PROFILE OF RANDOM TREES

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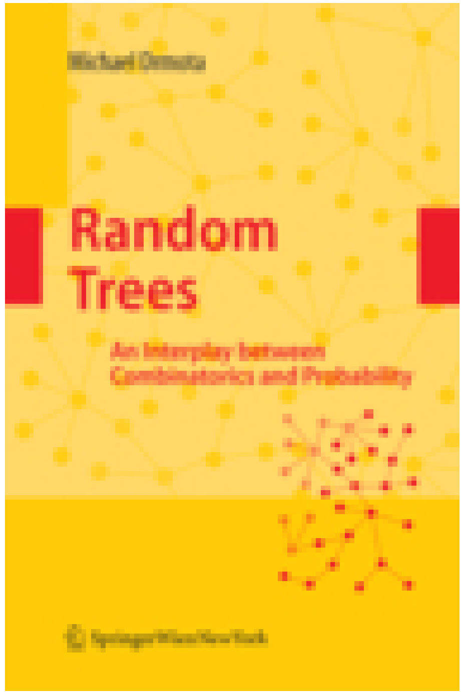
Contents

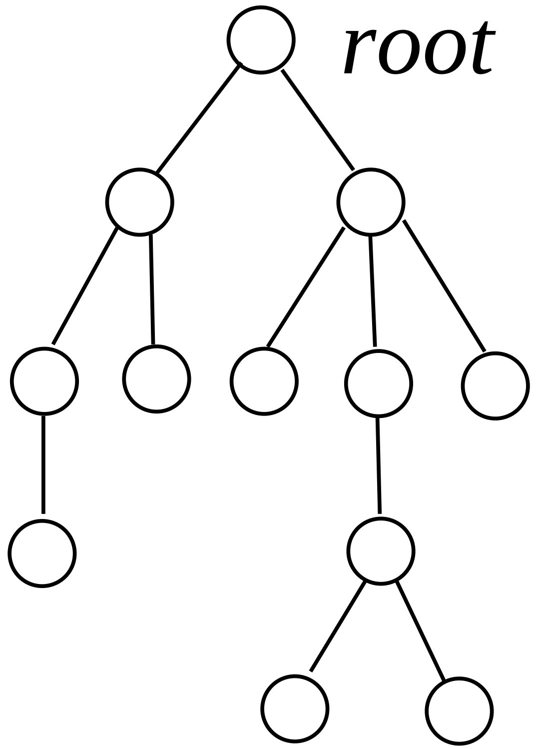
Book



Michael Drmota,
Random Trees, Springer, Wien-New York, 2009.

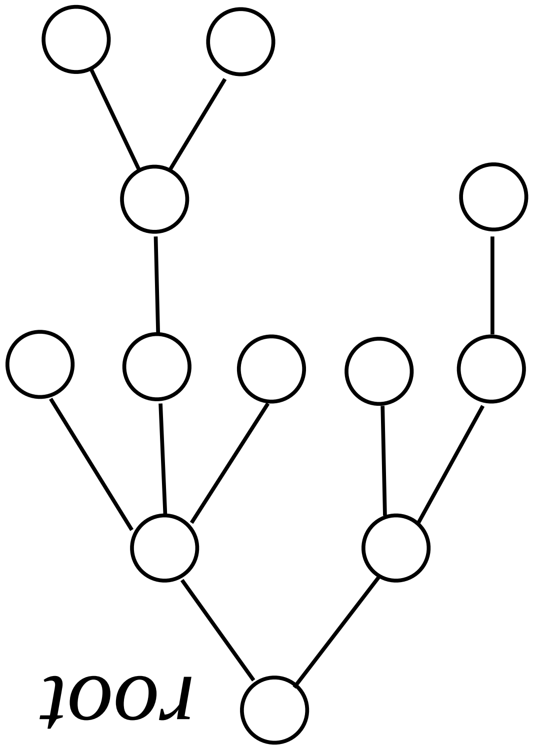
Profile of Trees

Rooted tree



Profile of Trees

Rooted tree



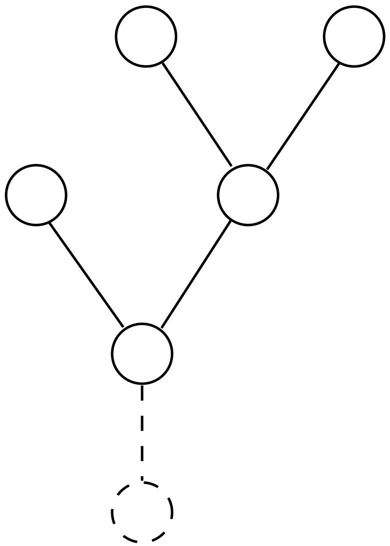
Profile of Trees

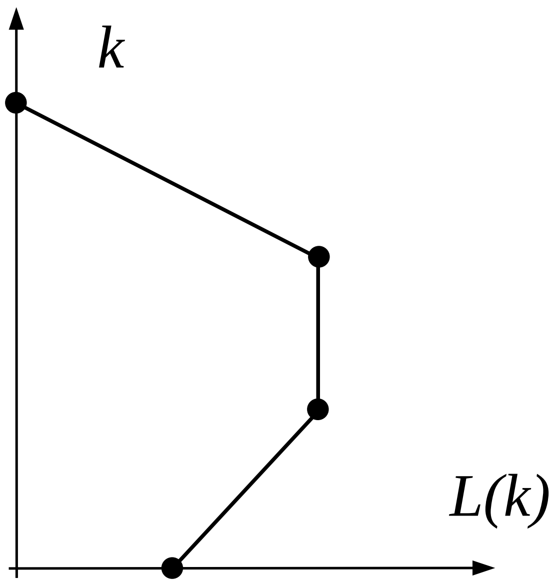
$I_T(k)$ ... number of nodes at distance k from the root

$(I_T(k))_{k\geq 0}$ ... profile of T

$(I_T(s), s \geq 0)$ ... linearly interpolated profile of T

$(I_{n,k})_{k\geq 0}$ ... profile in a **random tree** of size n





Profile of Trees

Parameters of interest:

- **Profile** $I_{n,k}$ (number of nodes at depth k)
- **Depth** of a random node: D_n
- **Internal path length**: L_n (sum of all distances to the root)
- **Height** H_n

Profile of Trees

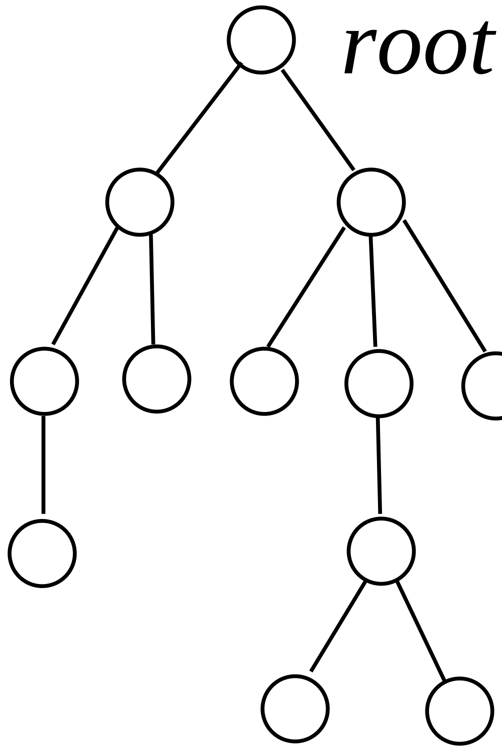
Relations to the profile $I_{n,k}$:

- $\mathbf{Pr}\{D_n = k\} = \dfrac{1}{n}\,\mathbf{E}\,I_{n,k}$
- $L_n = \sum_{k \ge 0} k I_{n,k}$
- $H_n = \max\{k \ge 0 : I_{n,k} > 0\}$
- The profile describes the **shape** of the tree.

Contents

Galton-Watson Trees

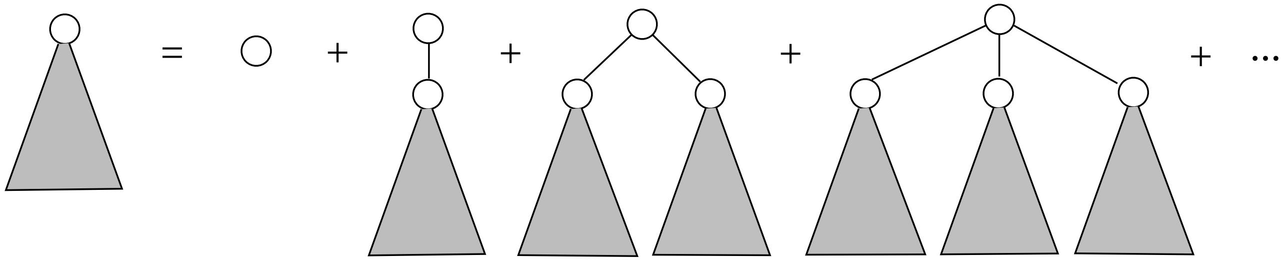
Catalan trees



rooted, ordered (or plane) tree

Galton-Watson Trees

Catalan trees. g_n = number of Catalan trees of size n; $G(x) = \sum_{n \geq 1} g_n x^n$



$$G(x) = x(1 + G(x) + G(x)^2 + \cdots) = \frac{x}{1 - G(x)}$$

$$G(x) = \frac{1 - \sqrt{1 - 4x}}{2} \quad \Longrightarrow \quad g_n = \frac{1}{n}\binom{2n-2}{n-1} \sim \frac{4^{n-1}}{\sqrt{\pi} \cdot n^{3/2}}$$

(Catalan numbers)

Galton-Watson Trees

Catalan trees with singularity analysis

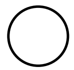
$$G(x) = \frac{1-\sqrt{1-4x}}{2} = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}$$

$$\Longrightarrow \quad g_n \sim -\frac{1}{2} \cdot \frac{4^n n^{-3/2}}{\Gamma(-\frac{1}{2})} = \frac{4^{n-1}}{\sqrt{\pi} \cdot n^{3/2}}$$

