An introduction to free probability
1. Free independence

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Villetaneuse, 04.03.2014

Classical probability:

$(\Omega, \mathcal{F}, P)$- probability space.
Random variable: measurable function $X : \Omega \to \mathbb{R}$.
$L^\infty(\Omega)$: commutative unital algebra of bounded measurable (equivalence classes of) functions $X : \Omega \to \mathbb{C}$.
The **unit**: $\mathbf{1}(\omega) = 1$ for all $\omega \in \Omega$.
Expectation: $EX := \int_\Omega X(\omega)dP(\omega)$ is a *state* on $L^\infty(\Omega)$:
E is a linear function $L^\infty(\Omega) \to \mathbb{C}$ such that

1. $E\mathbf{1} = 1$,
2. If $X(\omega) \geq 0$ for all $\omega \in \Omega$ then $E(X) \geq 0$.

Noncommutative probability:

Noncommutative probability space: $(\mathcal{A}, \phi)$
$\mathcal{A}$ is a unital, complex $*$-algebra
$\phi : \mathcal{A} \to \mathbb{C}$ is a linear map which satisfies
1. $\phi(\mathbf{1}) = 1$,
2. $\phi(x^* x) \geq 0$ for every $x \in \mathcal{A}$.

"random variables": elements of $\mathcal{A}$,
"expectation": ϕ.

$*$ is an *involution* on $\mathcal{A}$:

$$* : \mathcal{A} \to \mathcal{A}$$
$$(a + b)^* = a^* + b^*$$
$$(\alpha a)^* = \overline{\alpha} a^*$$
$$(ab)^* = b^* a^*$$

for $\alpha \in \mathbb{C}$, $a, b \in \mathcal{A}$.

Distribution of a self-adjoint element $a = a^* \in \mathcal{A}$

is a probability measure μ on $\mathbb{R}$ satisfying:

$$\phi(a^n) = \int_{\mathbb{R}} t^n \, d\mu(t), \qquad n = 1, 2, \ldots,$$

so that $\phi(a^n)$ are *moments* of μ.

Such measure exists, because the sequence $\phi(a^n)$ is positive definite: for a finite sequence of real numbers α_i we have

$$\sum_{i,j} \phi(a^{i+j})\alpha_i\alpha_j = \phi\left(\left(\sum{}_i \alpha_i a^i\right)^2\right) \geq 0.$$

Under some additional assumptions (for example that $\mathcal{A}$ is a C^*-algebra) μ is also unique.

Independence

Let $(\mathcal{A}, \phi)$ be a (noncommutative) probability space and let $\{\mathcal{A}_i\}$, $i \in I$, be a family of subalgebras, with $\mathbf{1} \in \mathcal{A}_i$.
We say that the subalgebras $\mathcal{A}_i$ are *independent* if
1. $ab = ba$ whenever $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $i, j \in I$, $i \neq j$,
2. $\phi(a_1 a_2 \ldots a_n) = \phi(a_1)\phi(a_2)\ldots\phi(a_m)$ whenever $a_1 \in \mathcal{A}_{i_1}, a_2 \in \mathcal{A}_{i_2}, \ldots, a_m \in \mathcal{A}_{i_m}$ and $i_1, i_2, \ldots, i_m \in I$ are distinct.

Let $(\Omega, \mathcal{F}, P)$ be the product probability space: $\Omega = \times_{i \in I} \Omega_i$, $\mathcal{F} = \times_{i \in I} \mathcal{F}_i$, $P = \times_{i \in I} P_i$. Then $\mathcal{A} := L^\infty(\Omega)$ is the tensor product of $\mathcal{A}_i := L^\infty(\Omega_i)$:
More generally, we can start with a family $(\mathcal{A}_i, \phi_i)$, $i \in I$, of noncommutative probability spaces, put $\mathcal{A} := \bigotimes_{i \in I} \mathcal{A}_i$ and define the natural state on $\mathcal{A}$:

$$\phi\,(a_1 \otimes a_2 \otimes \ldots \otimes a_m) := \phi_{i_1}(a_1)\phi_{i_2}(a_2)\ldots\phi_{i_n}(a_m)$$

for $a_1 \in \mathcal{A}_{i_1}, a_2 \in \mathcal{A}_{i_2}, \ldots, a_m \in \mathcal{A}_{i_m}$ and for $i_1, i_2, \ldots, i_m \in I$ distinct.
The family $\{\mathcal{A}_i\}$, $i \in I$, is independent in $(\mathcal{A}, \phi)$.

