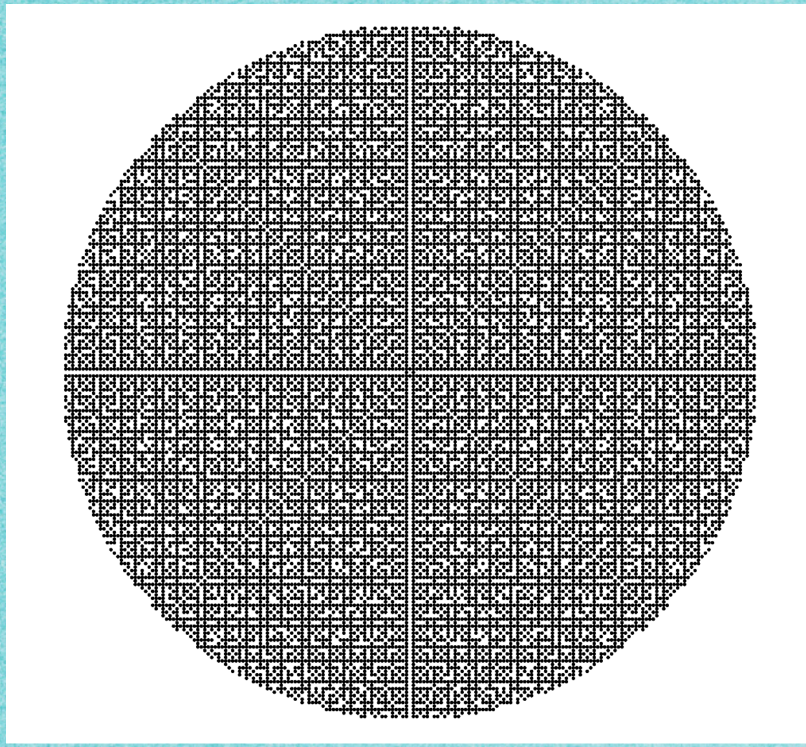
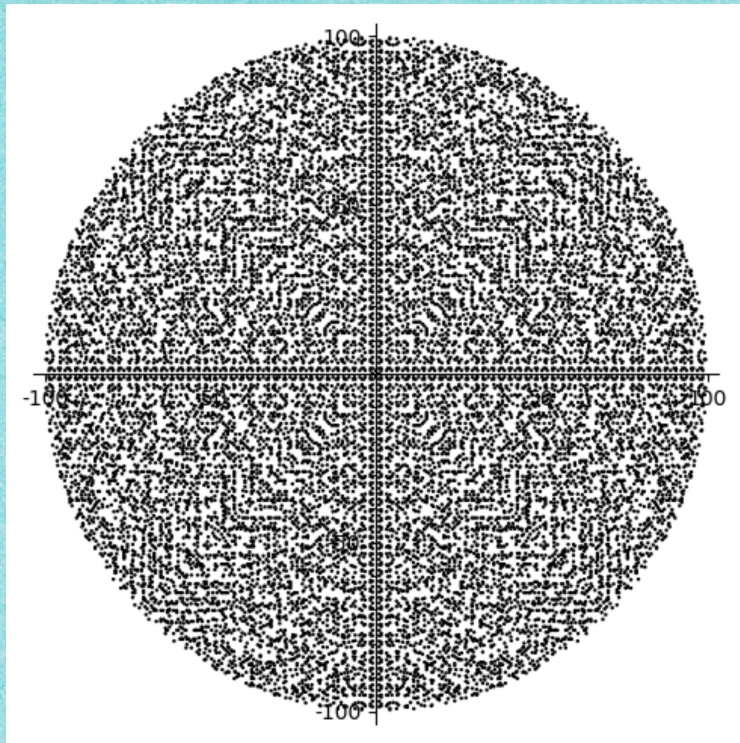
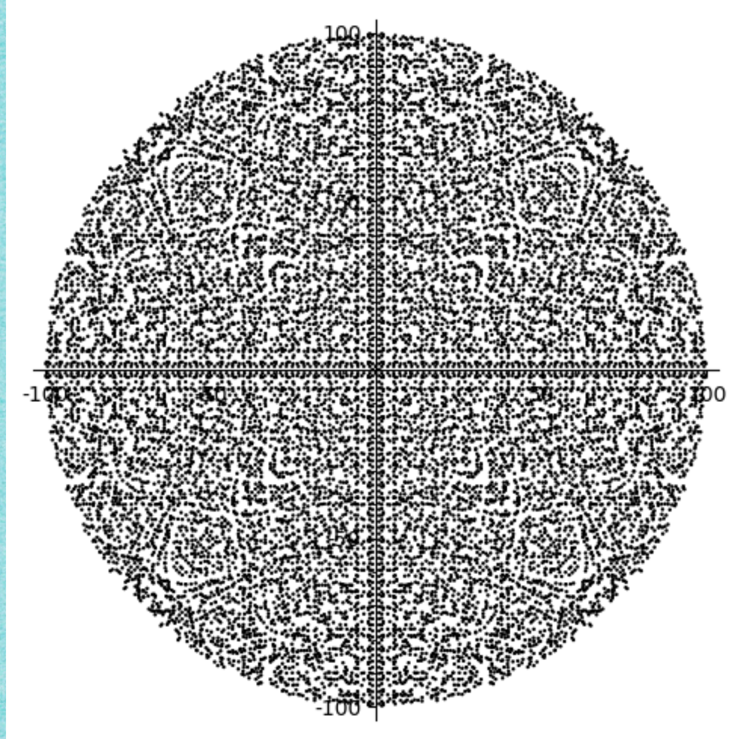






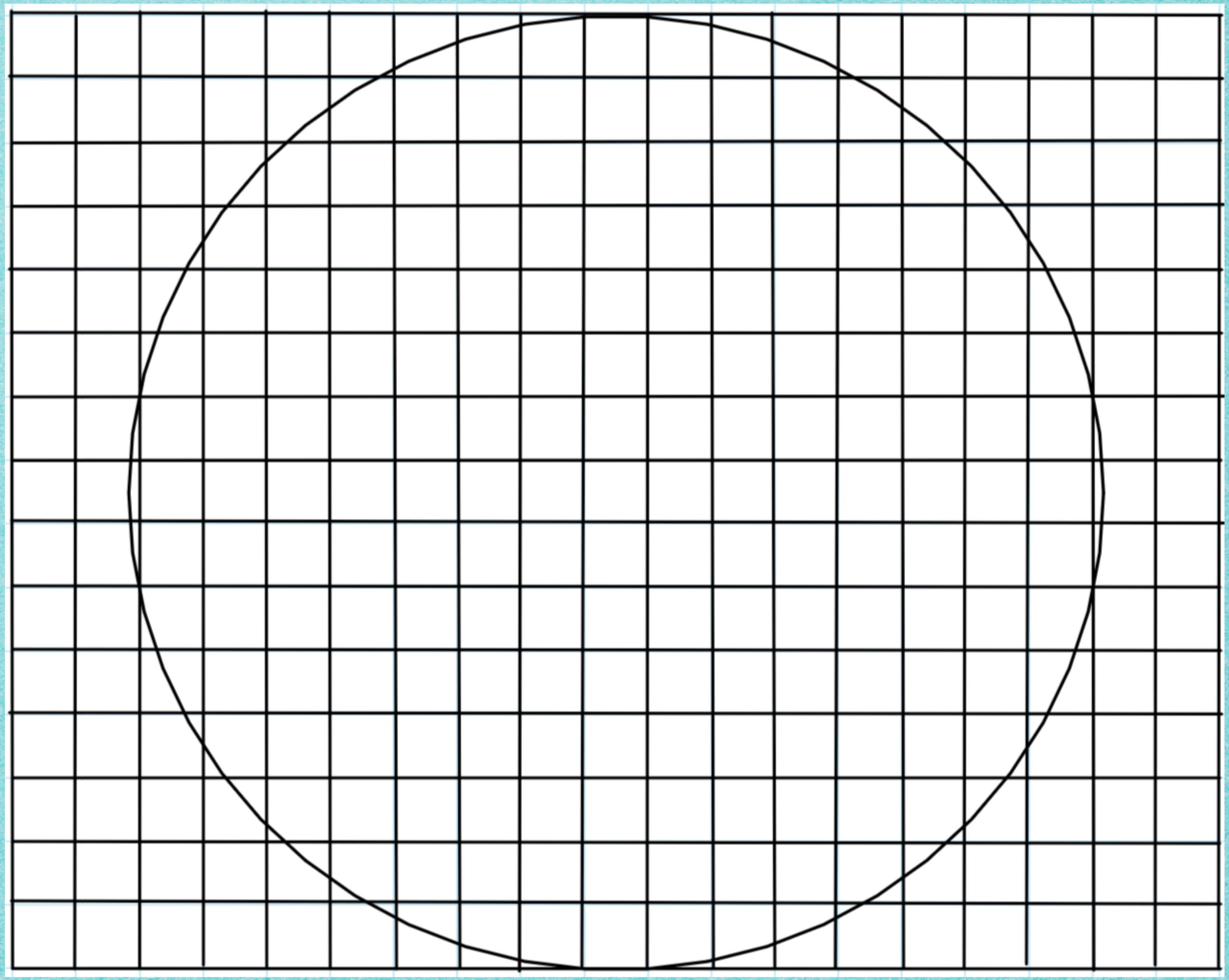
Counting K-tuples of closed geodesics on translation surfaces

