Extended Lévy exponent functions and infinite divisible distributions

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[1] Objects of interest in infinite divisibility

Classes of Function in **ID**

- Completely monotone ($\mathcal{CM}$): $f \in \mathcal{CM}$, if for $\lambda > 0$,

$$(-1)^k f^{(k)} \geq 0, \quad \forall k \geq 0 \Longleftrightarrow f(\lambda) = \int_{[0,\infty)} e^{-\lambda x} \mu(dx).$$

$$f \in \mathbf{ID} \text{ i.e., } \; f^t \in \mathcal{CM}, \; \forall t > 0 \Longleftrightarrow f = e^{-\phi}, \; \phi' \in \mathcal{CM}.$$

- Bernstein functions ($\mathcal{BF}$): $\varphi \in \mathcal{BF}$, if for $\lambda \geq 0$,

$$\varphi(\lambda) = \mathrm{d}\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \mu(dx).$$

- Laplace exponents :

$$\Psi(\lambda) = \mathrm{d}\lambda + \sigma^2\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda h(x)) \mu(dx), \quad (1)$$

$\mathrm{d}, \; \sigma, \; \lambda \geq 0, \; \int_{(0,\infty)} (1 \wedge x) h(x) \mu(dx) < \infty$, e.g. $\; h(x) = x \mathbb{1}_{x \leq 1}$.

Infinite divisibility

- $X \sim$ **ID**, if $\forall n \in \mathbb{N}^*$, there exists $X_{n,1}, \cdots, X_{n,n}$, *i.i.d.* s.t.

$$X \overset{d}{=} X_{n,1} + \cdots + X_{n,n}.$$

- $X \sim$ **ID** and $X \geq 0$, IFF

$$\phi_X(\lambda) := -\log \mathbb{E}[e^{-\lambda X}] \in \mathcal{BF}.$$

- $X \sim$ **ID** and is of the *spectrally negative type* (**SN**), IFF

$$\lambda \mapsto \Psi_X(\lambda) = \log \mathbb{E}[e^{\lambda X}] \text{ is a Lap. exp.}$$

- The subclass $\mathcal{LE}$ of Lap. exp. is formed by

$$\Psi(\lambda) = \mathrm{d}\lambda + \sigma^2\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x)\mu(dx).$$

has an interest: $X \sim \textbf{ID} \iff X \overset{d}{=} \lim X_1^n - X_2^n$,

$$X_1^n, X_2^n \text{ independent, and } \log \mathbb{E}[e^{\lambda X_i^n}] \in \mathcal{LE},$$

Examples

Let $\mathbb{S}_\alpha$, $\alpha \in (0,1)$, denote a positive stable r.v., i.e. for $\lambda \geq 0$,

$$\begin{aligned} \log \mathbb{E}[e^{-\lambda \mathbb{S}_\alpha}] &= \lambda^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1-e^{-\lambda x}) \, \frac{dx}{x^{\alpha+1}} \\ \log \mathbb{E}[\mathbb{S}_\alpha^{-\lambda}] &= \log \frac{\Gamma(1+\frac{\lambda}{\alpha})}{\Gamma(1+\lambda)} \\ &= \frac{(\alpha-1)\,\gamma}{\alpha}\lambda + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x) \frac{\rho_\alpha(x)}{x} dx, \end{aligned}$$

where $\rho_\alpha(x) = \frac{e^{-\alpha x}(1-e^{-(1-\alpha)x})}{(1-e^{-x})(1-e^{-\alpha x})}$. Observe that

$$\lambda \mapsto \log \mathbb{E}[e^{-\lambda \mathbb{S}_\alpha}] \in \mathcal{TBF} \Longrightarrow \mathbb{S}_\alpha \sim \mathbf{GGC} \subset \mathbf{ID}.$$

and

$$\lambda \mapsto \log \mathbb{E}[\mathbb{S}_\alpha^{-\lambda}] \in \mathcal{LE} \Longrightarrow \log \mathbb{S}_\alpha \sim \mathbf{SN} \subset \mathbf{ID}.$$

[2] Laplace exponents and their extension

The class $\mathcal{B}_2$ generalizes $\mathcal{LE}$

$$\mathcal{B}_2 := \left\{ \Psi_{\phi,\mu}(\lambda) = \phi(\lambda^2) + \int_{(0,\infty)} e_2(\lambda x)\mu(dx), \quad \lambda \geq 0 \right\},$$

where $\phi \in \mathcal{BF}$, $e_2(x) = e^{-x} - 1 + x$ and $\int_{(0,\infty)} (x \wedge 1)\, x\, \mu(dx) < \infty$.