Galton-Watson Trees

Galton-Watson branching process

ξ ... offspring distribution, $\varphi_k = \mathbb{P}\{\xi = k\}$, $\varphi_0 > 0$



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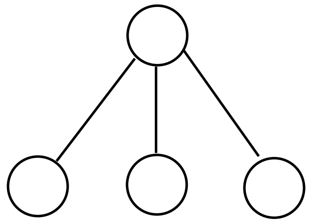

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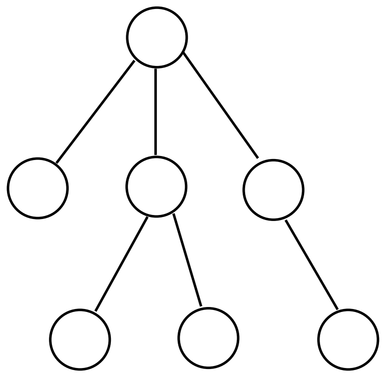

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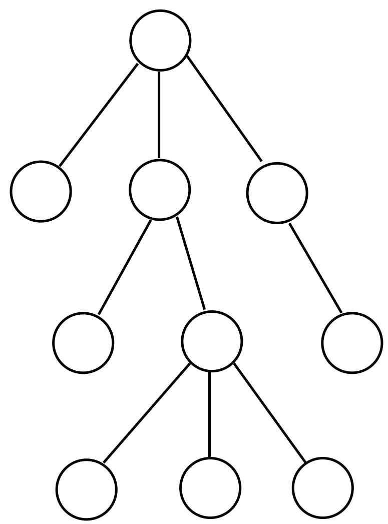

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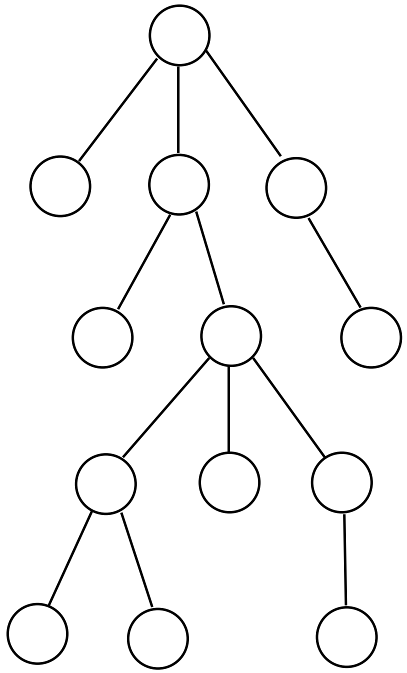

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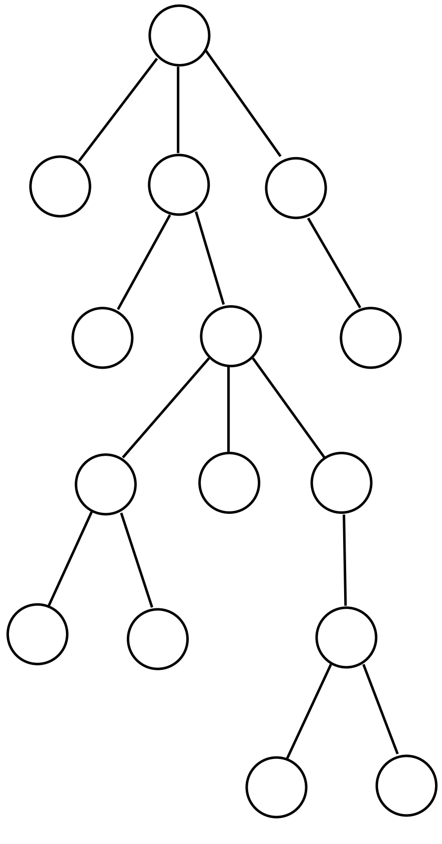


Galton-Watson Trees

Galton-Watson branching process. $(Z_k)_{k\geq 0}$

$Z_0 = 1$, and for $k \geq 1$

$$\boxed{Z_k = \sum_{j=1}^{Z_{k-1}} \xi_j^{(k)}},$$

where the $(\xi_j^{(k)})_{k,j}$ are iid random variables distributed as ξ.

Z_k ... number of nodes in k-th generation

$Z = Z_0 + Z_1 + Z_2 + \cdots$ ... total progeny

Galton-Watson Trees

Generating functions

$$y_n = \mathbb{P}\{Z = n\}, \qquad y(x) = \sum_{n\geq 1} y_n x^n$$

$$\Phi(w) = \mathbb{E}\, w^{\xi} = \sum_{k\geq 0} \varphi_k w^k$$

$$\Longrightarrow \quad \boxed{y(x) = x\,\Phi(y(x))}$$

Conditioned Galton-Watson tree

GW-branching process conditioned on the total progeny $Z = n$.

Galton-Watson Trees

Example. $\mathbb{P}\{\xi = k\} = 2^{-k-1}$, $\Phi(w) = 1/(2-w)$

$\Longrightarrow$ all trees of size n have the same probability

$\Longrightarrow$ conditioned GW-tree of size n is the same model as the **Catalan tree model** (with the uniform distribution on trees of size n)

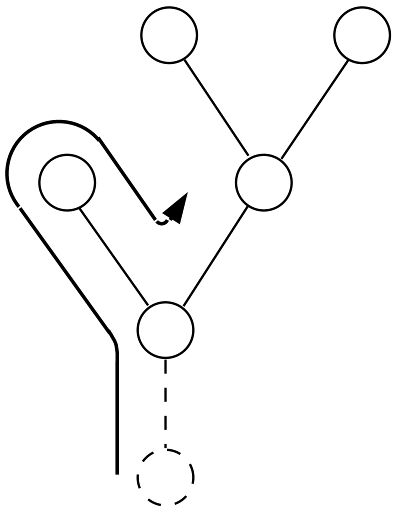
Example. $\Phi(w) = \frac{1}{2}(1+w)^2$: **binary trees** with n internal nodes.

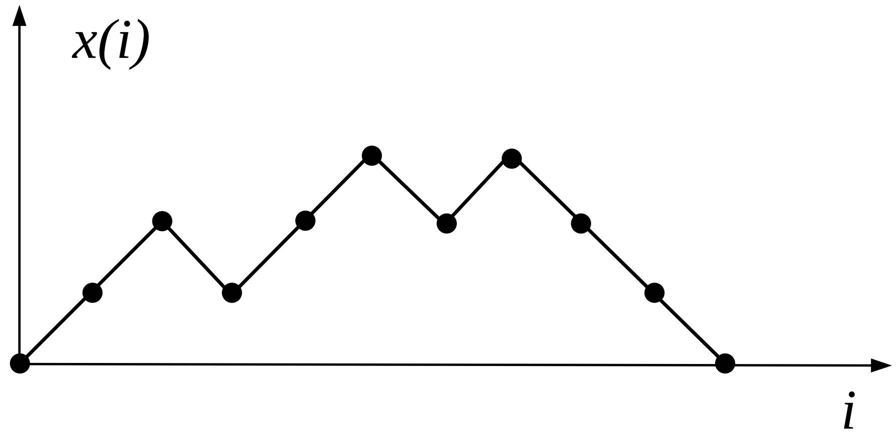
Example. $\Phi(w) = \frac{1}{3}(1+w+w^2)$: **Motzkin trees**

Example. $\Phi(w) = e^{w-1}$: **Cayley trees**

Galton-Watson Trees

Depth-First-Search – Rooted trees and discrete excursions





Bijection between

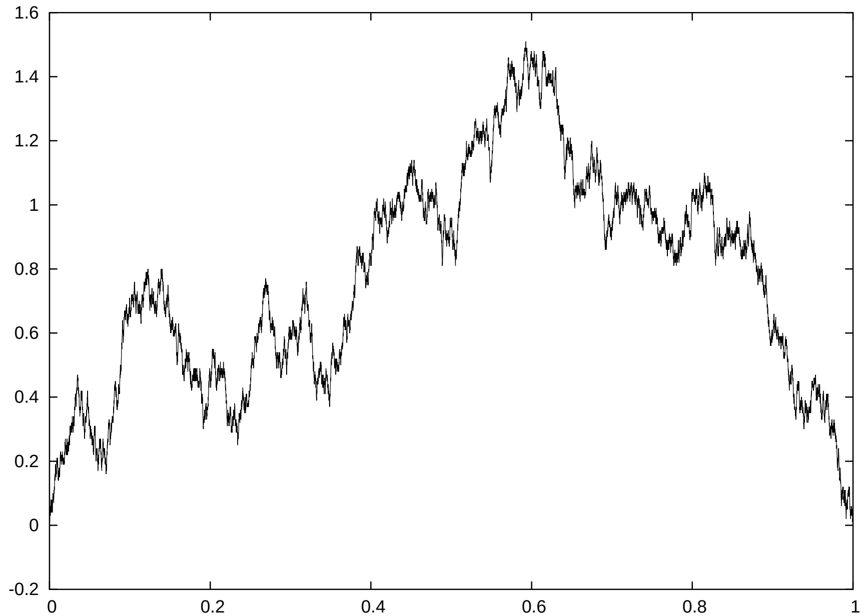
Catalan trees $\longleftrightarrow$ Dyck paths

random trees of size n $\longleftrightarrow$ **random** Dyck paths of length $2n$

Galton-Watson Trees

Depth-First-Search

Brownian excursion $(e(t), 0 \le t \le 1)$



Rescaled Brownian motion between 2 zeros.

Random function in $C[0,1]$.

Depth-First-Search

Kaigh's Theorem

$(X_n(t), 0 \le t \le 2n)$ ... random Dyck path of length $2n$.