Sam Fairchild
Max Planck Institute for Mathematics in the Sciences

Gauss Circle Problem

Want to understand

$$|\mathbb{Z}^2 \cap B(0,R)|$$



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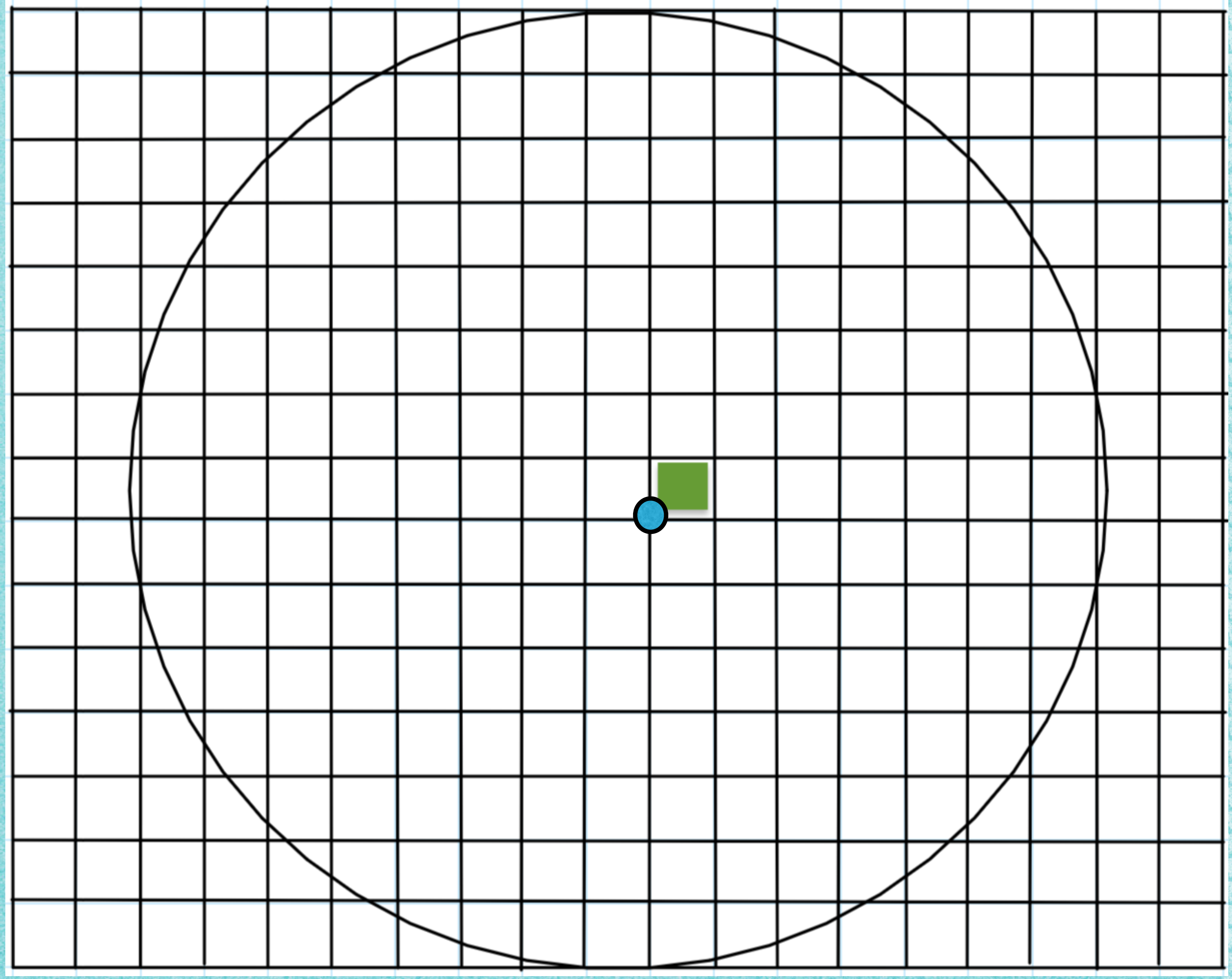
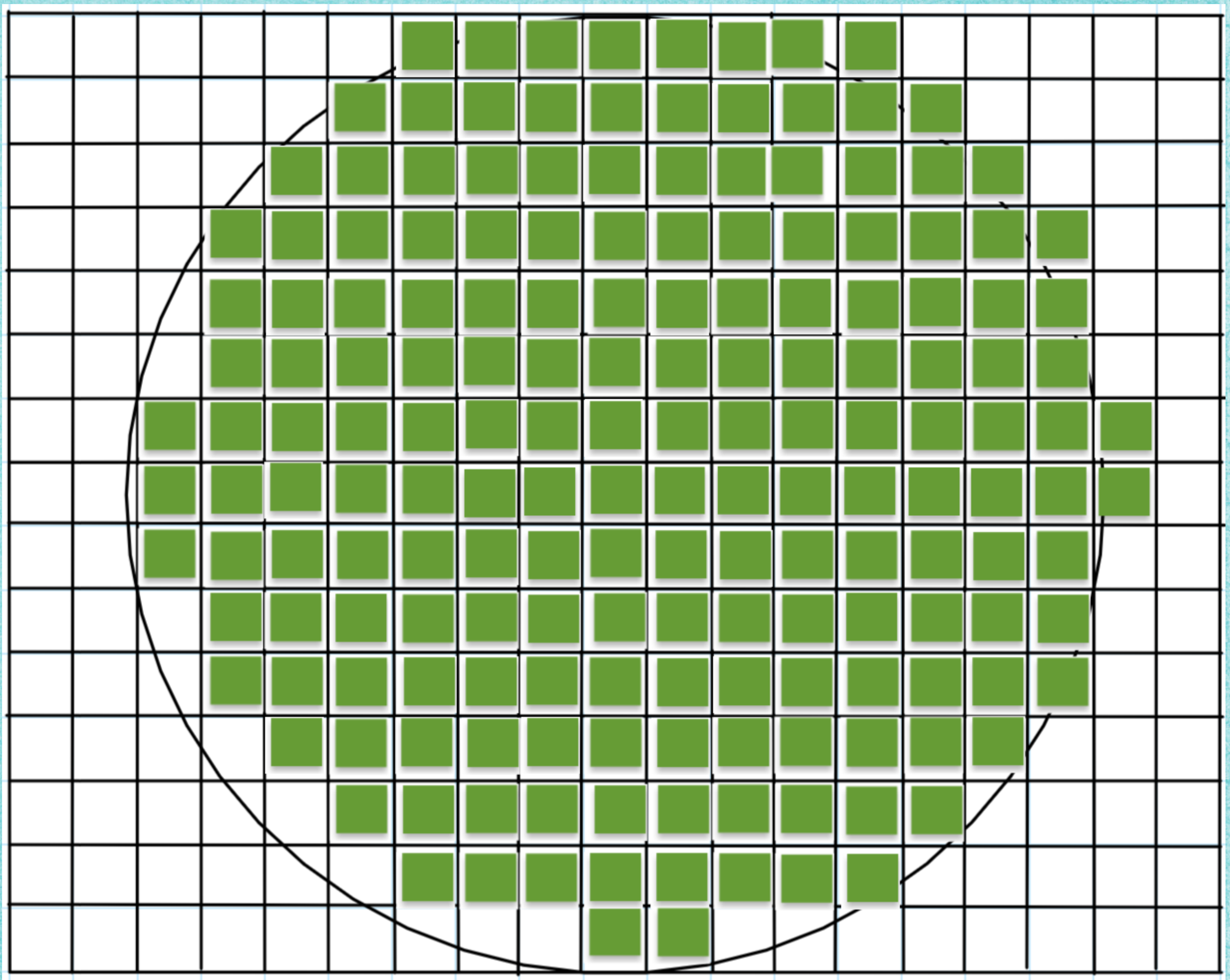