1) Fourati-Jedidi (2011):

$$\phi \in \mathcal{BF} \Longrightarrow \sqrt{\lambda}\phi(\sqrt{\lambda}) \in \mathcal{CBF}.$$

2) Basalim, Bridaa, Jedidi (2020):

$$\Psi = \Psi_{\phi,\mu} \in \mathcal{B}_2, \;\; \phi \in \mathcal{TBF} \Longrightarrow \Psi(\sqrt{\lambda}) \in \mathcal{TBF}.$$

Internality in Bertoin, Roynette, Yor (2004)

Definition

BRY (2004): $\varphi \in \mathcal{BF}$ is said to be *internal*, if

$$\Psi \circ \varphi \in \mathcal{BF}, \;\; \forall\, \Psi \in \mathcal{B}_2.$$

Theorem (Basalim, Bridaa, J, 2020)

For $\varphi \in \mathcal{BF}$, we have the equivalences:

1) φ *is internal;*
2) $\varphi^2 \in \mathcal{BF}$;
3) $\phi_1(\varphi)\,\phi_2(\varphi) \in \mathcal{BF}, \quad \forall \phi_1, \phi_2 \in \mathcal{BF}$;
4) $\exists \Psi \in \mathcal{B}_2 \;\; s.t \;\; \Psi(\lambda) \overset{0+}{\approx} \lambda^2 \;\; and \;\; \Psi(t\varphi) \in \mathcal{BF}, \;\; \forall t > 0.$
5) $\exists \phi_i \in \mathcal{BF} \;\; s.t \;\; \phi_i(\lambda) \overset{0+}{\approx} \lambda \;\; and \;\; \phi_1(t\varphi)\,\phi_2(t\varphi) \in \mathcal{BF}, \;\; \forall t > 0.$

Frame Title

Theorem (Basalim, Bridaa, J, 2020)

Let a subordinator $(X_t)_{t\geq 0}$ *with Bernstein function* $\phi \equiv (0, 0, \mu)$. *The following assertions are equivalent:*

1. ϕ *is internal;*
2. $t\,\mu(dx) - \mathbb{P}(X_t \in dx)$ *is positive;*
3. $x \mapsto t\,\overline{\mu}(x) - \mathbb{P}(X_t > x)$ *is nonnegative and non-increasing;*
4. $\exists$ *a subordinator* $(Y_t)_{t\geq 0}$ *and* $c \geq 0$ *s.t.*

$$\mu(dx) = c \int_0^\infty \mathbb{P}(Y_t \in dx)\, \frac{dt}{t^{\frac{3}{2}}}$$

and $(X_t)_{t\geq 0} \stackrel{d}{=} \left((Y \circ \mathbb{S}^{\frac{1}{2}})_t \right)_{t\geq 0}$.

The class $\mathcal{B}_\theta$, preparation to θ-internality

For $\theta \geq 0$, we consider

$$\mathcal{B}_\theta := \left\{ \Psi(\lambda) = \phi(\lambda^\theta) + \int_{(0,\infty)} e_\theta(\lambda x)\, \mu(dx), \quad \lambda \geq 0 \right\},$$

$\phi \in \mathcal{BF}$, $\int_{(0,\infty)} (x \wedge 1)\, x^{\theta-1}\, \mu(dx) < \infty$, $e_0(\lambda) = e^{-\lambda}$, and

$$e_\theta(\lambda) \quad = \quad \frac{\lambda^\theta}{\Gamma(\theta)} \int_0^1 e^{-\lambda x}\, (1-x)^{\theta-1}\, dx = \frac{\lambda^\theta}{\Gamma(\theta+1)}\, \mathbb{E}[e^{-\lambda \mathbb{B}_\theta}].$$

Why $\mathcal{B}_\theta$?