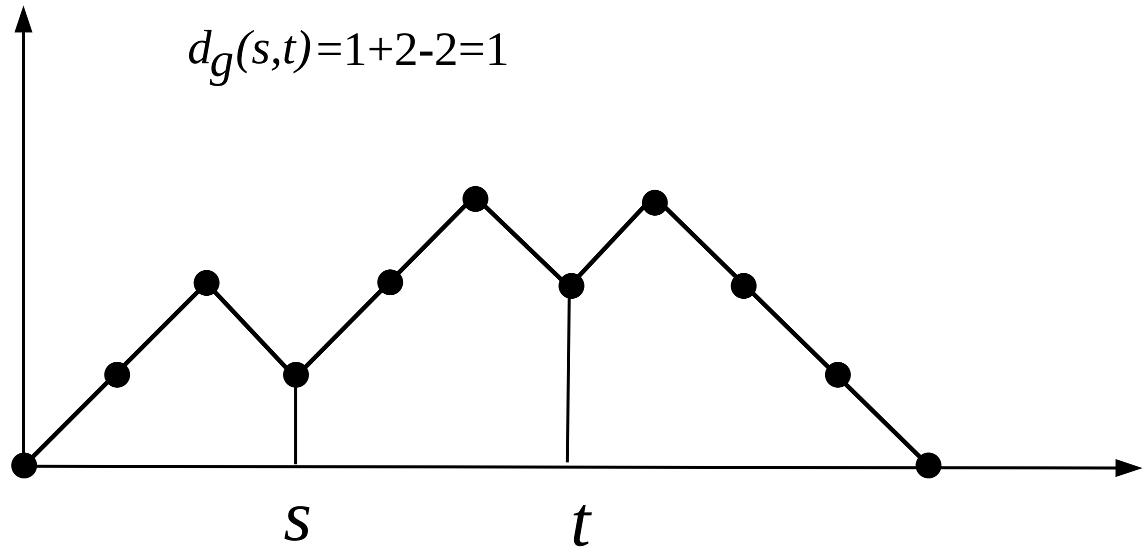
$$\Longrightarrow \quad \left(\frac{1}{\sqrt{2n}} X_n(2nt), 0 \le t \le 1\right) \xrightarrow{\mathrm{d}} (e(t), 0 \le t \le 1).$$

Remark. This theorem also holds for more general random walks with independent increments conditioned to be an excursion.

Galton-Watson Trees

$g : [0,1] \to [0,\infty)$ ... continuous, ≥ 0, $g(0) = g(1) = 0$

$$d_g(s,t) = g(s) + g(t) - 2 \inf_{\min\{s,t\} \leq u \leq \max\{s,t\}} g(u)$$



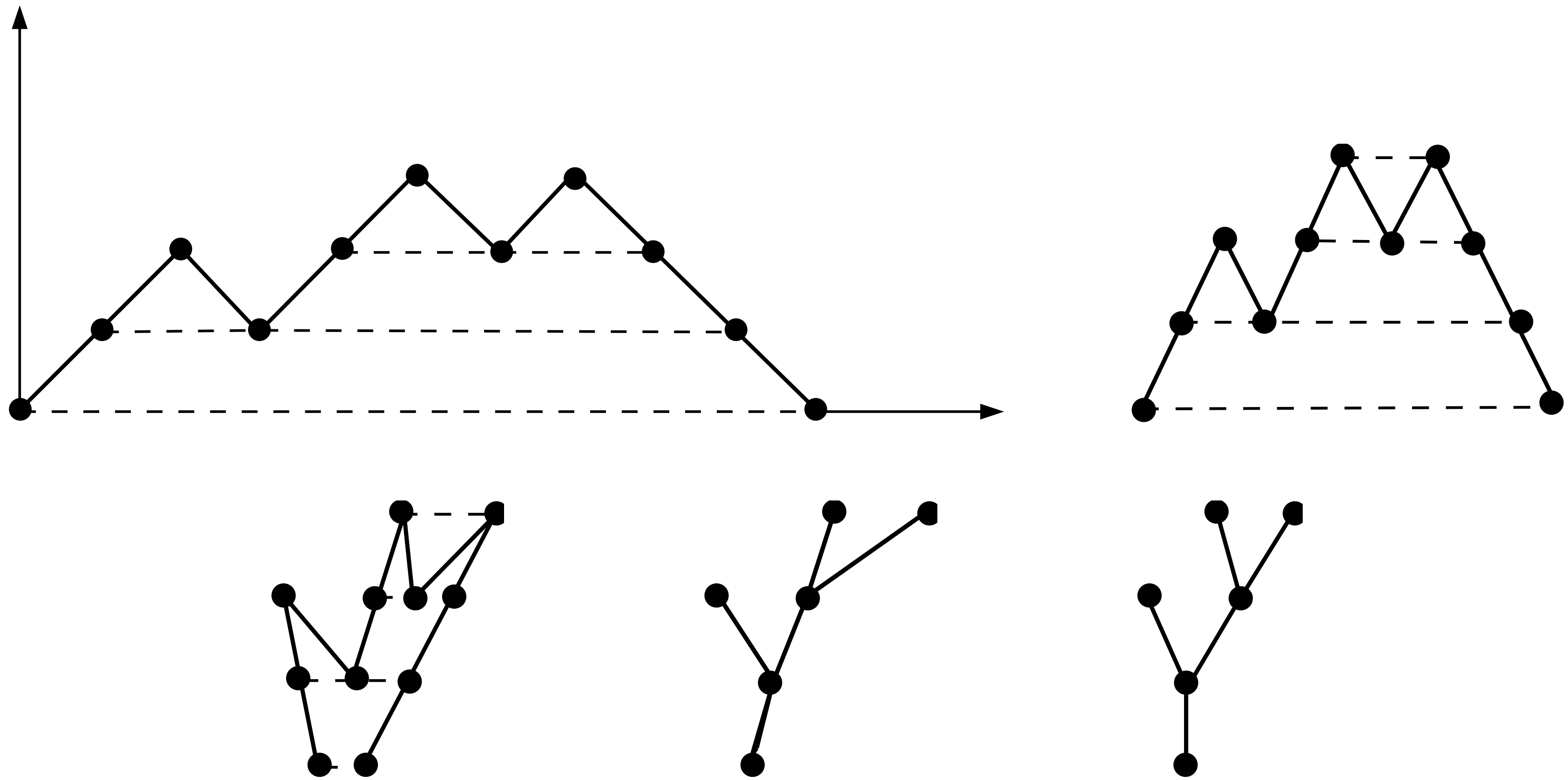
$$s \sim t \quad \Longleftrightarrow \quad d_g(s,t) = 0$$

$$\mathcal{T}_g = [0,1]/\sim$$

$\Longrightarrow$ $(\mathcal{T}_g, d_g)$ is a compact (so-called) real tree.

Galton-Watson Trees

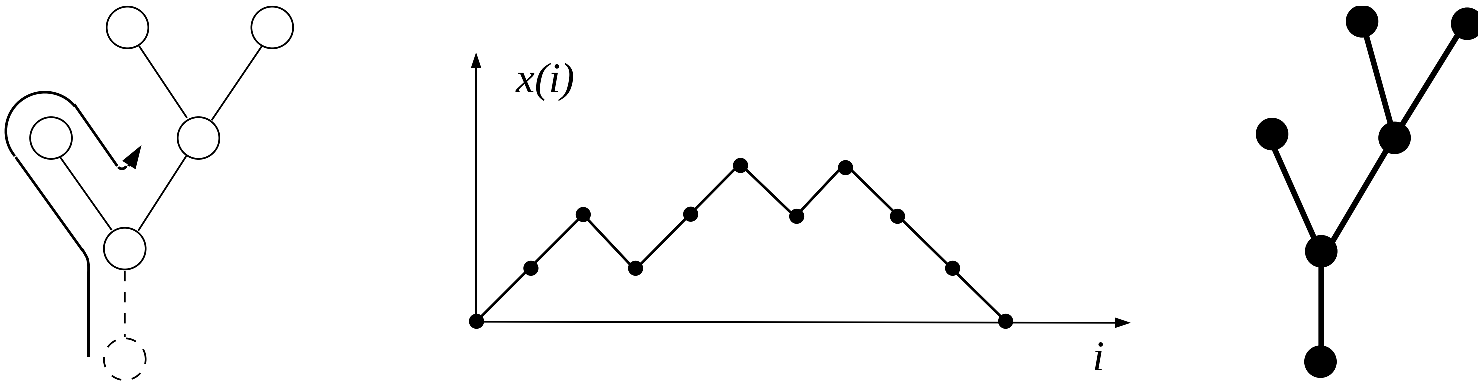
Construction of a real tree $\mathcal{T}_g$



The mapping $C[0,1] \to \mathbb{T}$, $g \mapsto \mathcal{T}_g$ is **continuous**.

Galton-Watson Trees

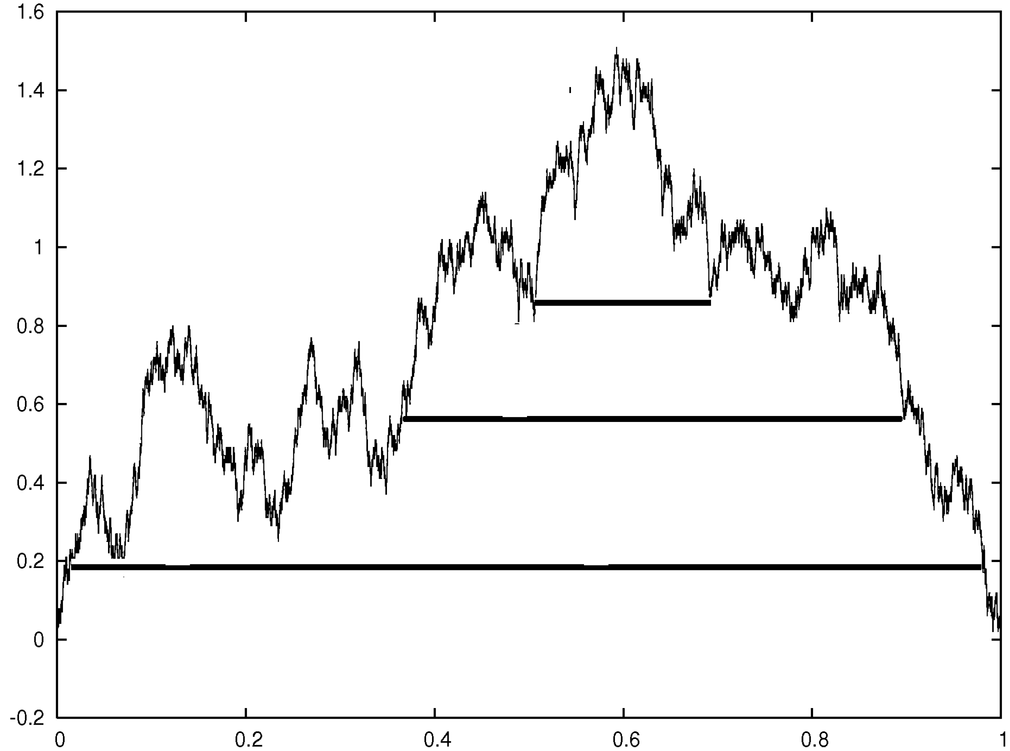
Catalan trees as real trees



T_n $\qquad$ $X_n = X_{T_n}$ $\qquad$ $\mathcal{T}_{X_n}$

Galton-Watson Trees

Continuum random tree $\mathcal{T}_{2e}$ (with Brownian excursion $e(t)$)



Galton-Watson Trees

Theorem

$(X_n(t), 0 \le t \le 2n)$ ... random Dyck paths of length $2n$
or the depth-first-search process of Catalan trees of size n.

$$\Longrightarrow \quad \boxed{\frac{1}{\sqrt{2n}}\mathcal{T}_{X_n} \xrightarrow{\text{d}} \mathcal{T}_{2e}}$$

In other words...