Gauss Circle Problem

Want to understand

$$|\mathbb{Z}^2 \cap B(0,R)|$$

$$\approx \pi R^2$$



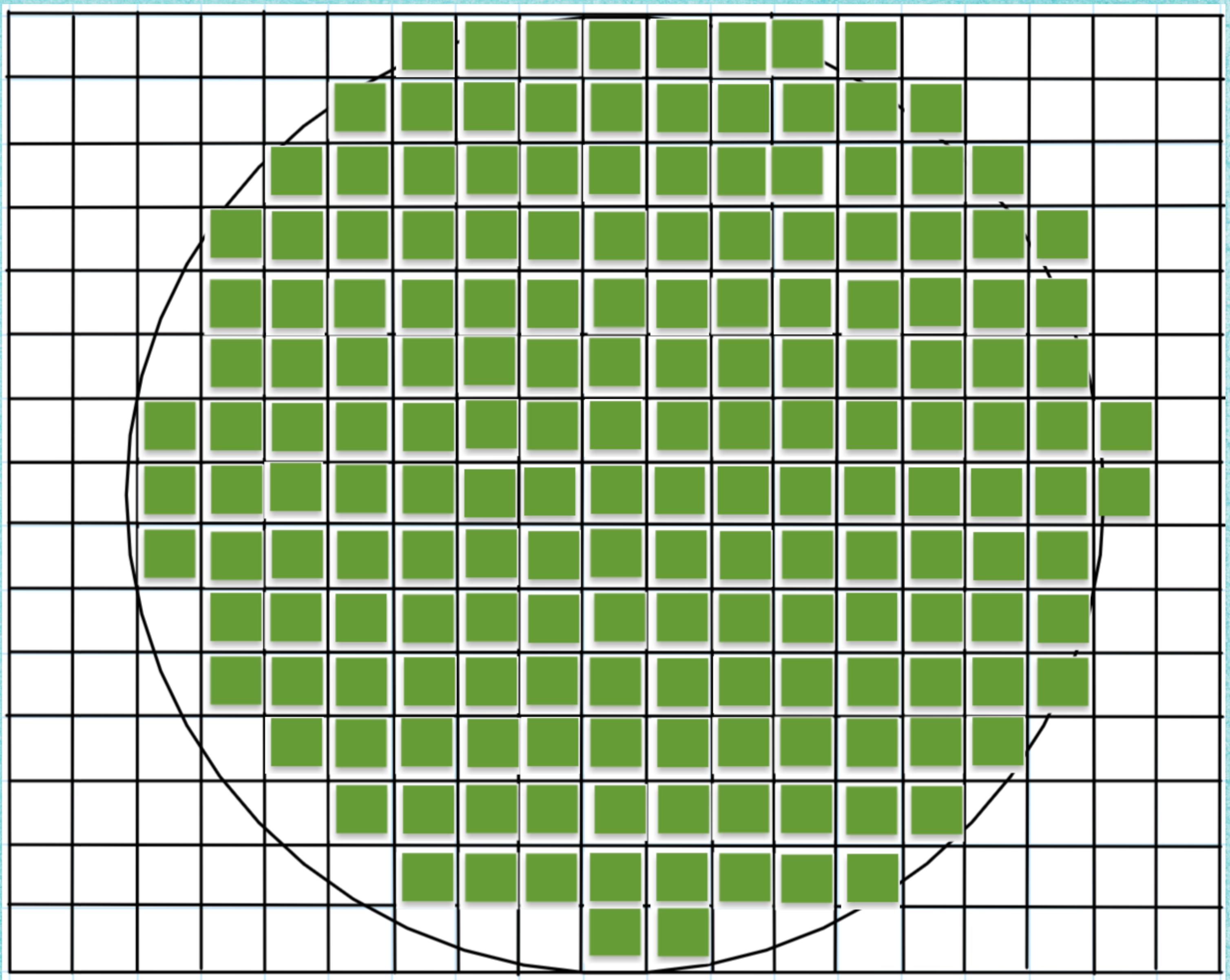
Gauss Circle Problem

Second Moment Information

$$\Big| \,|\mathbb{Z}^2 \cap B(0,R)| - \pi R^2 \Big| = O(R^\alpha)$$

Conjecture

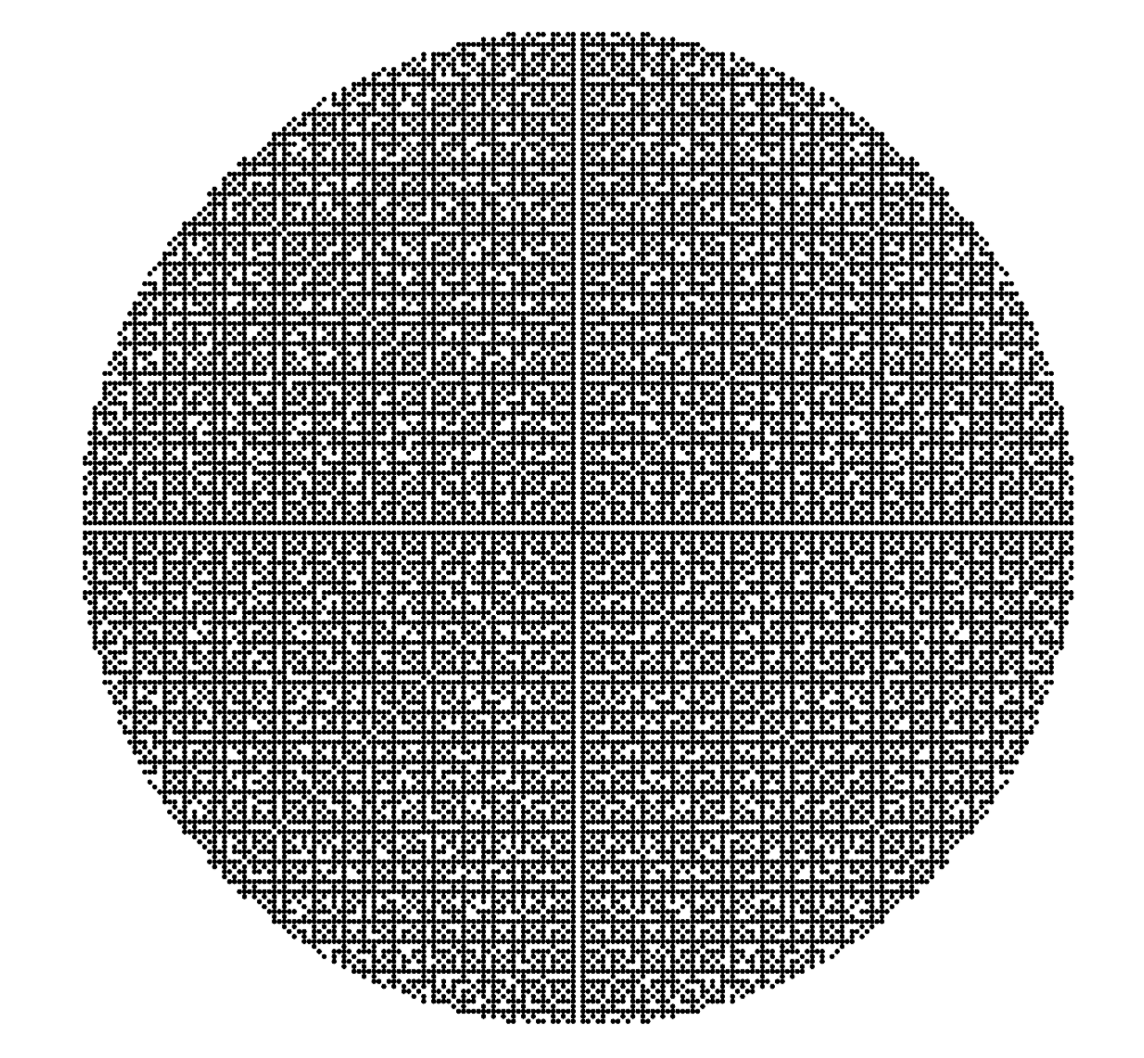
$$\alpha = \frac{1}{2} + \epsilon$$



Primitive Gauss Circle Problem

Want to understand

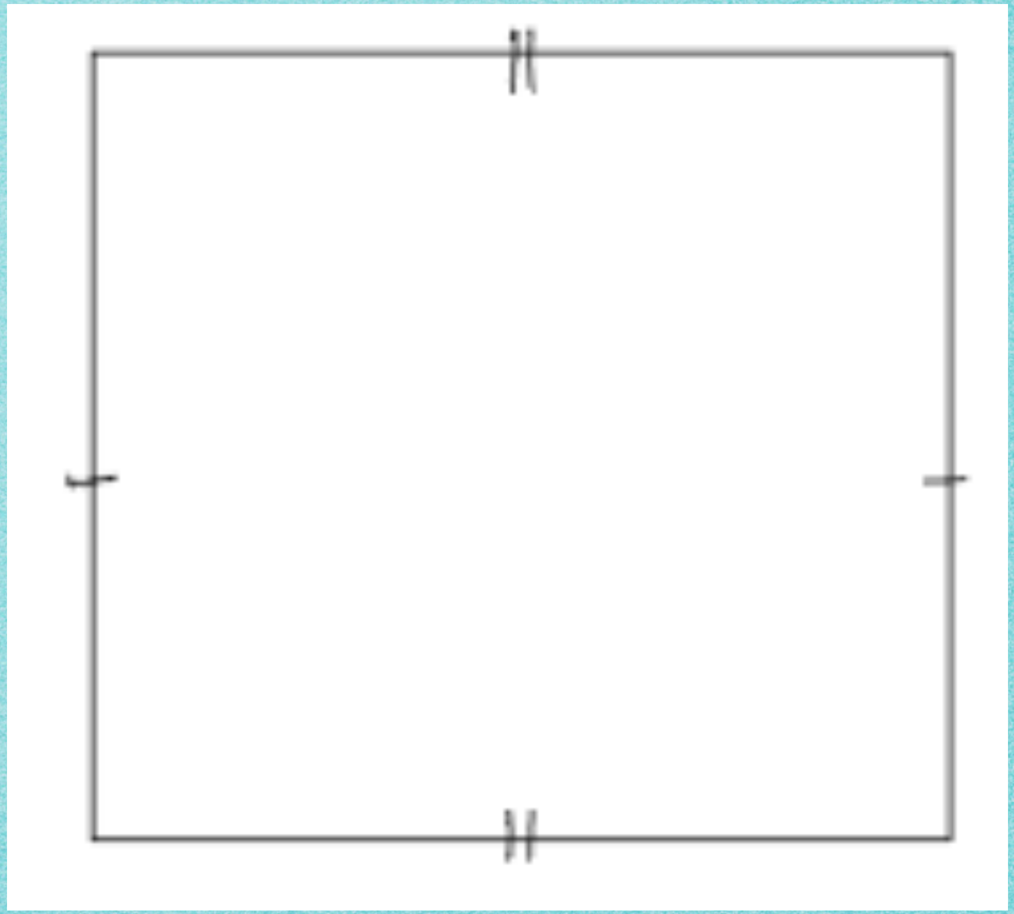
$$|\mathbb{Z}^2_{prim} \cap B(0,R)| \approx \frac{6}{\pi^2}\pi R^2$$



$$\mathbb{Z}^2_{prim} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2 : \gcd(a,b) = 1 \right\} = SL(2,\mathbb{Z}) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Saddle connections on Torus

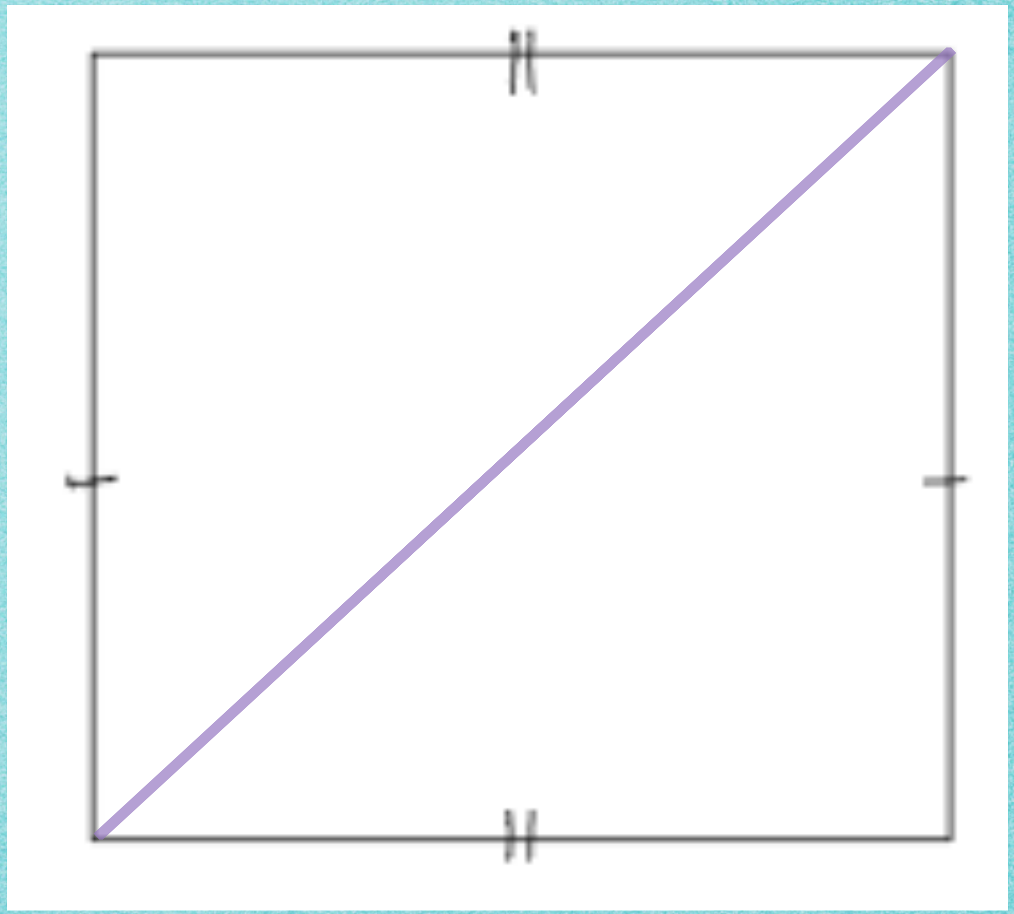
A ***saddle connection*** *is the integral of a simple closed geodesic which starts and ends at a vertex.*