- $\theta \in \mathbb{N} \Longrightarrow e_\theta(\lambda) = (-1)^\theta \sum_{k=\theta}^\infty \frac{(-\lambda)^k}{k!}$.
- $\mathcal{B}_0 = \mathcal{CM}$, $\mathcal{B}_1 = \mathcal{BF}$, $\mathcal{LE} \subset \mathcal{B}_2$.
- For $\theta \in (n, n+1)$, $n \geq 0$, we have

$$\lambda^\theta = \frac{\Gamma(\theta+1)}{\Gamma(\theta-n)\Gamma(n+1-\theta)} \int_0^\infty e_{n+1}(\lambda x) \frac{dx}{x^{\theta+1}} \in \mathcal{B}_{n+1}.$$

  If $\theta \in (1, 3]$, then

$$e^{\lambda^\theta} = \int_{\mathbb{R}} e^{-\lambda x}\, T_\theta(x)\, dx,$$

  where T_θ is the *trans-stable* function, which is NOT infinitely divisible!!! (cf. book of Zolotarev).
- If $\theta > 2$: No direct infinite divisible properties for $\mathcal{B}_\theta$!!!

[3] Motivation for $\mathcal{B}_\theta$, Barnes type functions

The G-Barnes function is the solution to

$$G(z+1) = \Gamma(z)\, G(z), \quad G(1) = 1$$

Barnes in Nikeghbali, Yor (2009)

Let $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_n$ be independent gamma random variables with respective parameters $n \geq 1$, then,

$$\lim_{N \to \infty} \frac{1}{N^{\frac{\lambda^2}{2}}} \mathbb{E}[e^{-\lambda(\sum_{n=1}^{N} \frac{\mathbb{G}_n}{n} - N)}] = \frac{\left(\frac{2\pi}{e}\right)^{\frac{\lambda}{2}} e^{-\frac{\lambda^2}{2}}}{G(1+\lambda)}, \quad Re(\lambda) > -1,$$

where the G-Barnes function has the representation

$$\log G(1+\lambda) = \frac{\log(2\pi) - 1}{2}\lambda - \frac{1+\gamma}{2}\lambda^2 + \int_0^\infty e_3(\lambda x)\, \frac{\sigma(x)}{x}\, dx,$$

with $\gamma =$ Euler constant, and

$$e_3(x) = e^{-x} - 1 + x - \frac{x^2}{2}, \quad \sigma(x) := \frac{1}{4\ \sinh^2(x/2)},$$

Barnes in Keating-Snaith conjecture (philosophy)

The moments of the Riemann zeta function ζ should satisfy

$$\lim_{T\to\infty} \frac{1}{T\,(\log T)^{\lambda^2}} \int_0^T \left|\zeta\left(\frac{1}{2}+iu\right)\right|^{2\lambda} du = M(\lambda)\; A(\lambda), \quad \mathrm{Re}(\lambda) > -1.$$

$M =$ *random factor*, $A =$ *arithmetic factor*,

$$\begin{aligned} M(\lambda) &:= \frac{\big(G(1+\lambda)\big)^2}{G(1+2\lambda)}, \\ A(\lambda) &:= \prod_{p\,\text{prime}} \left\{ \left(1-\frac{1}{p}\right)^{\lambda^2} \sum_{m=0}^{\infty} \frac{1}{p^m} \left(\frac{\Gamma(\lambda+m)}{m!\,\Gamma(\lambda)}\right)^2 \right\}, \end{aligned}$$

For $a>1$, we have

$$\begin{aligned} \Psi_{a,M}(\lambda) &:= a\log\big(G(1+\lambda)\big) - \log\big(G(1+a\lambda)\big) \in \mathcal{B}_2, \\ \Psi_{a,A}(\lambda) &:= \log\big(A(\lambda+a)\big) - \log\big(A(a)\big) \in \mathcal{B}_2. \end{aligned}$$

Barnes in J, Simon, Wang (2018)