Scaled Catalan trees (interpreted as "real trees") converge weakly to the continuum random tree.

Galton-Watson Trees

General assumption: $\boxed{\mathbb{E}\,\xi = 1}$, $\boxed{0 < \mathbb{V}\mathrm{ar}\,\xi = \sigma^2 < \infty}$

Theorem (Aldous)

$X_n(t)$ ... depth-first-search of conditioned GW-trees of size n

$$\Longrightarrow \quad \boxed{\left(\frac{\sigma}{2\sqrt{n}} X_n(2nt), 0 \le t \le 1\right) \xrightarrow{\mathrm{d}} (e(t), 0 \le t \le 1)}.$$

Corollary

$$\boxed{\frac{\sigma}{\sqrt{n}} \mathcal{T}_{X_n} \xrightarrow{\mathrm{d}} \mathcal{T}_{2e}}$$

Galton-Watson Trees

Corollary H_n ... height of conditioned GW-trees of size n:

$$\Longrightarrow \quad \boxed{\frac{1}{\sqrt{n}} H_n \xrightarrow{\mathrm{d}} \frac{2}{\sigma} \max_{0 \le t \le 1} e(t)}$$

Remark. Distribution function of $\max_{0 \le t \le 1} e(t)$:

$$\mathbb{P}\{\max_{0 \le t \le 1} e(t) \le x\} = 1 - 2 \sum_{k=1}^{\infty} (4x^2k^2 - 1) e^{-2x^2k^2}$$

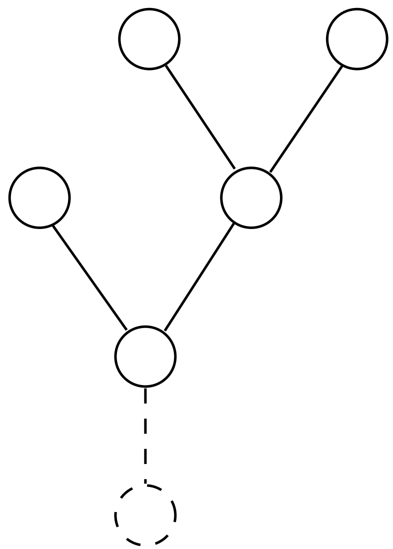
Galton-Watson Trees

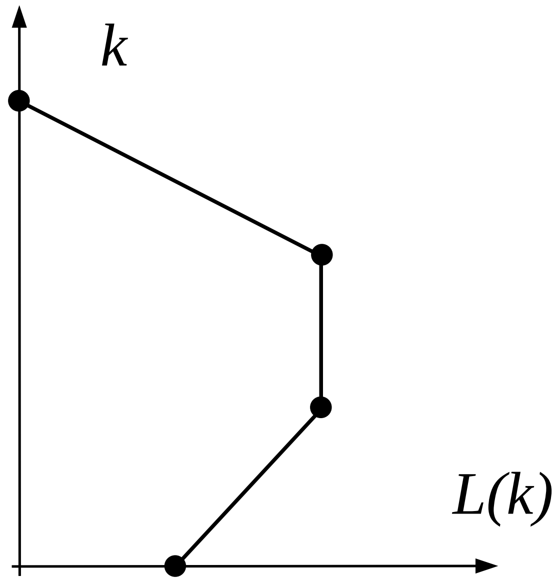
Profile

$I_T(k)$ ... number of nodes at distance k from the root

$(I_T(k))_{k\geq 0}$ ... profile of T

$(I_T(s), s \geq 0)$ ... linearly interpolated profile of T





Galton-Watson Trees

Value distribution

$$\mu_T = \frac{1}{|T|} \sum_{k \geq 0} I_T(k)\, \delta_k$$

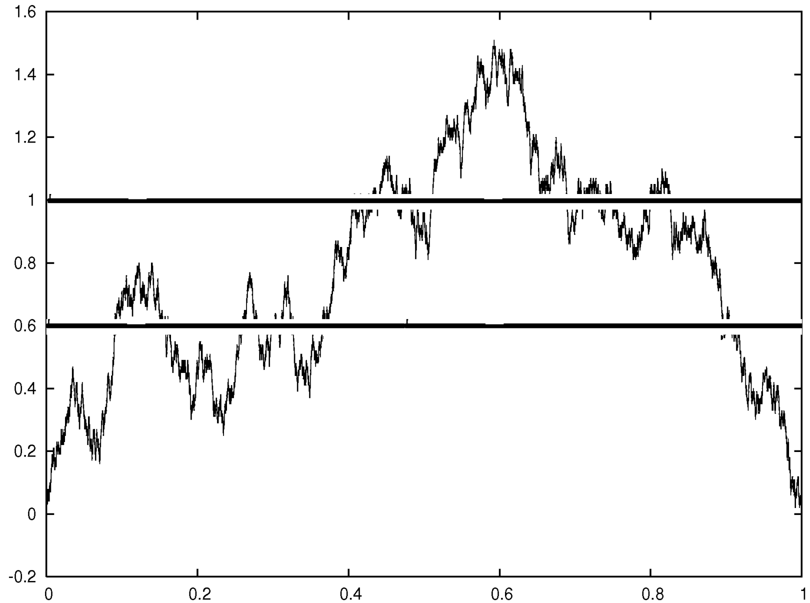
δ_x ... δ-distribution concentrated at x

Galton-Watson Trees

Occupation measure: random measure on $\mathbb{R}$

$$\mu(A) = \int_0^1 \mathbf{1}_A(e(t)\,dt$$

measure how long $e(t)$ stays in set A



Galton-Watson Trees

Theorem (Aldous)

$(I_{n,k}, k \ge 0)$ ... random profile of conditioned GW-trees of size n

$$\Longrightarrow \quad \boxed{\frac{1}{n}\sum_{k\ge 0} I_{n,k}\,\delta_{(\sigma/2)k/\sqrt{n}} \xrightarrow{\mathrm{d}} \mu}$$

Galton-Watson Trees

Local time of the Brownian excursion: random density of μ

$$\boxed{l(s) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^1 \mathbf{1}_{[s,s+\varepsilon]}(e(t))\, dt}$$

Theorem (D.+Gittenberger)

$(l_n(s), s \ge 0)$ ... random profile of conditioned GW-trees of size n

$$\Longrightarrow \quad \boxed{\left(\frac{1}{\sqrt{n}} l_n(s\sqrt{n}),\ s \ge 0\right) \xrightarrow{\text{d}} \left(\frac{\sigma}{2} l\left(\frac{\sigma}{2}s\right),\ s \ge 0\right)}$$

Proof with asymptotics on generating functions (very involved)!!!

Galton-Watson Trees

Width

$$W = \max_{k \ge 0} L(k) = \max_{t \ge 0} L(t),$$

maximal number of nodes in a level.

Corollary

$$\boxed{\frac{1}{\sqrt{n}} W_n \xrightarrow{\text{d}} \frac{\sigma}{2} \sup_{0 \le t \le 1} l(t)}$$

Remark. $\sup_{t \ge 0} l(t) = 2 \sup_{0 \le t \le 1} e(t)$ (in distribution)

Contents

(Binary) Search Trees

Storing of data

4,6,3,5,1,8,2,7

(Binary) Search Trees

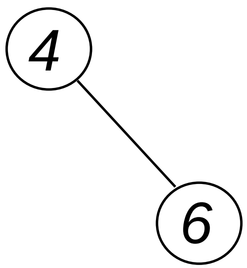
Storing of data

6,3,5,1,8,2,7

(Binary) Search Trees

Storing of data

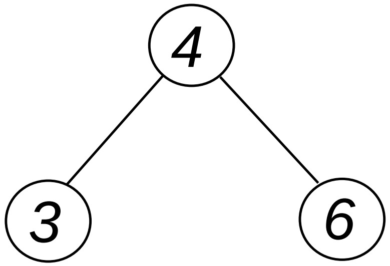
3,5,1,8,2,7



(Binary) Search Trees

Storing of data

5,1,8,2,7



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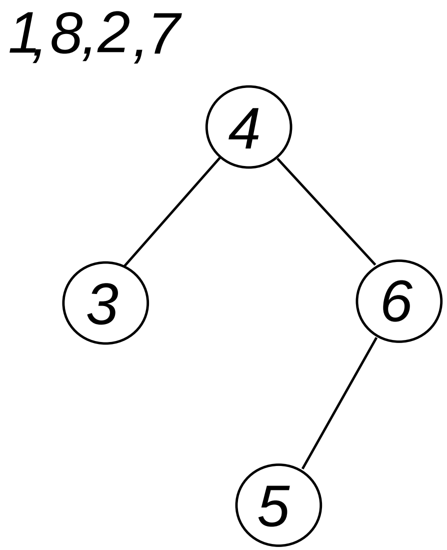