Saddle connections on Torus

*A **saddle connection** is the integral of a simple closed geodesic which starts and ends at a vertex.*

$$\vec{v}_1 = \int_\gamma dz = 1 + i$$

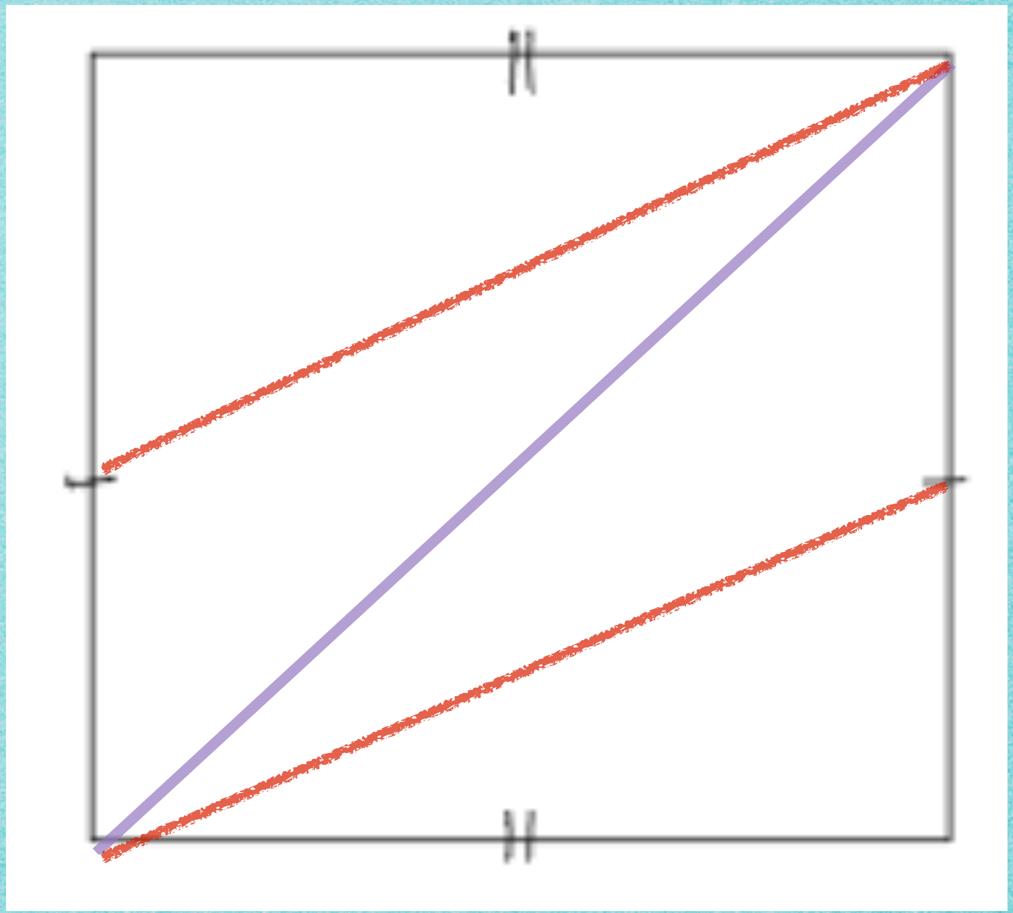


Saddle connections on Torus

*A **saddle connection** is the integral of a simple closed geodesic which starts and ends at a vertex.*

$$\vec{v}_1 = \int_{\gamma_1} dz = 1 + i$$

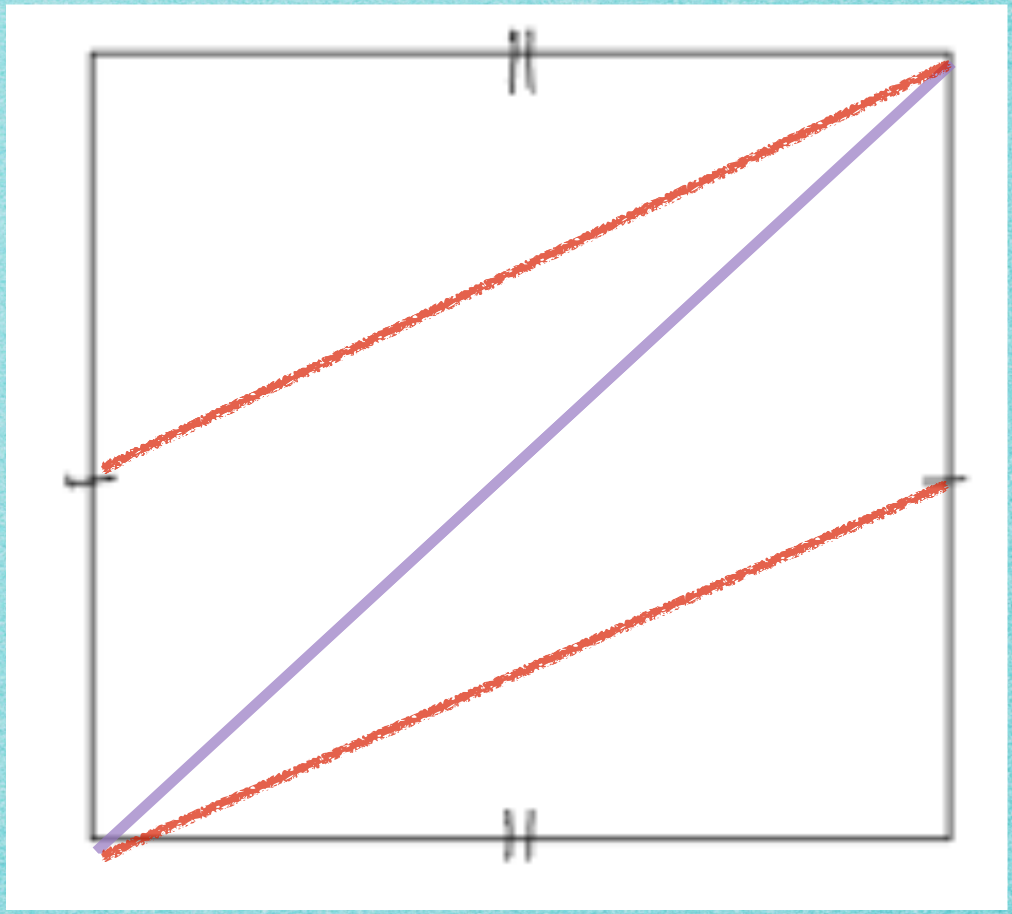
$$\vec{v}_2 = \int_{\gamma_2} dz = 2 + i$$



Saddle connections on Torus

The set of saddle connections on the torus are

$$SL(2,\mathbb{Z}) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbb{Z}^2_{prim}$$



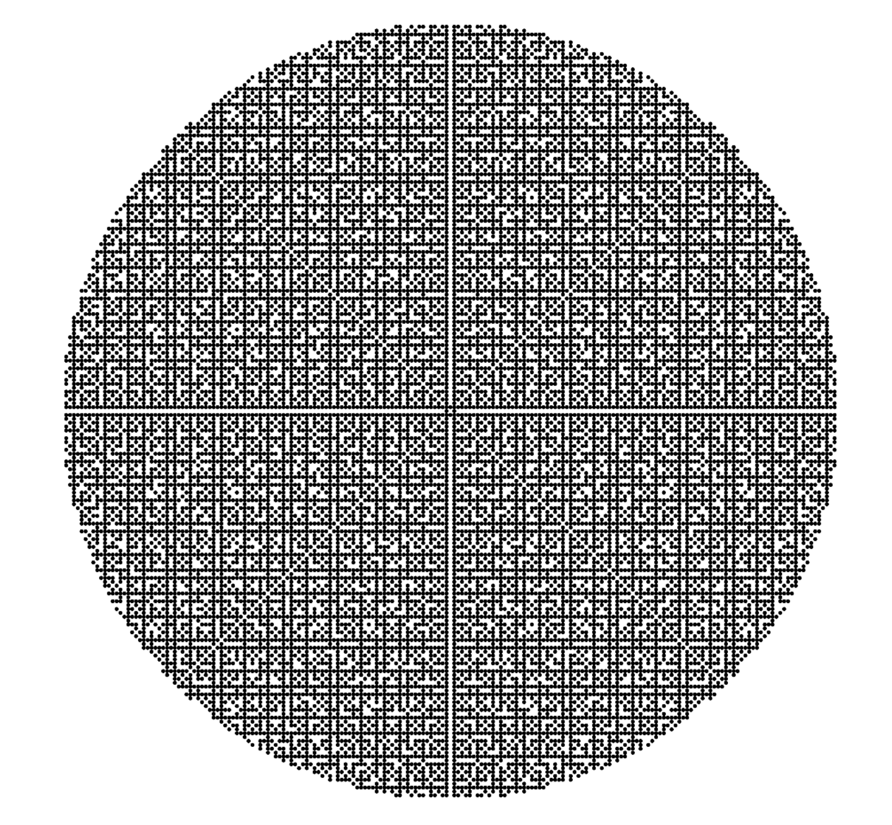
Primitive Gauss circle problem = counting saddle connections on torus

- Expected Value (Siegel '45):

$$\mathbb{E}\left(\left\{\overrightarrow{v} \in gSL(2,\mathbb{Z})e_1 : \|\overrightarrow{v}\|_2 < R\right\}\right) \sim \frac{6}{\pi^2}\pi R^2$$

- Second Moment (Schmidt '60)

$$\mathbb{E}\left(\left\{\overrightarrow{v} \in gSL(2,\mathbb{Z})e_1 : \|\overrightarrow{v}\|_2 < R\right\}\right) = \frac{6}{\pi^2}\pi R^2 + O\left(R^{1+\epsilon}\right)$$