Example

For $\alpha > 0$, $a = m - \alpha$, the integro-differential equation

$$x^m f(x) \;=\; \frac{1}{\Gamma(\alpha)} \int_0^x (x-v)^{\alpha-1} f(v)\, dv$$

has a density solution $\iff m > \alpha$. The associated R.V. X is s.t.

$$\mathbb{E}[X^\lambda] = \frac{G(\frac{m}{a}+\lambda, \frac{1}{a})G(1, \frac{1}{a})}{G(1+\lambda, \frac{1}{a})G(\frac{m}{a}, 1)}, \quad \lambda > -1. \tag{2}$$

$G(z+1;\tau) = \Gamma(z\tau^{-1})G(z;\tau)$, $G(z+\tau;\tau) = (2\pi)^{\frac{\tau-1}{2}}\tau^{\frac{1}{2}-z}\Gamma(z)G(z;\tau)$.
Consider

$$\Psi_{m,a}(\lambda) := \log \mathbb{E}[X^\lambda] \in \mathcal{LE} \Longrightarrow \log X \sim \mathbf{ID} \cap \mathbf{SN} \;\text{ and }\; X \sim \mathbf{GGC}.$$

Barnes-type in Jacod, Kowalski, Nikeghbali (2011)

Definition

A sequence (Z_N) converges in the mod-Gaussian (CMG), if

$$\exp(\lambda\beta_N - \frac{\lambda^2}{2}\gamma_N)\,\mathbb{E}[e^{-\lambda Z_N}] \longrightarrow \Phi(\lambda), \quad \lambda \in \mathbf{i}\,\mathbb{R}.$$

Theorem (JKN (2011))

1) If $(X_i^n)_{1\le i\le n}$ *is a triangular array, then*

$$Z_N = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{n}(X_1^n + \cdots + X_n^n) \text{ CMG with } \beta_N = 0,\, \gamma_N = \log N.$$

2) If furthermore $X_i^n \sim \mathbf{ID}$ + *integrability on the Lévy measures, then*

$$\Psi(\lambda) := \log\Phi(\lambda) = \int_{\mathbb{R}} e_3(\lambda x)\,\mu(dx), \quad \lambda \in \mathbf{i}\,\mathbb{R}$$

where $\mu(A) = \sum_{n\ge1} n\mu_n(n\,A)$.

[4] Analytic properties of the class $\mathcal{B}_\theta$

$$\mathcal{B}_\theta := \left\{ \Psi(\lambda) = \phi(\lambda^\theta) + \int_{(0,\infty)} e_\theta(\lambda x)\, \mu(dx), \quad \lambda \geq 0 \right\},$$

Properties of e_θ

1) **Stirling-like formula:** $e_\theta(\theta\lambda) \sim \frac{\lambda^\theta}{1+\lambda} \frac{e^\theta}{\sqrt{2\pi\theta}}, \quad$ as $\theta \to \infty$.

2) **Differentiation:** For $\theta \geq 1$,

$$e'_\theta(\lambda) = e_{\theta-1}(\lambda).$$

3) **Recursive equation:** For $\theta > 0,\ \lambda \geq 0$,

$$\begin{aligned} \frac{\lambda^\theta}{\Gamma(\theta+1)} &= e_{\theta+1}(\lambda) + e_\theta(\lambda) \\ e_\theta(\lambda) &= \sum_{1}^{[\theta]} (-1)^{n-1} \frac{\lambda^{\theta-n}}{\Gamma(\theta+2-n)} + (-1)^{[\theta]}\, e_{\{\theta\}}(\lambda). \end{aligned}$$

4) **Difference:** For $\theta = 2, 3\ldots,\ a \geq 0$,

$$e_\theta(\lambda+a) - e_\theta(\lambda) - e_\theta(a) = (1-e^{-a})\, e_{\theta-1}(\lambda) + \sum_{j=1}^{\theta-2} e_{\theta-j}(a)\, \frac{\lambda^j}{j!}.$$

Linking $\mathcal{B}_\theta$ with θ-monotone functions, $\theta > 0$

let $\overline{\mu}_\theta$ the θ-monotone function

$$\overline{\mu}_\theta(x) = \frac{1}{\Gamma(\theta)} \int_x^\infty (t-x)^{\theta-1} \mu(dt), \quad x > 0.$$