However the tensor product of algebras is very commutative: elements form distinct $\mathcal{A}_i$ do commute.

Unital free product

Let $(\mathcal{A}_i, \phi_i)$, $i \in I$, noncommutative probability spaces. Put $\mathcal{A}_i^0 := \mathrm{Ker}\phi_i$. Then the unital free product $\mathcal{A} = *_{i\in I}\mathcal{A}_i$ can be represented as

$$\mathcal{A} := \mathbb{C}\mathbf{1} \oplus \bigoplus_{\substack{m\geq 1 \\ i_1,\ldots,i_m\in I \\ i_1\neq i_2\neq\ldots\neq i_m}} \mathcal{A}_{i_1}^0 \otimes \mathcal{A}_{i_2}^0 \otimes \ldots \otimes \mathcal{A}_{i_m}^0 = \mathbb{C}\mathbf{1} \oplus \mathcal{A}^0. \tag{1}$$

The notation "$i_1 \neq i_2 \neq \ldots \neq i_m$" means that

$$i_1 \neq i_2, \quad i_2 \neq i_3, \ldots, i_{m-1} \neq i_m.$$

$\mathcal{A}$ is the unique unital algebra containing all $\mathcal{A}_i$ as subalgebras, such that for given unital homomorphisms $h_i : \mathcal{A}_i \to \mathcal{B}$, there is a unique homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that $h|_{\mathcal{A}_i} = h_i$ for all $i \in I$ (**coproduct**).

Multiplication: if

$$\mathbf{a} = a_1 \otimes a_2 \otimes \ldots \otimes a_m, \qquad \mathbf{b} = b_1 \otimes b_2 \otimes \ldots \otimes b_n, \tag{2}$$

with $m, n \geq 1$, $a_1 \in \mathcal{A}^0_{i_1}, \ldots, a_m \in \mathcal{A}^0_{i_m}$ and $b_1 \in \mathcal{A}^0_{j_1}, \ldots, b_n \in \mathcal{A}^0_{j_n}$ then the product is defined by:

$$\mathbf{a} \cdot \mathbf{b} := \begin{cases} a_1 \otimes \ldots \otimes a_m \otimes b_1 \otimes \ldots \otimes b_n & \text{if } i_m \neq j_1, \\ a_1 \otimes \ldots \otimes a_{m-1} \otimes c \otimes b_2 \otimes \ldots \otimes b_n & \\ +\alpha(a_1 \otimes \ldots \otimes a_{m-1}) \cdot (b_2 \otimes \ldots \otimes b_n) & \text{if } i_m = j_1 := i, \end{cases}$$

where $\alpha := \phi_i(a_m b_1)$, $c := a_m b_1 - \phi_i(a_m b_1)\mathbf{1}$, so that $a_m b_1 = c + \alpha \mathbf{1}$, $c \in \mathcal{A}^0_i$.

For the expression

$$(a_1 \otimes \ldots \otimes a_{m-1}) \cdot (b_2 \otimes \ldots \otimes b_n)$$

we proceed inductively. By definition, for **a** as in (2) we have

$$\mathbf{a} = a_1 \cdot a_2 \cdot \ldots \cdot a_m.$$

What is natural state on $\mathcal{A}$?

The one satisfying: $\phi(\mathbf{1}) = 1$ and $\phi(\mathbf{a}) = 0$ for $\mathbf{a}$ as in (2), with $m \geq 1$,

so that in (1) $\mathcal{A}^0$, the second summand, is the kernel of ϕ.
This justifies the following definition

Definition: Let $(\mathcal{A}, \phi)$ be a probability space.

A family $\{\mathcal{A}_i\}_{i \in I}$ of unital (i.e. $\mathbf{1} \in \mathcal{A}_i$) subalgebras is called *free* if

$$\phi(a_1 a_2 \ldots a_m) = 0$$

whenever $m \geq 1$, $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$, $i_1, \ldots, i_m \in I$, $i_1 \neq i_2 \neq \ldots \neq i_m$ and $\phi(a_1) = \ldots = \phi(a_m) = 0$.