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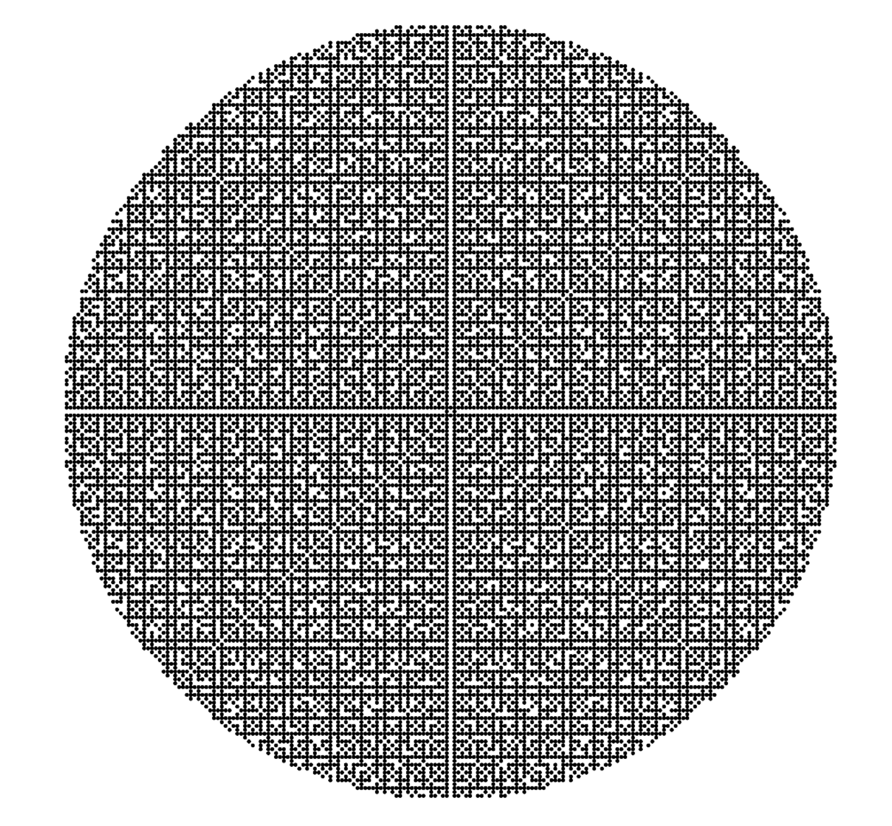
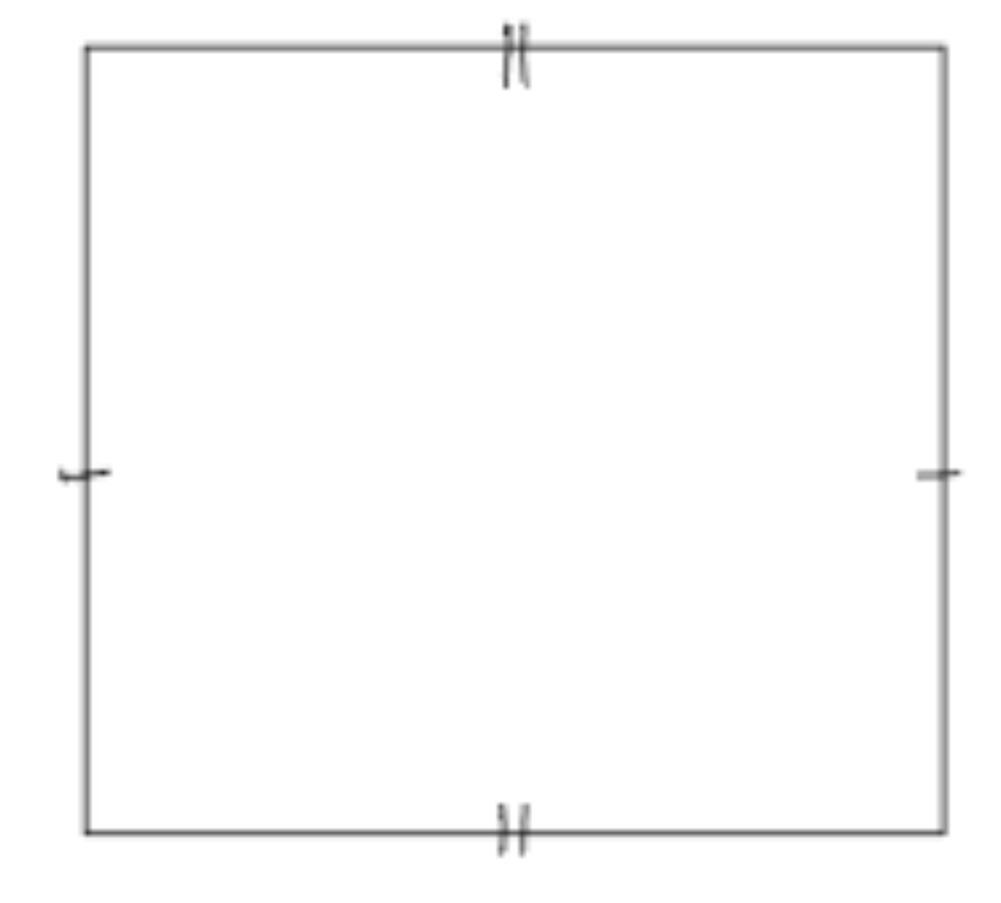


$\mathbb{E}$ is expected value over probability space $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$

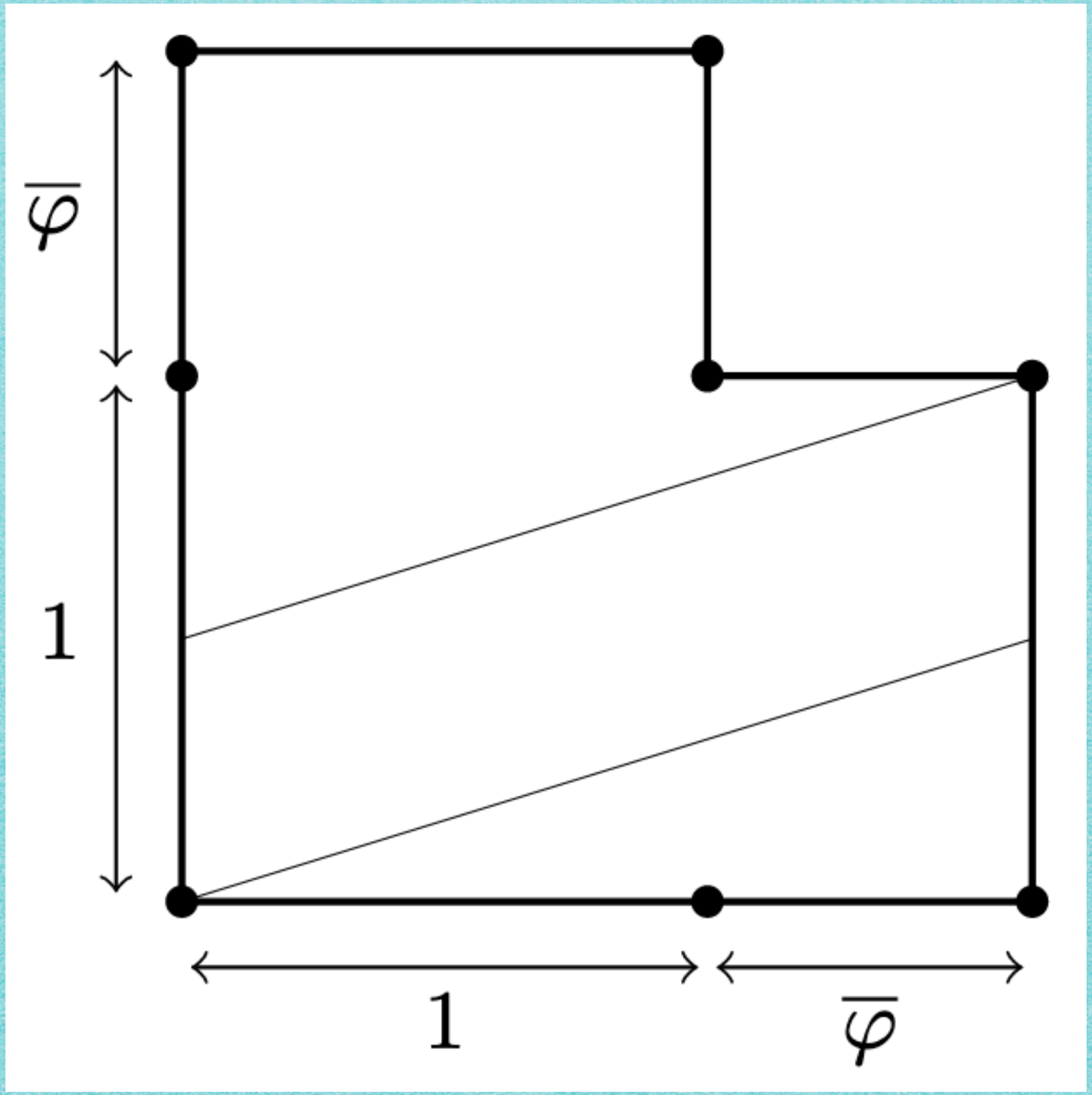
Translation Surface

A polygon with:

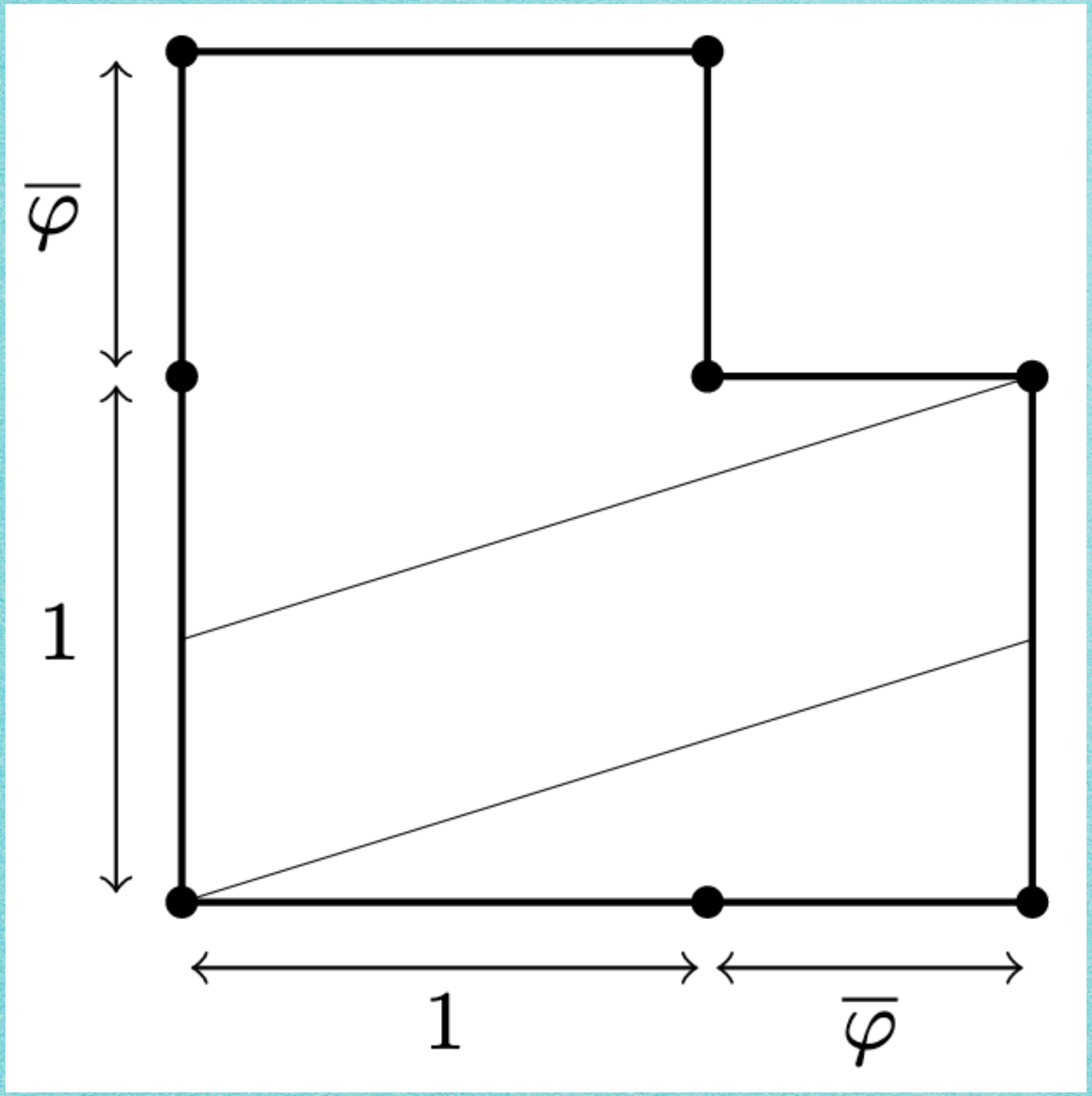
- *Opposite sides identified by translation*
- *Up to cut and paste relationship*