If $\int_{(0,\infty)} x^\theta \, \mu(dx) < \infty$, then

$$x \mapsto \int_x^\infty \overline{\mu}_\theta(y)\, dy = \int_x^\infty \frac{(t-x)^\theta}{\theta} \mu(dt) \text{ is } (\theta+1)\text{-monotone.}$$

Proposition

The classes $\mathcal{B}_\theta$ are convex cones, closed by pointwise limits, and

$$\Psi \in \mathcal{B}_\theta \Longleftrightarrow \Psi(\lambda) = \phi(\lambda^\theta) + \lambda^\theta \int_0^\infty e^{-\lambda x}\, \overline{\mu}_\theta(x)\, dx.$$

Corollary ($\Psi = \Psi_{0,\mu},\ \theta \geq 1$)

$\Psi \in \mathcal{B}_\theta \Longrightarrow \lambda \mapsto \Psi(\lambda^{1/\theta}) \in \mathcal{BF}$ *(resp. $\mathcal{TBF}$ if $\theta \geq 2$).*

Corollary ($\Psi = \Psi_{0,\mu},\ \theta \geq 1$)

We have the equivalence:

1) $\Psi \in \mathcal{B}_\theta$;
2) (i) *The left R-L fractional derivative of order θ*

$$D^\theta(\Psi)(\lambda) = \frac{1}{\Gamma(n-\theta)} \frac{d^n}{d\lambda^n} \int_0^\lambda (\lambda - u)^{n-\theta-1}\ \Psi(u)\, du$$

is completely monotone and integrable at 0;

(ii) $\lim_{\lambda\to\infty} \Psi(\lambda)/\lambda^\theta = 0$;

(iii) $\lim_{\lambda\to 0+} \lambda^{1-\theta}\Psi(\lambda)$ *exists.*

Corollary ($\theta \in \mathbb{N}$)

$\Psi \in \mathcal{B}_\theta \Longleftrightarrow \Psi^{(k)} \geq 0,\ \ \forall k = 0, 1, \ldots \theta$ *and* $\Psi^{(\theta)} \in \mathcal{CM}$.

$\mathcal{B}_\theta$ "mimicks" $\mathcal{BF}$ and $\mathcal{LE}$

Proposition ($\Psi = \Psi_{\phi,\mu}$)

1) **Differentiation**

$$\theta > 1 \text{ and } \lambda\, \phi'(\lambda^{\theta/(\theta-1)}) \in \mathcal{BF} \implies \Psi' \in \mathcal{B}_{\theta-1}.$$

2) **Composition by power functions:**

$$0 < \alpha < 1 \implies \Psi(\lambda^\alpha) \in \mathcal{B}_{\alpha\theta}.$$

3) **Lowering the exponent:** *Let* $\kappa \in (0, \theta)$.

$$\phi(\lambda) = \lambda^{1-\frac{\kappa}{\theta}}\, \varphi(\lambda^{\frac{\kappa}{\theta}}), \;\; \varphi \in \mathcal{BF} \Longrightarrow \lambda \mapsto \lambda^{\kappa-\theta}\, \Psi(\lambda) \in \mathcal{B}_\kappa.$$

4) **Infinite divisibility:** $\theta \leq 1 \Longrightarrow 1/\Psi \in \mathbf{ID}$.

Injecting $\mathcal{B}_\theta$ into $\mathcal{BF}$

Proposition

1) *Let* $\theta \geq 1$, $0 \leq \alpha \leq 1$, *and* $\phi = 0$ *in case* $\alpha\theta > 1$. *Then*

$$\Psi \in \mathcal{B}_\theta \Longrightarrow \lambda^{1-\alpha\theta}\ \Psi(\lambda^\alpha) \in \mathcal{BF}.$$

2) *Let* $\theta_i \geq 1$, $\alpha_i \in (0,1)$, *s.t.* $\sum_{i=1}^n \alpha_i\theta_i \leq 1$. *Then*

$$\Psi_i \in \mathcal{B}_{\theta_i} \Longrightarrow \lambda \mapsto \prod_{i=1}^n \Psi_i(\lambda^{\alpha_i}) \in \mathcal{BF}.$$