Hence in the construction (1) the algebras $\mathcal{A}_i$ are free in $(\mathcal{A}, \phi)$.

Now about positivity of ϕ.

Assume that all ϕ_i admit GNS representation, i.e.

$$\phi_i(a) = \langle \pi_i(a)\xi_i, \xi_i \rangle, \qquad a \in \mathcal{A}_i$$

where

$$\pi_i : \mathcal{A}_i \to \mathcal{B}(\mathcal{H}_i)$$

is a $*$-representation of $\mathcal{A}_i$ on a Hilbert space $\mathcal{H}_i$, ξ_i is a unit vector in $\mathcal{H}_i$.

Now we are going to construct the GNS representation for ϕ, which will prove positivity of ϕ. Define $\mathcal{H}_i^0 := \xi_i^\perp$, the orthocomplement of ξ_i in $\mathcal{H}_i$, so that $\mathcal{H}_i = \mathbb{C}\xi_i \oplus \mathcal{H}_i$. Put

$$\mathcal{H} := \mathbb{C}\xi_0 \oplus \bigoplus_{\substack{m \geq 1 \\ i_1, \ldots, i_m \in I \\ i_1 \neq i_2 \neq \ldots \neq i_m}} \mathcal{H}_{i_1}^0 \otimes \mathcal{H}_{i_2}^0 \otimes \ldots \otimes \mathcal{H}_{i_m}^0.$$

Now for every $i \in I$ we define a representation σ_i of $\mathcal{A}_i$ acting on $\mathcal{H}$.

Namely, we decompose $\mathcal{H}$ as

$$\mathcal{H} = \left(\mathbb{C}\xi_0 \oplus \mathcal{H}_i^0\right) \otimes \mathcal{H}(i),$$

(we identify ξ_0 with ξ_i) where

$$\mathcal{H}(i) = \mathbb{C}\xi_0 \oplus \bigoplus_{\substack{m \geq 1 \\ i_1, \ldots, i_m \in I \\ i \neq i_1 \neq i_2 \neq \ldots \neq i_m}} \mathcal{H}_{i_1}^0 \otimes \mathcal{H}_{i_2}^0 \otimes \ldots \otimes \mathcal{H}_{i_m}^0.$$

Then we put

$$\sigma_i(a) := \pi_i(a) \otimes \mathrm{Id}_{\mathcal{H}(i)}.$$

In this way we have constructed a $*$-representation σ_i of $\mathcal{A}_i$ acting on $\mathcal{H}$.

By the coproduct property we extend to a $*$-representation π of whole $\mathcal{A} = *_{i\in I}\mathcal{A}_i$. We are going to show, that for every $\mathbf{c} \in \mathcal{A}$

$$\langle \pi(\mathbf{c})\xi_0, \xi_0\rangle = \phi(\mathbf{c}).$$

Namely: for $\mathbf{a} = a_1 a_2 \dots a_m$, with $m \geq 1$, $a_1 \in \mathcal{A}^0_{i_1}, \dots, a_m \in \mathcal{A}^0_{i_m}$, $i_1, \dots, i_m \in I$ and $i_1 \neq i_2 \neq \dots \neq i_m$, we have

$$\pi(\mathbf{a}) = \sigma_{i_1}(a_1)\sigma_{i_2}(a_2)\dots\sigma_{i_m}(a_m)$$

Moreover, since $\phi_{i_k}(a_k) = 0$ we have

$$\sigma_{i_k}(a_k)\xi_0 = \pi_{i_k}(a_k)\xi_0 \in \mathcal{H}^0_{i_k}.$$

By induction it is easy to check, that

$$\pi(\mathbf{a})\xi_0 = \pi_{i_1}(a_1)\xi_0 \otimes \pi_{i_2}(a_2)\xi_0 \otimes \dots \otimes \pi_{i_m}(a_m)\xi_0$$

$$\pi(\mathbf{a})\xi_0 \in \mathcal{H}^0_{i_1} \otimes \mathcal{H}^0_{i_2} \otimes \dots \otimes \mathcal{H}^0_{i_m},$$

which means that $\phi(\mathbf{a}) = 0$

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