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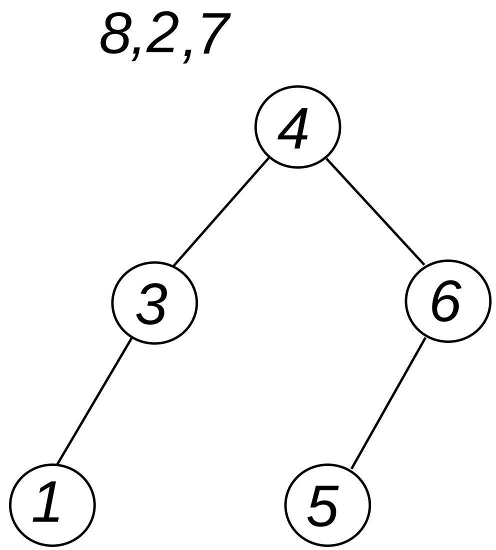

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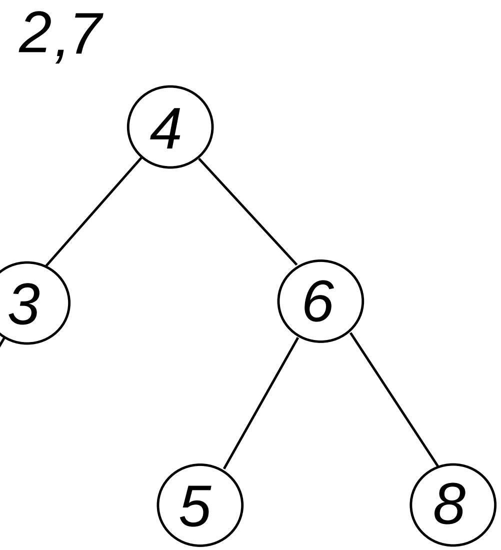

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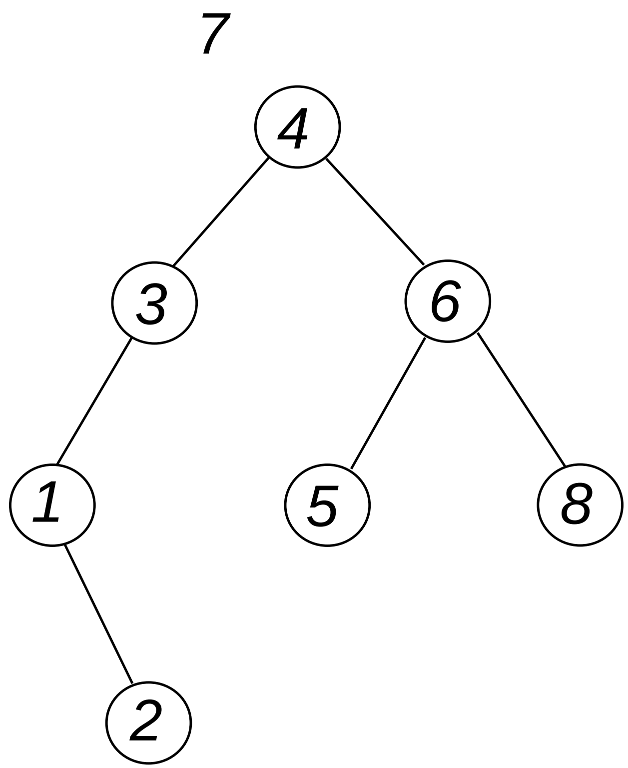

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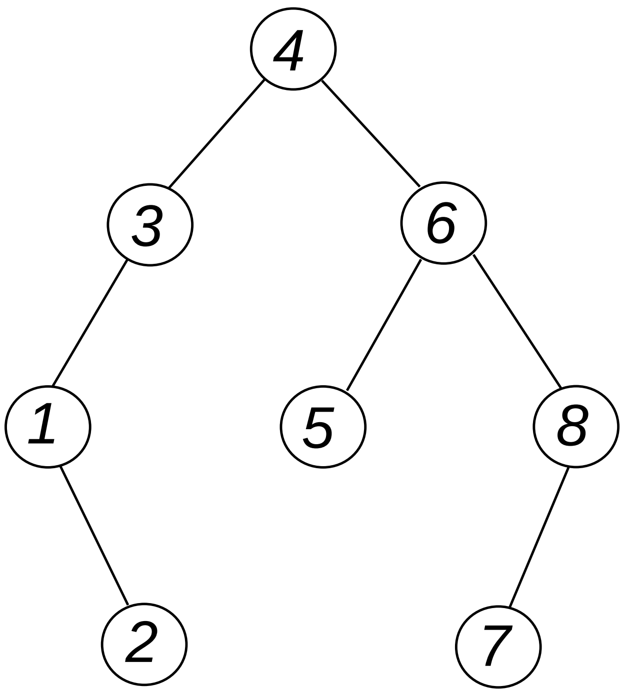

(Binary) Search Trees

Quicksort – Sorting of data

4,6,3,5,1,8,2,7

(Binary) Search Trees

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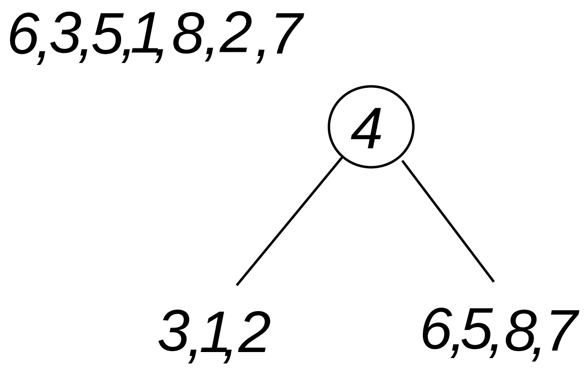

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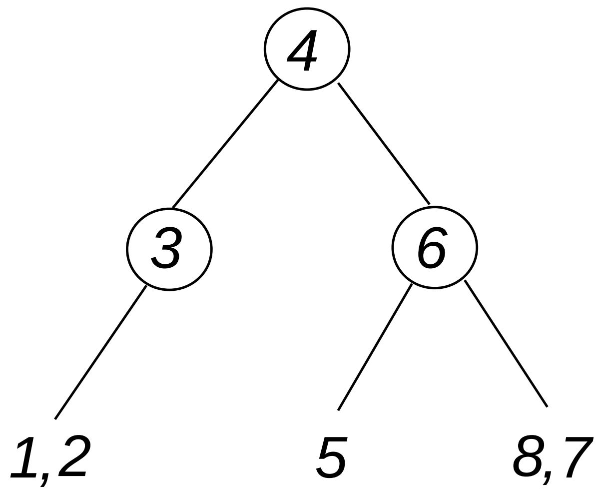

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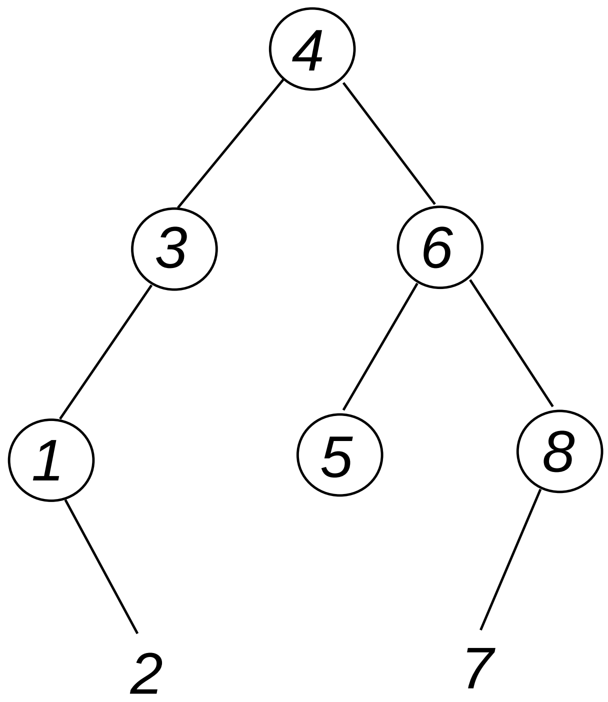

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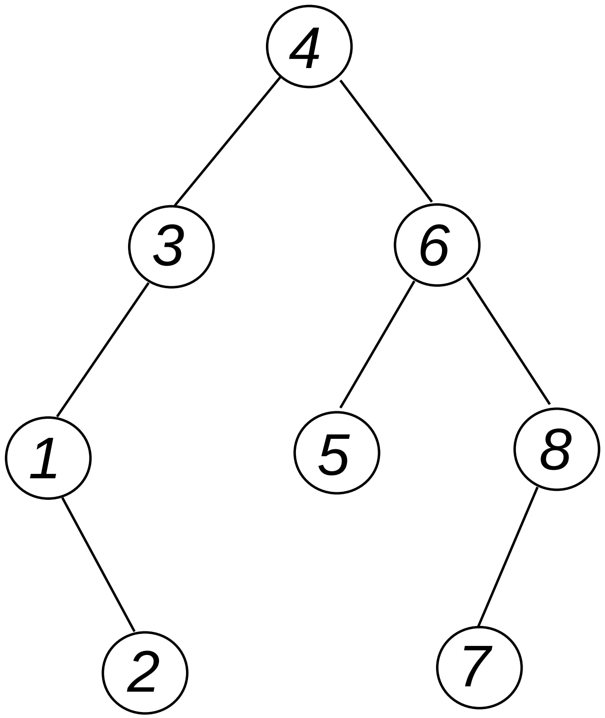