The Golden L



The Golden L



$$2 - 2g = V - E + F$$

$$= 1 - 4 + 1$$

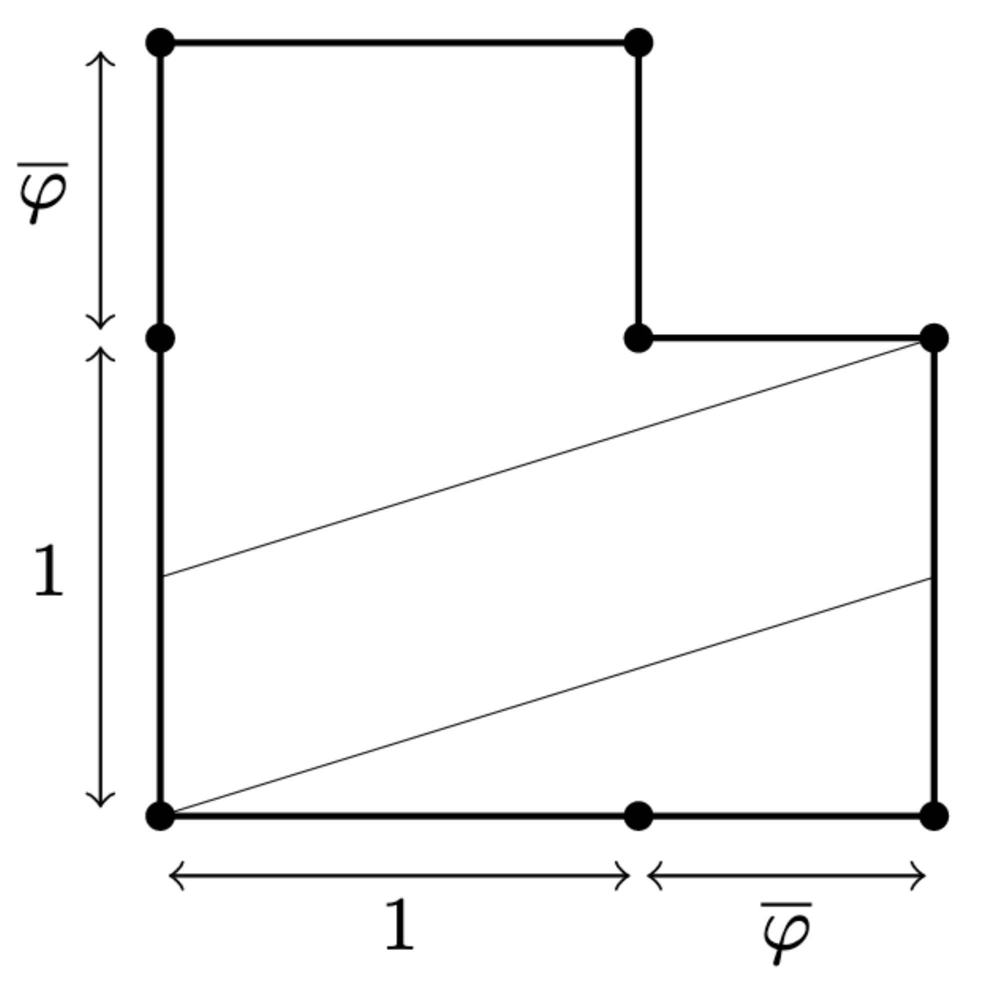
$$\Rightarrow g = 2$$

Saddle connections on Golden L

The set of saddle connections on the Golden L are given by

$$\Gamma e_1 \cup \overline{\varphi}\Gamma e_1$$

$$\Gamma = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2\cos(\pi/5) \\ 0 & 1 \end{bmatrix} \right\rangle$$



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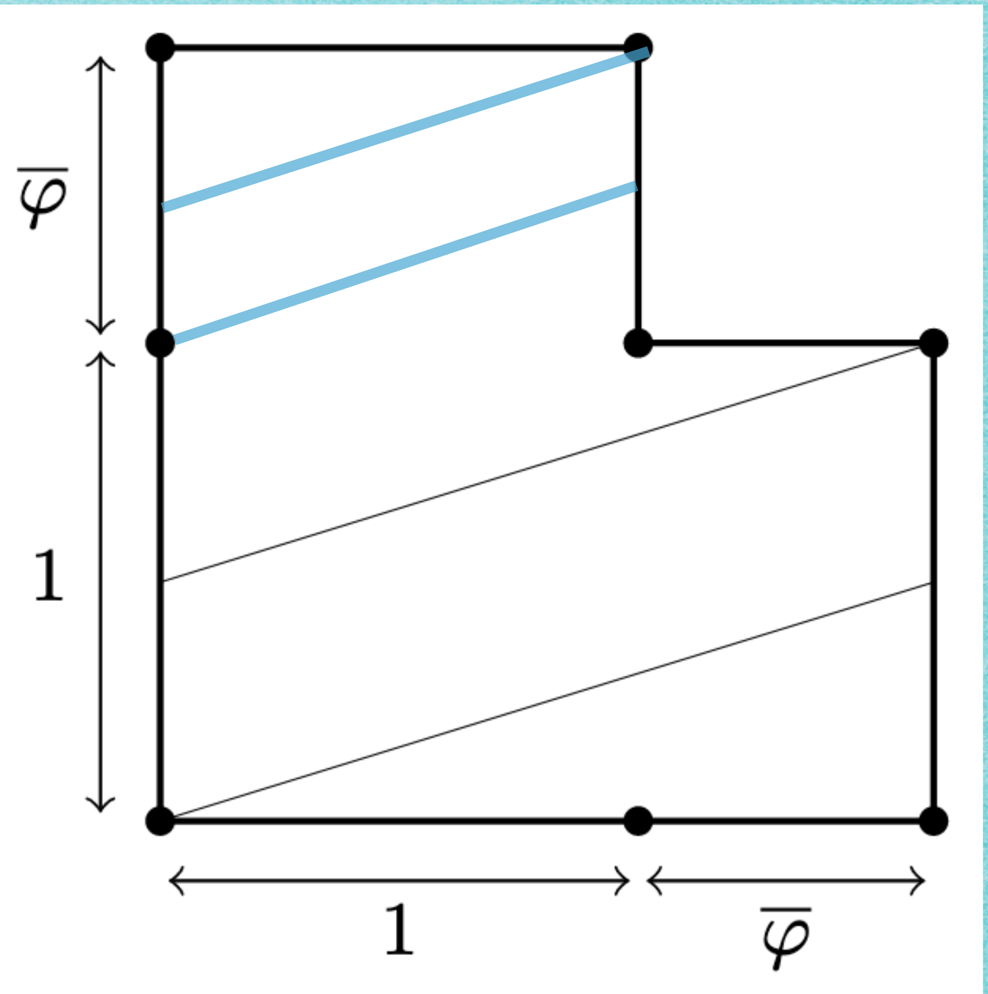

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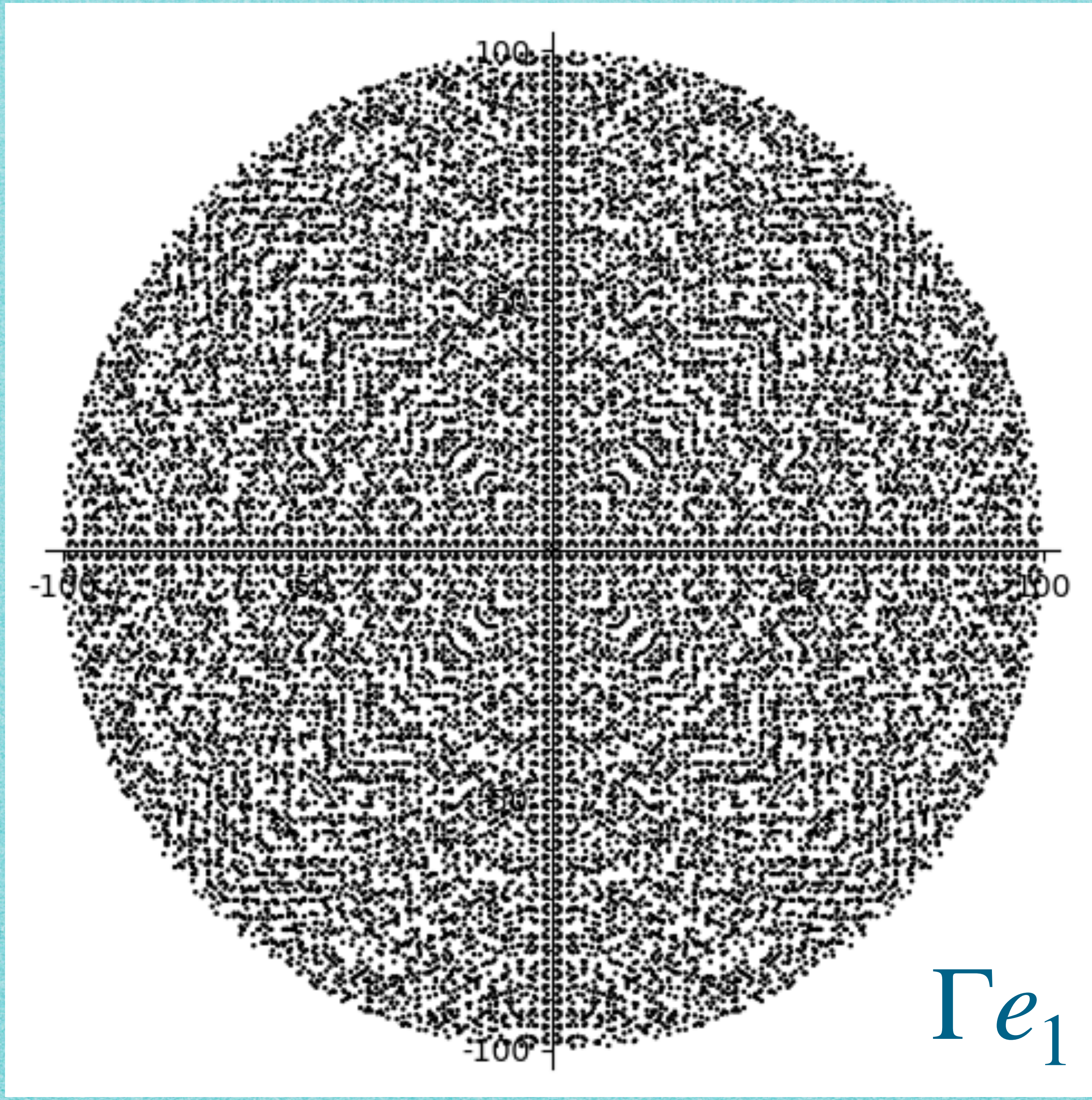
Saddle connections on Golden L

