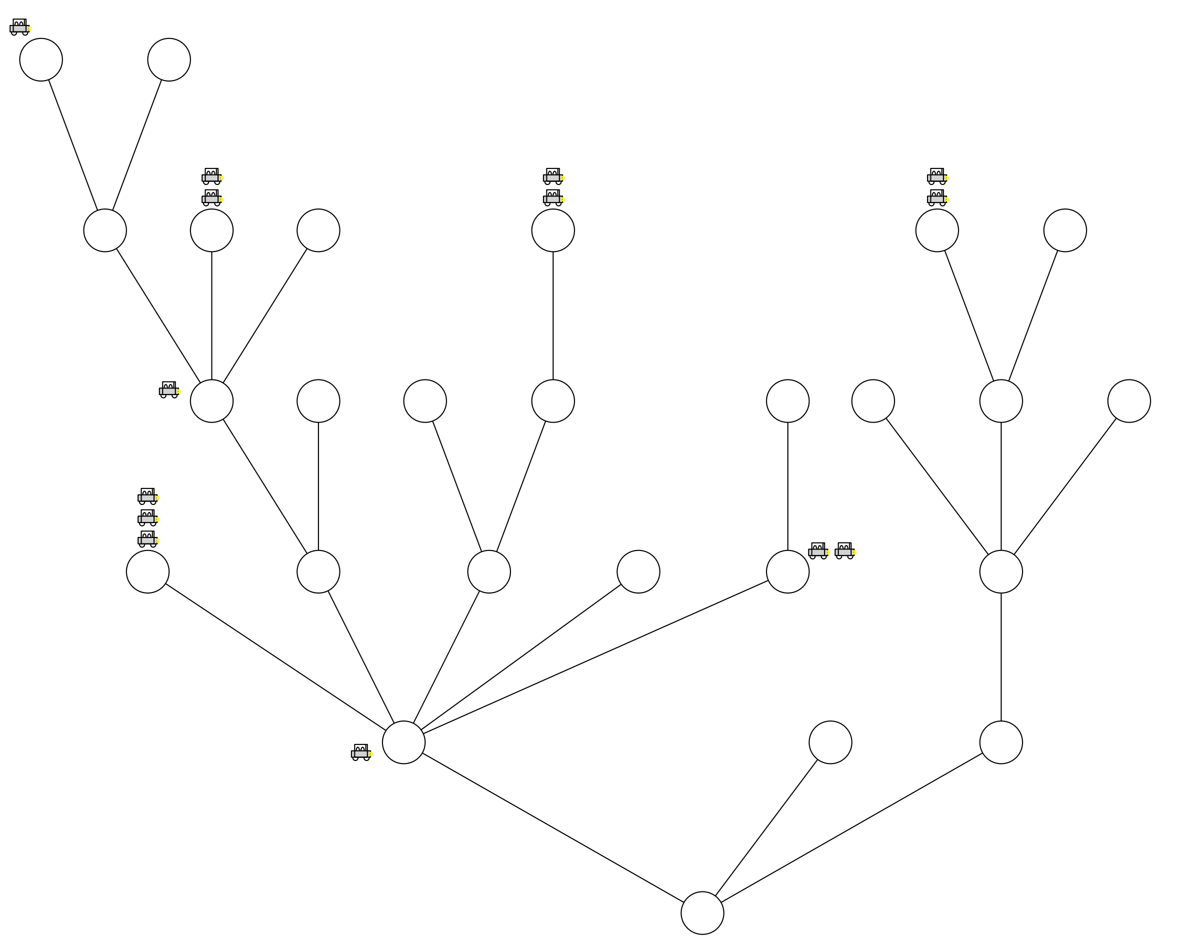


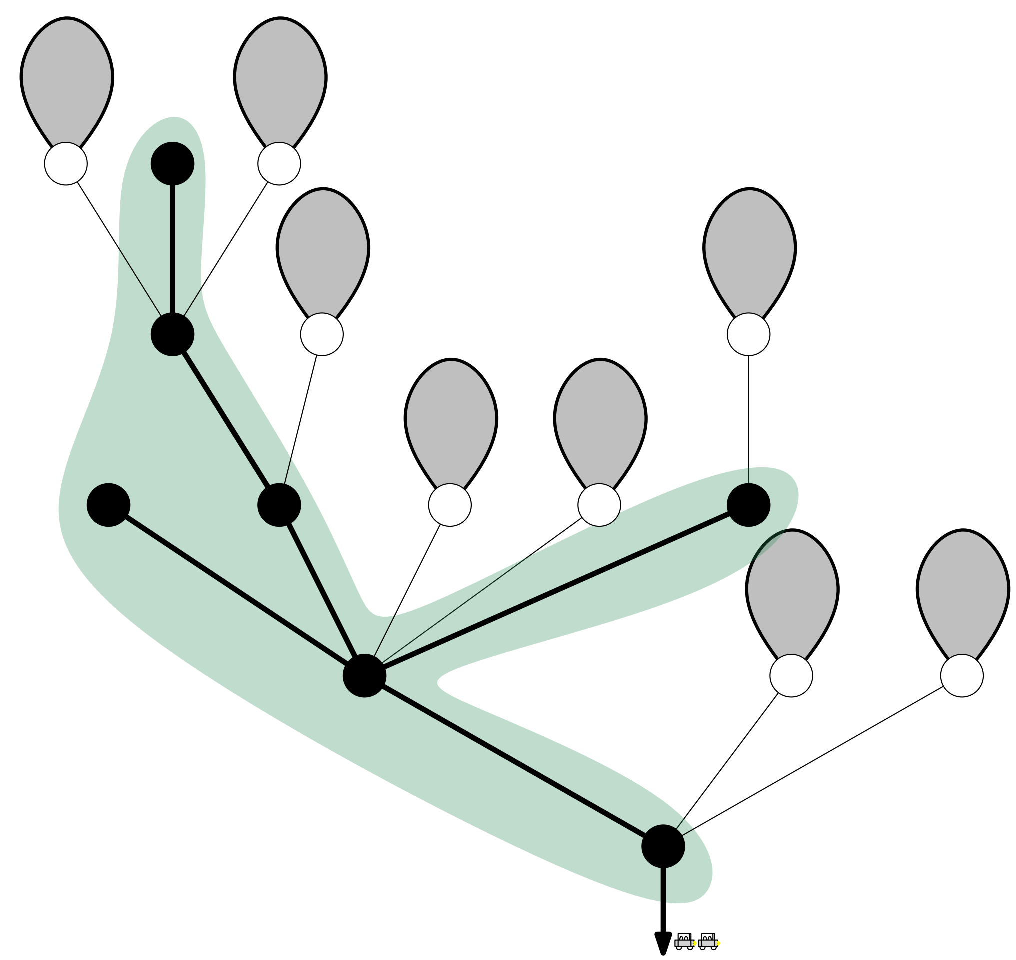


- Similarly, we obtain a collection of equations.

$$p_\circ = \frac{(1-q)G(0)}{1-q(p_\circ + p_\bullet)}$$

$$p_\bullet = \frac{1-qp_\circ}{q} F\left(\frac{q(1-q)}{(1-qp_\circ)^2}, 0\right)$$





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- Solution: in the subcritical regime, we have $\mathbb{P}(X < \infty) = 1$

The parking process is subcritical if and only if there exists a positive solution $p_\circ$ to the equation

$$\frac{1 - qp}{q} \cdot F\left(\frac{q(1-q)}{(1-qp)^2}, 1\right) + p = 1$$

Theorem (Chen, C., 2024)

Suppose that there exists t_c such that

$$t_c := \inf\{t > 0, (G(t) - tG'(t))^2 = 2t^2 G(t)G''(t)\}.$$

Then the parking process is subcritical if and only if

$$t_c > 1 \quad \text{and} \quad \frac{t_c G(t_c)}{\varphi(t_c)^2} \leq q(1-q),$$

where $\varphi(y) = (y+1)G(y) - y(y-1)G'(y)$.

Thank you for your attention !