(Binary) Search Trees

Quicksort – Median of 3

4,6,3,5,1,8,2,7

(Binary) Search Trees

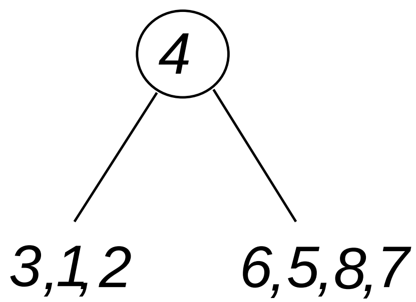
Quicksort – Median of 3

↓

4,6,3,5,1,8,2,7

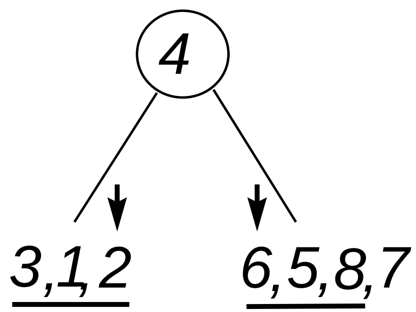
(Binary) Search Trees

Quicksort – Median of 3



(Binary) Search Trees

Quicksort – Median of 3



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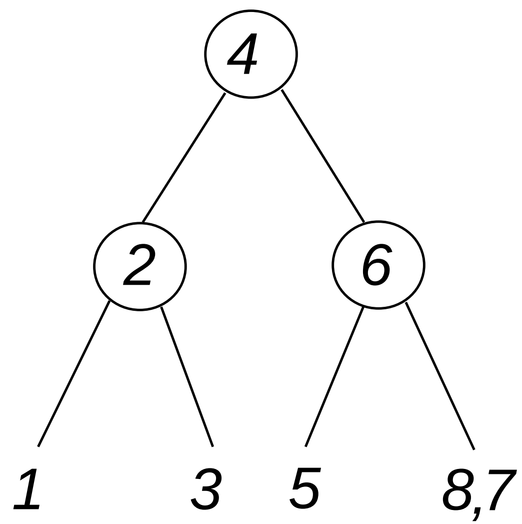

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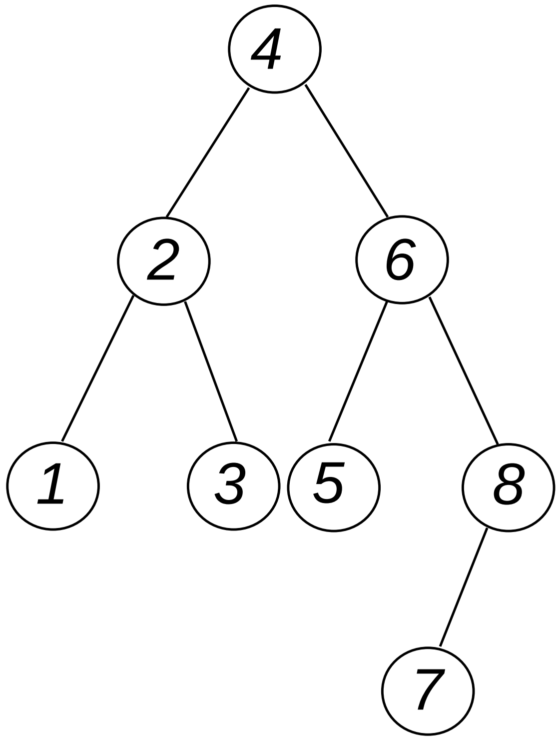


(Binary) Search Trees

Probabilistic Model

Every permutation on the data $\{1, 2, \ldots, n\}$ ist equally likely

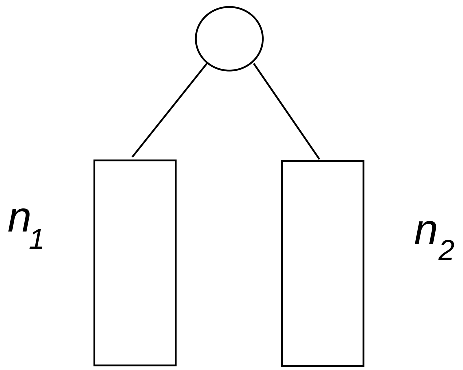
$\longrightarrow$ probability distribution on binary (m-ary) trees of size n

$\longrightarrow$ all tree parameters are **random variables**

(Binary) Search Trees

Probabilistic Model – Recursive structure

Subtrees have the same structure:
$(n = n_1 + n_2 + 1)$.



Splitting probabilities: p_{n_1,n_2}

Quicksort: $$p_{n_1,n_2} = \frac{1}{n}$$

Median of 3: $$p_{n_1,n_2} = \frac{n_1 n_2}{\binom{n}{3}}$$

Search Trees

General Model

$m \geq 2$, $t \geq 0$ ... given integers $\qquad$ n keys (data)

- If $n \geq m$, we **randomly select** $m-1$ **pivots** $\boxed{x_1 < x_2 < \cdots < x_{m-1}}$.

- The pivots are stored in the **root**.

- The remaining $n-m+1$ keys are divided into m **subsets** $I_1, \ldots, I_m$:

  $I_1 := \{x_i : x_i < x_1\},\ I_2 := \{x_i : x_1 < x_i < x_2\},\ \ldots, I_m := \{x_i : x_{m-1} < x_i\}.$

- Apply this procedure **recursively** to $I_1, I_2, \ldots, I_m$.

Search Trees

General Splitting Probabilities

$\mathbf{V}_n = (V_{n,1}, V_{n,2}, \ldots, V_{n,m})$.. random splitting vector

$V_{n,k} := |I_k|$ ... number of keys in the kth subset
(= the number of nodes in the kth subtree of the root)

$$V_{n,1} + V_{n,2} + \cdots + V_{n,m} = n - (m-1) = n + 1 - m$$

$$\boxed{\mathbb{P}\{\mathbf{V}_n = (n_1, \ldots, n_m)\} = \frac{\binom{n_1}{t} \cdots \binom{n_m}{t}}{\binom{n}{mt+m-1}}}$$

$$(n_1 + n_2 + \cdots + n_m = n - m + 1)$$

Quicksort: $m = 2$, $t = 0$ **Median of 3:** $m = 2$, $t = 1$

Search Trees

Recursive relation for the profile:

$$\boxed{I_{n,k} \stackrel{\mathrm{d}}{=} I^{(1)}_{V_{n,1},k-1} + I^{(2)}_{V_{n,2},k-1} + \cdots + I^{(m)}_{V_{n,m},k-1}}$$

$(I^{(j)}_{n,k})_{k\ge 0}$, $j = 1, \ldots, m$ ... independent copies of $X_{n,k}$

Search Trees

Expected Profile

$$F(\theta) := \frac{t!}{m(mt+m-1)!}(\theta+t)(\theta+t+1)\cdots(\theta+mt+m-2),$$

$\boxed{\lambda_1(z)}$, $\lambda_2(z)$, ..., $\lambda_{(m-1)(t+1)}(z)$ ... roots of $\boxed{F(\theta)=z}$:
$\boxed{\Re(\lambda_1(z))} \geq \Re(\lambda_2(z)) \geq \dots$.

$\beta(\alpha) > 0$ ... defined by $\boxed{\beta(\alpha)\lambda_1'(\beta(\alpha)) = \alpha}$.

$$\alpha_0 := \left(\frac{1}{t+1} + \frac{1}{t+2} + \cdots + \frac{1}{(t+1)m-1}\right)^{-1}$$

Search Trees

Expected Profile $\boxed{k = \alpha \log n}$

Theorem [D.+Janson+Neininger]

- $0 < \alpha = k/\log n < \alpha_0$:

$$\boxed{\mathbb{E}\, I_{n,k} \sim (m-1)m^k}.$$

- $\alpha = k/\log n > \alpha_0$:

$$\boxed{\mathbb{E}\, I_{n,k} \sim \frac{E(\beta(\alpha))n^{\lambda_1(\beta(\alpha))-\alpha\log(\beta(\alpha))-1}}{\sqrt{2\pi(\alpha+\beta(\alpha)^2\lambda_1''(\beta(\alpha)))\log n}}}$$

for some continuous function $E(z)$

Note: $m^k = n^{\alpha \log m}$

Search Trees

Expected Profile

$$\alpha_{\mathsf{max}} := \left(\frac{1}{t+2} + \frac{1}{t+3} + \cdots + \frac{1}{(t+1)m}\right)^{-1}$$

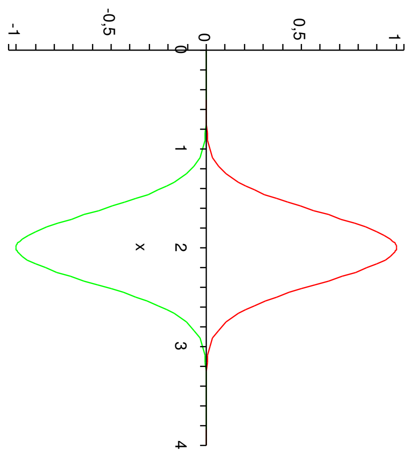
$$\boxed{\mathbb{E}\, I_{n,k} \sim \frac{n}{\sqrt{2\pi(\alpha_{\mathsf{max}} + \lambda_1''(1)) \log n}} \exp\left(-\frac{(k - \alpha_{\mathsf{max}} \log n)^2}{2(\alpha_{\mathsf{max}} + \lambda_1''(1)) \log n}\right)}$$

($\Longrightarrow$ CLT for depth D_n)

Search Trees

The average profile: $m = 2$, $t = 0$ (special case)

$$\boxed{\mathbf{E}\, I_{n,k} = \frac{n}{\sqrt{4\pi \log n}} \left(e^{-\frac{(k-2\log n)^2}{4\log n}} + \mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right)\right)}.$$



Search Trees

Theorem 1 [D.+Janson+Neininger]

$m \geq 2$, $t \geq 0$ ... given integers
$(I_{n,k})_{k \geq 0}$ ... random profile

$I = \{\beta > 0 : 1 < \lambda_1(\beta^2) < 2\lambda_1(\beta) - 1\}$,
$I' = \{\beta \lambda_1'(\beta) : \beta \in I\}$

$\boxed{\beta(\alpha)\lambda_1'(\beta(\alpha)) = \alpha}$.

$\Longrightarrow$

$$\boxed{\left(\frac{I_{n,\lfloor \alpha \log n \rfloor}}{\mathbb{E}\, I_{n,\lfloor \alpha \log n \rfloor}}, \alpha \in I'\right) \xrightarrow{\text{d}} \Big(Y(\beta(\alpha)), \alpha \in I'\Big)}$$

in $D(I')$ (Skorohod topology).