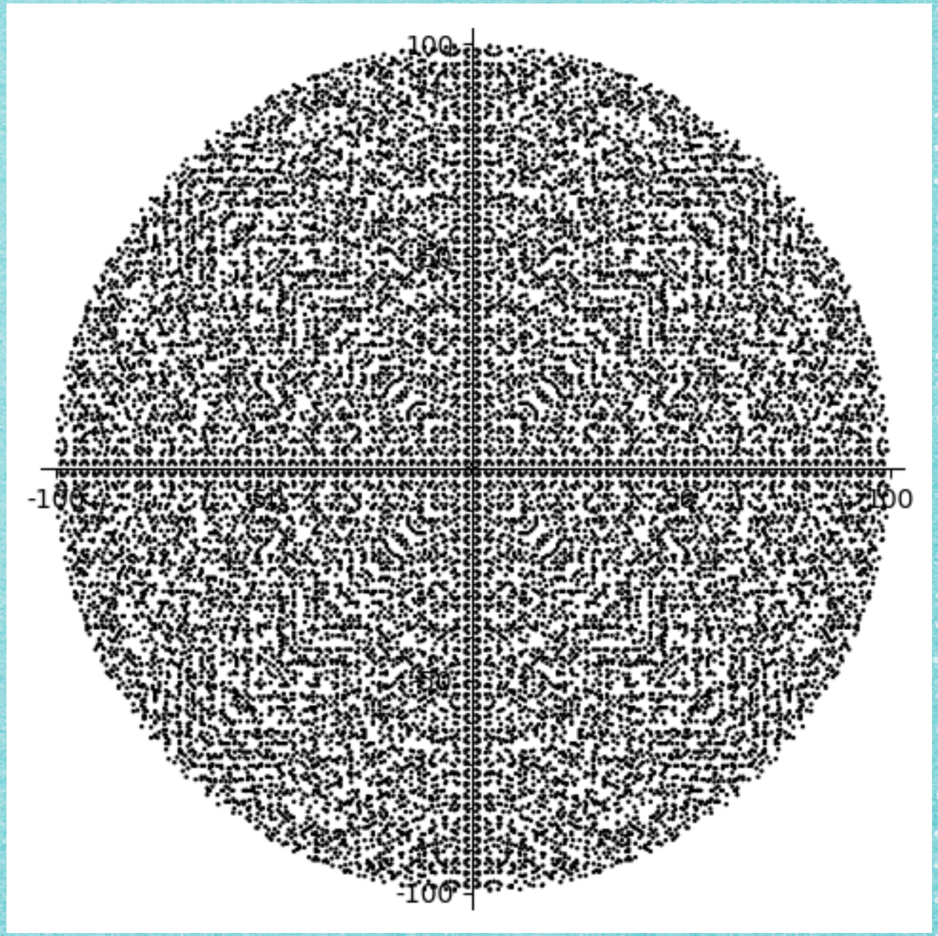


Saddle connections on the Golden L

- Expected Value (Veech '98):

$$\mathbb{E}\left(\left\{\overrightarrow{v} \in g\Gamma e_1 : \|\overrightarrow{v}\|_2 < R\right\}\right) \sim \frac{10}{3\pi^2}\pi R^2$$

-



$\mathbb{E}$ = expected value over $SL_2(\mathbb{R})/\Gamma$

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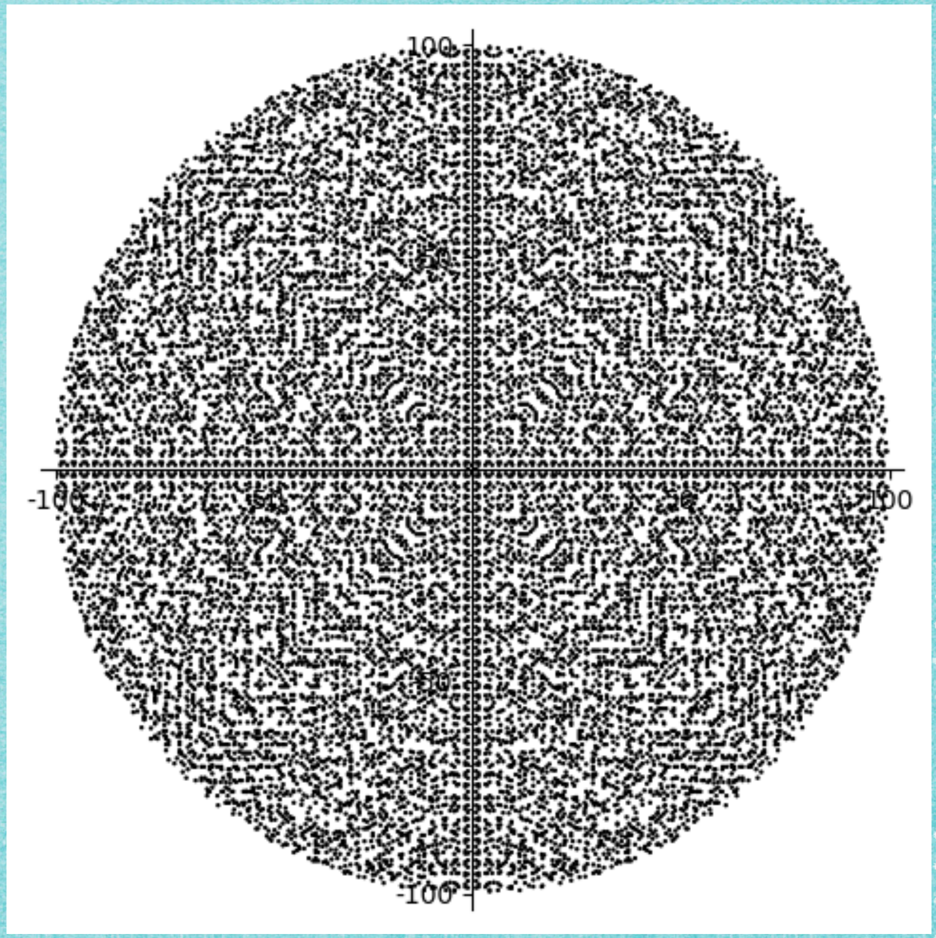
- Second Moment (BNRW '19, BF (in preparation))

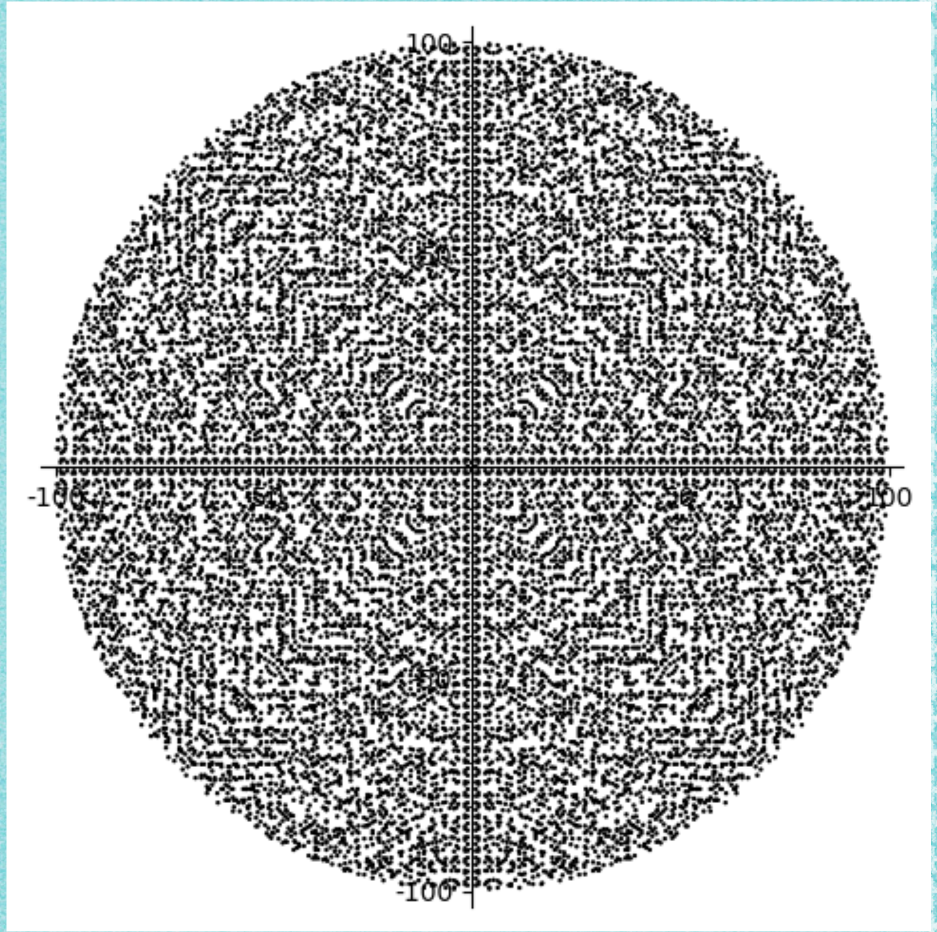
$$\mathbb{E}\left(\left\{\overrightarrow{v} \in g\Gamma e_1 : \|\overrightarrow{v}\|_2 < R\right\}\right) = \frac{10}{3\pi^2}\pi R^2 + O\left(R^{1+1/3}\right)$$



$\mathbb{E}$ = expected value over $SL_2(\mathbb{R})/\Gamma$

Pair Correlations for closed geodesics on the Golden L

- (Burrin—F (in preparation))



$$\exists C \text{ s.t. } \forall \epsilon > 0, \limsup_{R \to \infty} \frac{|\vec{v} \in \Gamma \cdot e_1 \cap B_R : \exists \vec{w} \in B_\epsilon(\vec{v})|}{|\Gamma \cdot e_1 \cap B_R|} \leq C |B_\epsilon|$$

Counting K-tuples

Let $f \in B_c((\mathbb{R}^2)^k)$

for $g \in SL_2(\mathbb{R})/\Gamma$ define the SV transform

$$f_{SV}(g) = \sum_{v_1,\ldots,v_k \in \Gamma \cdot e_1} f(gv_1, \ldots, gv_k)$$

Counting 1-tuples

Let $f \in B_c((\mathbb{R}^2)^k)$

for $g \in SL_2(\mathbb{R})/\Gamma$ define the SV transform

$$f_{SV}(g) = \sum_{v_1,\ldots,v_k \in \Gamma \cdot e_1} f(gv_1, \ldots, gv_k)$$

Mean Value Formula (Veech '98) $k = 1$

$$\int_{SL_2(\mathbb{R})/\Gamma} f_{SV}(g)\, d\mu(g) = \frac{1}{vol(SL_2(\mathbb{R})/\Gamma)} \int_{\mathbb{R}^2} f(x)\, dx$$

Counting 2-tuples

Higher Moment Formula (F '21) $k = 2$

$$\int_{SL_2(\mathbb{R})/\Gamma} f_{SV}(g)\, d\mu(g)$$

$$= \sum_{n \in \mathcal{N}} \frac{\varphi(n)}{vol(SL_2(\mathbb{R})/\Gamma)} \int_{SL_2(\mathbb{R})} f\left(g \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\right) dg$$

$$+ \frac{2}{vol(SL_2(\mathbb{R})/\Gamma)} \int_{\mathbb{R}^2} f(x, x) + f(x, -x)\, dx$$