Search Trees

Random analytic functions

$B \subseteq \mathbb{C}$, $(I \subseteq B)$ $\qquad$ $Y(z)$ ... random analytic function on B

$$\boxed{Y(z) \overset{\mathrm{d}}{=} zV_1^{\lambda_1(z)-1}Y^{(1)}(z) + zV_2^{\lambda_1(z)-1}Y^{(2)}(z) \cdots + zV_m^{\lambda_1(z)-1}Y^{(m)}(z)}$$

$Y^{(j)}(z)$ ... independent copies of $Y(z)$

$\mathbf{V} = (V_1, V_2, \ldots, V_m)$ ... random vector supported on the simplex $\Delta = \{(s_1, \ldots, s_m) : s_j \ge 0, s_1 + \cdots + s_m = 1\}$ with density

$$f(s_1, \ldots, s_m) = \frac{((t+1)m-1)!}{(t!)^m}(s_1 \cdots s_m)^t.$$

$\mathbf{V}, Y^{(1)}(z), \ldots, Y^{(m)}(z)$ ... independent.

Search Trees

Profile Polynomials

$$\boxed{W_n(z) := \sum_k I_{n,k} z^k}$$

$$I_{n,k} \stackrel{\mathrm{d}}{=} I^{(1)}_{V_{n,1},k-1} + I^{(2)}_{V_{n,2},k-1} + \cdots + I^{(m)}_{V_{n,m},k-1},$$

$\Longrightarrow$

$$\boxed{W_n(z) \stackrel{\mathrm{d}}{=} zW^{(1)}_{V_{n,1}}(z) + zW^{(2)}_{V_{n,2}}(z) + \cdots + zW^{(m)}_{V_{n,m}}(z) + m - 1}$$

for $n \geq m$

Search Trees

Profile Polynomials

Theorem 2 [D.+Janson+Neininger]

B ... complex region, $(1/m, \beta(\alpha_+)) \in B$,
$$\lambda_1(\beta(\alpha_+)) - \alpha_+ \log(\beta(\alpha_+)) - 1 = 0.$$

$\Longrightarrow$

$$\boxed{\left(\frac{W_n(z)}{\mathbb{E}\, W_n(z)}, z \in B\right) \xrightarrow{\mathrm{d}} (Y(z), z \in B)}$$

in $\mathcal{H}(B)$.

Remark Theorem 2 $\Longrightarrow$ Theorem 1

Search Trees

Profile Polynomials

$(I_{n,k})$ ... random profile

$\Longrightarrow W_n(z) := \sum_{k\ge 0} I_{n,k} z^k$ ... random analytic function

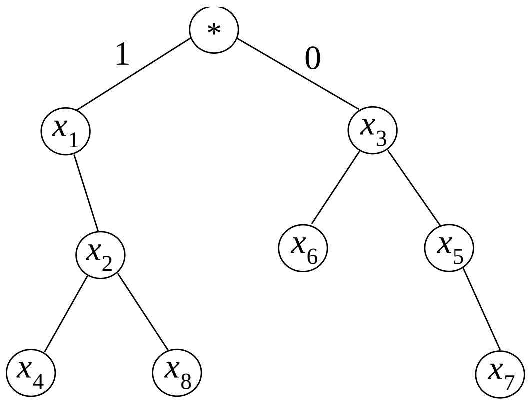
$\Longrightarrow \dfrac{W_n(z)}{\mathbb{E}\, W_n(z)}$ ... random analytic function

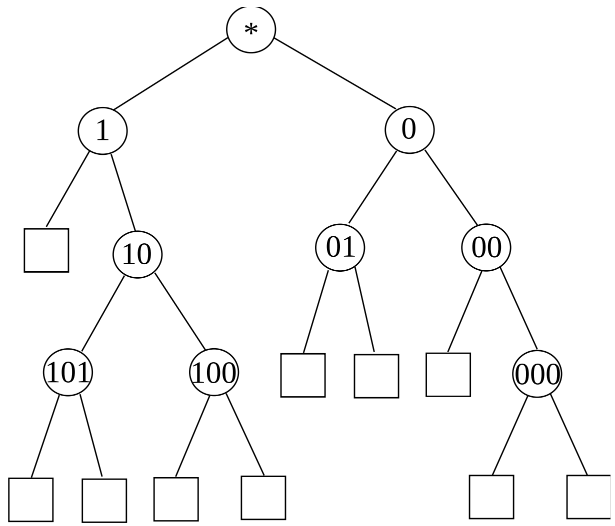
Contents

Digital Trees

Digital Search Trees

$$
\begin{aligned}
x_1 &= 110011\cdots \\
x_2 &= 100110\cdots \\
x_3 &= 010010\cdots \\
x_4 &= 101110\cdots \\
x_5 &= 000110\cdots \\
x_6 &= 010111\cdots \\
x_7 &= 000100\cdots \\
x_8 &= 100101\cdots .
\end{aligned}
$$





Digital Trees

Digital Search Trees

Bernoulli model

The input is a sequence of n independent and identically distributed random variables, each being composed of an infinite sequence of Bernoulli random variables with mean p, where $0 < p < 1$ is the probability of a 1 and $q = 1 - p$ is the probability of a 0.

Digital Trees

Profile

$B_{n,k}$ ... number of external nodes at level k after n insertions

$I_{n,k}$ ... number of internal nodes at level k after n insertions

Digital Trees

Expected Profile $p = q = \frac{1}{2}$

$$E_k(x) := \sum_{n \ge 0} \mathbb{E}\, B_{n,k} \frac{x^n}{n!}$$

$$E_k'(x) = 2e^{x/2} E_{k-1}\left(\frac{x}{2}\right)$$

$$E_0(x) = 1 \text{ and } E_k(0) = 0 \text{ for } k \ge 1$$

$$\Longrightarrow \quad \boxed{E_k(x) = 2^k e^x \sum_{m=0}^{k} \frac{(-1)^m 2^{-\binom{m}{2}}}{\gamma_m \gamma_{k-m}} e^{-x 2^{m-k}}}$$

$$\gamma_\ell = \prod_{j=1}^{\ell} \left(1 - \frac{1}{2^j}\right) \qquad (\ell > 0).$$

Digital Trees

Expected Profile

Theorem $p = q = \frac{1}{2}$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, B_{n,k} = 2^k \sum_{m=0}^{k} \frac{(-1)^m 2^{-\binom{m}{2}}}{\gamma_m \gamma_{k-m}} \left(1 - \frac{1}{2^{k-m}}\right)^n}$$

$$F(z) = \sum_{m \ge 0} \frac{(-1)^m 2^{-\binom{m}{2}}}{\gamma \gamma_m} e^{-z 2^m},$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, B_{n,k} = 2^k F(n2^{-k}) + F'(n2^{-k}) + \mathcal{O}\left(n2^{-k}\right)}$$

Digital Trees

Variance of the Profile

Theorem [D.+Fuchs+Hwang+Neiniger] $p = q = \frac{1}{2}$

$$\mathrm{Var}(B_{n,k}) \begin{cases} \sim \dfrac{2^k}{Q_k}\left(1-2^{-k}\right)^n, & \text{if } n/2^k \to \infty; \\ = 2^k H(n/2^k) + \mathcal{O}(1), & \text{if } n/4^k \to 0, \end{cases}$$

$$H(x) \sim 2F(x), \qquad (x \to 0).$$

Remark

$\mathbb{E}(B_{n,k}) \to \infty$ iff $\mathrm{Var}(B_{n,k}) \to \infty$

Digital Trees

Central Limit Theorem for the Profile

Theorem [D.+Fuchs+Hwang+Neininger] $p = q = \frac{1}{2}$

If $\mathbb{E}(B_{n,k}) \to \infty$, we have

$$\boxed{\frac{B_{n,k} - \mathbb{E}(B_{n,k})}{\sqrt{\mathrm{Var}(B_{n,k})}} \xrightarrow{d} N(0,1)}.$$

Digital Trees

Theorem [D.+Szpankowski]

$\boxed{p \neq q}$, $\frac{1}{\log \frac{1}{p}} + \varepsilon \leq \frac{k}{\log n} \leq \frac{1}{\log \frac{1}{q}} - \varepsilon$ (for some $\varepsilon > 0$)

$$\boxed{\mathbb{E}\, B_{n,k} = G\left(\rho_{n,k}, \log_{p/q} p^k n\right) \frac{(p^{-\rho_{n,k}} + q^{-\rho_{n,k}})^k n^{-\rho_{n,k}}}{\sqrt{2\pi\beta(\rho_{n,k})k}} \left(1 + O\left(k^{-1/2}\right)\right)}$$

$G(\rho, x)$ is a non-zero periodic function with period 1, $\rho_{n,k} = \rho(k/\log n)$.

$$\rho(\alpha) = \frac{1}{\log(p/q)} \log \frac{1 - \alpha \log(1/p)}{\alpha \log(1/q) - 1}.$$

$$\beta(\rho) = \frac{p^{-\rho} q^{-\rho} \log(p/q)^2}{(p^{-\rho} + q^{-\rho})^2},$$

Digital Trees

$$\alpha = \frac{k}{\log n}, \qquad \frac{1}{\log \frac{1}{p}} < \alpha < \frac{1}{\log \frac{1}{q}}$$

$$\boxed{\mathbb{E}\, B_{n,k} \approx \frac{n^{\kappa(\alpha)}}{\sqrt{\log n}}}$$

$$\kappa(\alpha) = \alpha \log \left(p^{-\rho(\alpha)} + q^{-\rho(\alpha)} \right) - \rho(\alpha)$$

$$\left[\rho(\alpha) = \frac{1}{\log(p/q)} \log \frac{1 - \alpha \log(1/p)}{\alpha \log(1/q) - 1} \right]$$

Digital Trees

Generating Functions for External Profile

$$P_{n,k}(u) = \mathbb{E}\, u^{B_{n,k}} = \sum_{\ell \geq 0} \mathbb{P}\{B_{n,k} = \ell\}\, u^{\ell}$$

$$\Longrightarrow \quad \boxed{P_{n+1,k+1}(u) = \sum_{\ell=0}^{n} \binom{n}{\ell} p^{\ell} q^{n-\ell} P_{n,\ell}(u) P_{n,n-\ell}(u)}$$

$$G_k(x,u) = \sum_{n \geq 0} P_{n,k}(u) \frac{x^n}{n!}$$

$$\Longrightarrow \quad \boxed{\frac{\partial}{\partial x} G_k(x,u) = G_{k-1}(px,u) G_{k-1}(qx,u)}, \qquad (k \geq 1),$$

$$G_0(x,u) = u + e^x - 1 \text{ and } G_k(0,u) = 1 \ (k \geq 1)$$

Digital Trees

Generating Functions for Internal Profile

$$G_k^{[I]}(x,u) = \sum_{n\ge 0} \mathbb{E}\, u^{I_{n,k}} \frac{x^n}{n!}$$

$$\Longrightarrow \quad \boxed{\frac{\partial}{\partial x} G_k^{[I]}(x,u) = G_{k-1}^{[I]}(px,u)G_{k-1}^{[I]}(qx,u)}, \qquad (k \ge 1),$$

$$G_0^{[I]}(x,u) = 1 + u(e^x - 1) \text{ and } G_k^{[I]}(0,u) = 1 \ (k \ge 1)$$

The analysis of the internal profile is very similar to that of the external one and will not be discussed.

Digital Trees

Generating Functions for External Profile

$$E_k(x) = \sum_{n\ge 0} \mathbb{E}\, B_{n,k} \frac{x^n}{n!} = \left[\frac{\partial G_k(x,u)}{\partial u}\right]_{u=1}$$

$$\boxed{E_k'(x) = e^{qx} E_{k-1}(px) + e^{px} E_{k-1}(qx)}$$

$E_0(x) = 1$ and $E_k(0) = 0$ $(k \ge 1)$

$$\Delta_k(x) := e^{-x} E_k(x)$$

$$\boxed{\Delta_k'(x) + \Delta_k(x) = \Delta_{k-1}(px) + \Delta_{k-1}(qx)}$$

Digital Trees

Generating Functions for External Profile

$$
\begin{aligned}
E_0(x) &= 1,\\
E_1(x) &= \frac{e^{(1-p)x}}{1-p} - \frac{1}{1-p} + \frac{e^{(1-q)x}}{1-q} - \frac{1}{1-q},\\
E_2(x) &= \frac{e^{(1-p^2)x}-1}{(1-p)(1-p^2)} - \frac{e^{(1-p)x}-1}{(1-p)^2} + \frac{e^{(1-pq)x}-1}{(1-q)(1-pq)} - \frac{e^{(1-p)x}-1}{(1-p)(1-q)}\\
&\quad + \frac{e^{(1-pq)x}-1}{(1-p)(1-pq)} - \frac{e^{(1-q)x}-1}{(1-p)(1-q)} + \frac{e^{(1-q^2)x}-1}{(1-q)(1-q^2)} - \frac{e^{(1-q)x}-1}{(1-q)^2}
\end{aligned}
$$

Digital Trees

Mellin transform for $\boxed{\Delta_k(x) := e^{-x} E_k(x)}$

$$\Delta_k^*(s) = \int_0^\infty \Delta_k(x) x^{s-1}\, dx.$$

$$\boxed{\Delta_k^*(s) - (s-1)\Delta_k^*(s-1) = p^{-s}\Delta_{k-1}^*(s) + q^{-s}\Delta_{k-1}^*(s)}$$

Inverse Mellin transform

$$\Delta_k(x) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \Delta_k^*(s) x^{-s}\, ds$$

Digital Trees

"Simplified version"

Original version

$$E_k'(x) = e^{qx}E_{k-1}(px) + e^{px}E_{k-1}(qx)$$

"simplified" to

$$\boxed{E_k(x) = e^{qx}E_{k-1}(px) + e^{px}E_{k-1}(qx)}$$

$\Delta_k(x) = e^{-x}E_k(x),\ \Delta_k^*(s) = \int_0^\infty \Delta_k(x)x^{s-1}\,dx$

$$\Delta_k^*(s) = (p^{-s} + q^{-s})\Delta_{k-1}^*(s)$$

$$\Longrightarrow \quad \boxed{\Delta_k^*(s) = \Gamma(s)(p^{-s} + q^{-s})^k}$$

Digital Trees

"Simplified version"

Inverse Mellin transform for $x = n$

$$\begin{aligned}\Delta_k(n) &= \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \Delta_k^*(s) n^{-s}\, ds \\ &= \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \Gamma(s)\,(p^{-s}+q^{-s})^k n^{-s}\, ds\end{aligned}$$

$$(p^{-s}+q^{-s})^k n^{-s} = e^{k\log(p^{-s}+q^{-s}) - s\log n}$$

Saddle point:

$$\frac{\partial}{\partial s}\Big(k\log(p^{-s}+q^{-s}) - s\log n\Big) = 0$$

Digital Trees

"Simplified version"

... infinitely many saddle points for on the line $\Re(s) = \rho_{n,k} = \rho(k/\log n)$:

$$\Longrightarrow \quad \boxed{s_j = \rho_{n,k} + \frac{2\pi i j}{\log \frac{p}{q}}}$$

... with usual saddle point analysis:

$$\Longrightarrow \quad \boxed{\Delta_k(n) \sim G\left(\rho_{n,k}, \log_{p/q} p^k n\right) \frac{(p^{-\rho_{n,k}} + q^{-\rho_{n,k}})^k n^{-\rho_{n,k}}}{\sqrt{2\pi\beta(\rho_{n,k})k}}}$$

Recall: $\mathbb{E}\, B_{n,k} \sim \Delta_k(n)$ (by the Poisson heuristics)

Digital Trees

Mellin transform (for the original problem)

$$\Delta_k^*(s) = \Gamma(s) F_k(s),$$

$$\boxed{F_k(s) - F_k(s-1) = (p^{-s} + q^{-s}) F_{k-1}(s)}$$

$$F_0(x) = 1,$$

$$F_1(x) = \frac{p^{-s}}{1-p} - \frac{1}{1-p} + \frac{q^{-s}}{1-q} - \frac{1}{1-q},$$

$$F_2(x) = \frac{p^{-2s}-1}{(1-p)(1-p^2)} - \frac{p^{-s}-1}{(1-p)^2} + \frac{p^{-s}q^{-s}-1}{(1-q)(1-pq)} - \frac{p^{-s}-1}{(1-p)(1-q)}$$
$$+ \frac{p^{-s}q^{-s}-1}{(1-p)(1-pq)} - \frac{q^{-s}-1}{(1-p)(1-q)} + \frac{q^{-2s}-1}{(1-q)(1-q^2)} - \frac{q^{-s}-1}{(1-q)^2}$$

Digital Trees

Remark.

The Mellin transform $\Delta_k^*(s)$ exists for $\Re(s) > -k$

$$\Delta_k^*(s) = \Gamma(s) F_k(s)$$

$$\Longrightarrow \quad \boxed{F_k(0) = 0} \quad (k > 0)$$

Digital Trees

A linear operator

Set $\boxed{T(s) = p^{-s} + q^{-s}}$ and define

$$\boxed{\mathbf{A}[f](s) = \sum_{j\ge 0} f(s-j)T(s-j)}$$

Furthermore set $\boxed{R_k(s) = \mathbf{A}^k[1](s)}$:

$$
\begin{aligned}
R_0(s) &= 1, \\
R_1(s) &= \frac{p^{-s}}{1-p} + \frac{q^{-s}}{1-q}, \\
R_2(s) &= \frac{p^{-2s}}{(1-p)(1-p^2)} + \frac{p^{-s}q^{-s}}{(1-p)(1-pq)} \\
&\quad + \frac{p^{-s}q^{-s}}{(1-q)(1-pq)} + \frac{q^{-2s}}{(1-q)(1-q^2)}.
\end{aligned}
$$

Digital Trees

Lemma 1

$$\boxed{F_k(s) = \mathbf{A}[F_{k-1}](s) - \mathbf{A}[F_{k-1}](0)}$$

$$\boxed{\sum_{k\geq 0} F_k(s) w^k = \frac{\sum_{\ell\geq 0} R_\ell(s) w^\ell}{\sum_{\ell\geq 0} R_\ell(0) w^\ell}}$$

Digital Trees

Proof

One has to show

$$\sum_{\ell=0}^{k} F_\ell(s) R_{k-\ell}(0) = R_k(s), \qquad (k \geq 0),$$

or equivalently

$$F_k(s) = R_k(s) - \sum_{\ell=0}^{k-1} F_\ell(s) R_{k-\ell}(0), \qquad (k \geq 0).$$

Digital Trees

Proof

The case $k = 0$ is obvious.

General induction step:

$$\begin{aligned} F_{k+1}(s) &= \mathbf{A}[F_k](s) - \mathbf{A}[F_k](0) \\ &= \mathbf{A}[R_k](s) - \mathbf{A}[R_k](0) \\ &\quad - \sum_{\ell=0}^{k-1} (\mathbf{A}[F_\ell](s) - \mathbf{A}[F_\ell](0)) R_{k-\ell}(0) \\ &= R_{k+1}(s) - R_{k+1}(0) - \sum_{\ell=0}^{k-1} F_{\ell+1}(s) R_{k-\ell}(0) \\ &= R_{k+1}(s) - \sum_{\ell=0}^{k} F_\ell(s) R_{k+1-\ell}(0). \end{aligned}$$

Digital Trees

$$g(w,s) = \sum_{\ell \geq 0} R_\ell(s) w^\ell$$

$$\boxed{g(w,s) = 1 + w\mathbf{A}[g(w,\cdot)](s) = 1 + \sum_{j \geq 0} g(w,s-j)T(s-j)}$$

Digital Trees

Lemma 2

$$\boxed{g(w,s) = \frac{h(w,s)}{1 - wT(s)}}$$

with

$$h(w,s) = 1 + \sum_{j\ge 1} h(w,s-j)\frac{wT(s-j)}{1-wT(s-j)}.$$

which is analytic for $w = T(s)$.

Digital Trees

Corollary

$$\boxed{F_k(s) = f(s)T(s)^k \left(1 + O\left(e^{-\eta k}\right)\right)}$$

$\longrightarrow F_k(s)$ behaves as in the "simplified" case. Hence, the inverse Mellin transform (with infinitely many saddle points) works and the Poisson heuristics applies as well. *QED*

Thank You!