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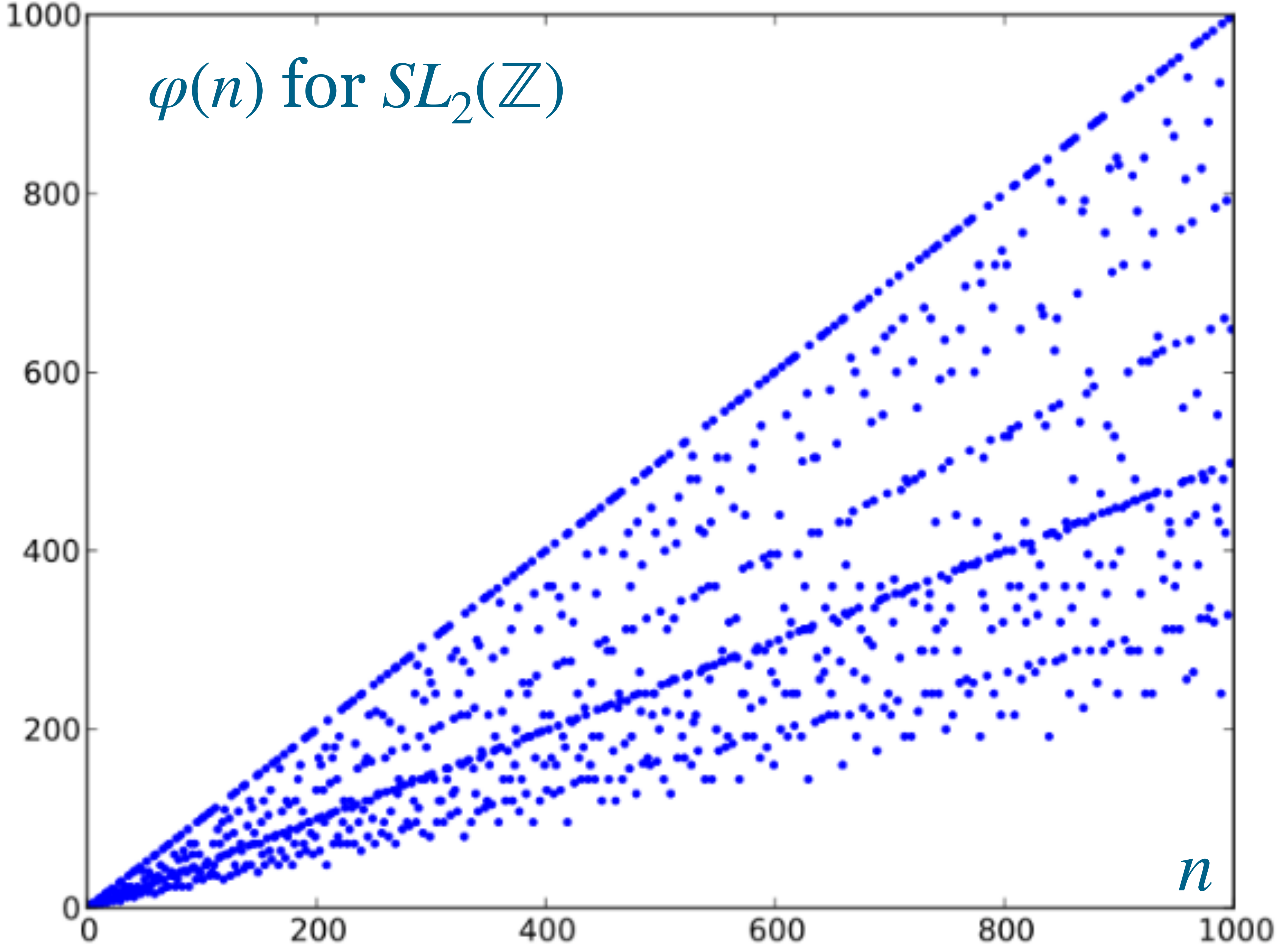
$\mathcal{N}$ = set of possible determinants

$\varphi(n)$ = #Γ-orbits on determinant n subset of $(\Gamma \cdot e_1)^2$

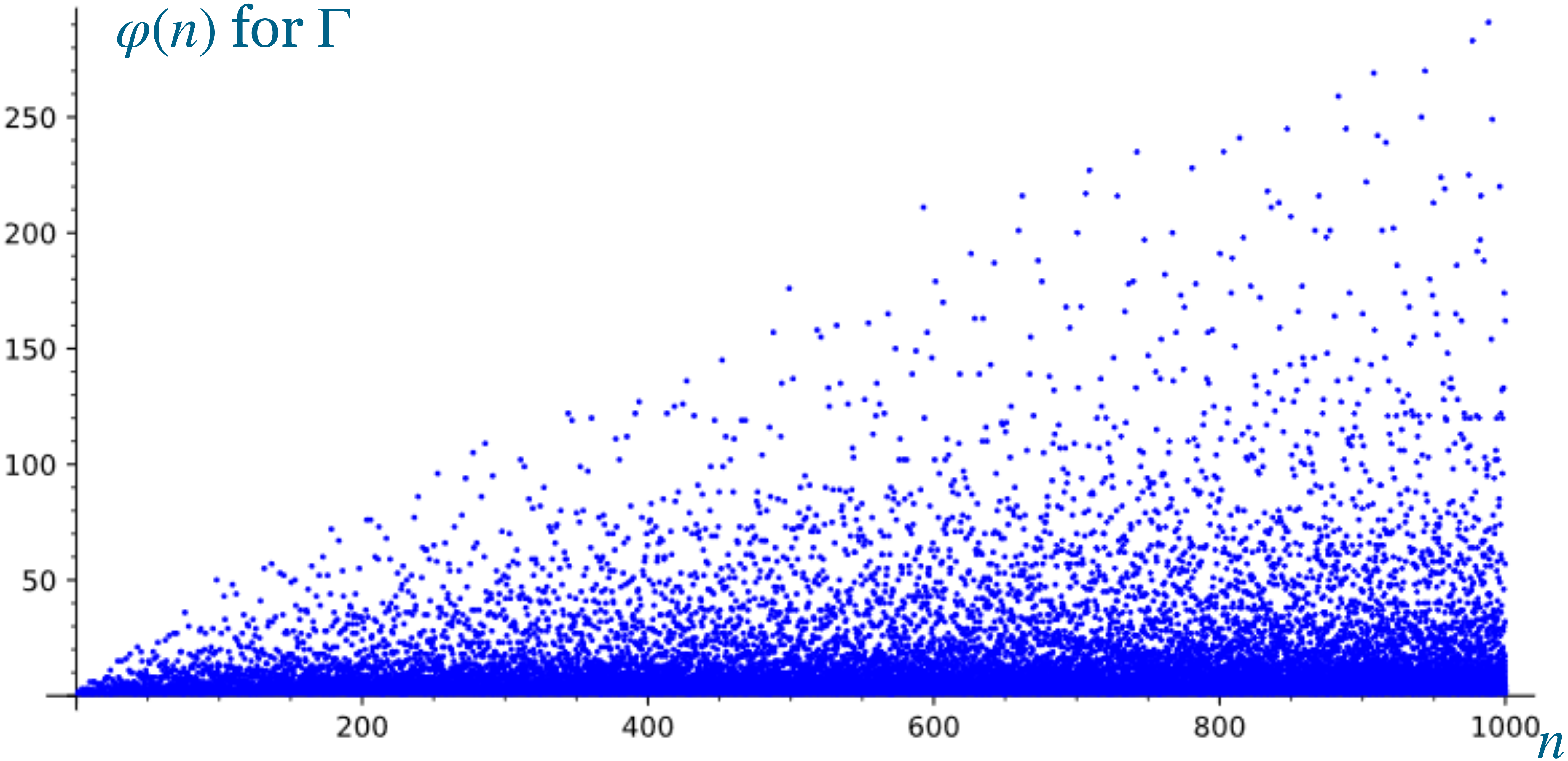
Understanding orbit counting for k=2

$\varphi(n) = \#\Gamma$-orbits on determinant n subset of $(\Gamma \cdot e_1)^2$

$$= \left\{ 0 < a \leq |n| : \begin{pmatrix} a \\ n \end{pmatrix} \in \Gamma \cdot e_1 \right\}$$



$\varphi(n)$ for $SL_2(\mathbb{Z})$
n
0
200
400
600
800
1000


$\varphi(n)$ for Γ
250
200
150
100
50
200
400
600
800
1000
n

Understanding orbit counting for k=2

$\varphi(n) = \#\Gamma$-orbits on determinant n subset of $(\Gamma \cdot e_1)^2$

$$= \left\{ 0 < a \leq |n| : \begin{pmatrix} a \\ n \end{pmatrix} \in \Gamma \cdot e_1 \right\}$$

Counting double cosets

$$\sum_{\substack{0 \leq n \leq N \\ n \in \mathcal{N}}} \varphi(n) = \frac{1}{\pi \cdot vol} R^2 + \sum_{j=1}^{k} \tau_j T^{2\rho_j} + O(R^{4/3})$$

Understanding orbit counting for k=3

$$\int_{SL_2(\mathbb{R})/\Gamma} f_{SV}\, d\mu(g)$$

$$= \sum_{\lambda \in \{(1,\pm 1,\pm 1)\}} \frac{1}{vol} \int_{\mathbb{R}^2} f(\lambda x)\, dx$$

$$+ \sum_{n \in \mathcal{N}} \sum_{1 \le m \le |n|} \sum_{\lambda = \pm 1} \frac{1}{vol} \int_{SL_2(\mathbb{R})} f\left(ge_1, g\lambda e_1, g \begin{pmatrix} m \\ n \end{pmatrix} \right) dg$$

$$+ \sum_{n \in \mathcal{N}} \sum_{1 \le m \le |n|} \sum_{\alpha,\beta} \frac{1}{vol} \int_{SL(2,\mathbb{R})} f\left(ge_1, g \begin{pmatrix} m \\ n \end{pmatrix}, g \begin{pmatrix} \alpha + m\beta \\ n\beta \end{pmatrix} \right) dg$$

where we have $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in J_{n,m}^{-1} \Gamma e_1$.

Understanding orbit counting for k=3

$$\int_{SL_2(\mathbb{R})/\Gamma} f_{SV}\, d\mu(g)$$

$$= \sum_{\lambda\in\{(1,\pm1,\pm1)\}} \frac{1}{vol}\int_{\mathbb{R}^2} f(\lambda x)\, dx$$

$$+ \sum_{n\in\mathcal{N}} \sum_{1\le m\le|n|} \sum_{\lambda=\pm1} \frac{1}{vol}\int_{SL_2(\mathbb{R})} f\left(ge_1, g\lambda e_1, g\begin{pmatrix} m \\ n\end{pmatrix}\right) dg$$

$$+ \sum_{n\in\mathcal{N}} \sum_{1\le m\le|n|} \sum_{\alpha,\beta} \frac{1}{vol}\int_{SL(2,\mathbb{R})} f\left(ge_1, g\begin{pmatrix} m \\ n\end{pmatrix}, g\begin{pmatrix} \alpha+m\beta \\ n\beta\end{pmatrix}\right) dg$$

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Any Ideas?