3) $\theta \geq 2$ *and* $0 \leq \alpha\theta \leq 2 \Longrightarrow \Psi(\sqrt{\lambda})^\alpha \in \mathcal{CBF}$;
4) $\theta \geq 2$, $0 \leq \alpha \leq 1/2$ *and* $\phi ... \Longrightarrow \lambda^{1-\alpha\theta}\ \Psi(\lambda^\alpha) \in \mathcal{TBF}$.
5) *Let* $\theta_i \geq 1$ $(\theta_i \geq 2)$, $\alpha_i \in (0,1/2]$, *s.t.* $\sum_{i=1}^n \alpha_i\theta_i \leq 1$, *and* $\Psi_i = \Psi_{\phi_i,\mu_i} \in \mathcal{B}_{\theta_i}$. *Then*

$$\phi_i \in \mathcal{CBF}\ (\mathcal{TBF}) \Longrightarrow \prod_{i=1}^n \Psi_i(\lambda^{\alpha_i}) \in \mathcal{CBF}\ (\mathcal{TBF}).$$

Inverses of functions in $\mathcal{B}_\theta$

Proposition ($\Psi = \Psi_{\phi,\mu} \in \mathcal{B}_\theta,\ \theta \geq 2$)

If $\lambda\,\phi'(\lambda^{\theta/(\theta-1)}) \in \mathcal{BF}$, *then the inverse function* φ *of* Ψ *(in the sense of the composition) is s.t.* $\varphi^{\theta-1} \in \mathcal{SBF}$, *i.e.*

$$\varphi^{\theta-1} \in \mathcal{BF} \quad \textit{and} \quad \lambda \mapsto \frac{\lambda}{\varphi^{\theta-1}(\lambda)} \in \mathcal{BF}.$$

Internality within the class $\mathcal{B}_\theta$

Proposition

Let $\theta \geq 1$ and $\varphi : [0, \infty) \to [0, \infty)$. We have the equivalences:

1) $\varphi^\theta \in \mathcal{BF}$;
2) φ *is* θ*-internal, i.e.*

$$\Psi \circ \varphi \in \mathcal{BF}, \quad \textit{for every } \Psi \in \mathcal{B}_\theta.$$

Conjecture (True if $\theta = 2$)

Let $(X_t)_{t\geq 0}$ be a subordinator associated to a Bernstein function φ and Lévy measure Π and $\theta \geq 2$. Then, the following are equivalent:

(i) φ *is* θ*-internal;*

(ii) $t\Pi(dx) - (\theta - 1)\mathbb{P}(X_t \in dx)$ *is a positive measure.*

Differences in the class $\mathcal{B}_\theta$

It is straightforward that

$$\phi \in \mathcal{BF} \Longrightarrow -\Delta_c(\phi)(\lambda) := \phi(\lambda) + \phi(c) - \phi(\lambda + c) \in \mathcal{BF}.$$

Lemma

For $\theta > 1$ and $\phi \in \mathcal{BF}$, we have the equivalences:

(i) $\lambda \mapsto \lambda^{1-\frac{1}{\theta}} \phi'(\lambda) \in \mathcal{BF}$;

(ii) $\forall c > 0, \ \exists \varphi_{c,\theta} \in \mathcal{BF}$ *s.t.*

$$\Delta_c\big(\lambda \mapsto \phi(\lambda^{\theta})\big) = \phi\left((\lambda + c)^{\theta}\right) - \phi\left(\lambda^{\theta}\right) - \phi\left(c^{\theta}\right) = \varphi_{c,\theta}(\lambda^{\theta}).$$

Differences in the class $\mathcal{B}_\theta$

$$\begin{aligned}\Delta_a\Psi(\lambda) &= \Psi(\lambda+a)-\Psi(\lambda)-\Psi(a)\\ \Omega_a\Psi(\lambda) &= \Psi(\lambda)-\Psi(a\lambda)\\ \chi_a\Psi(\lambda) &= \Psi(a\lambda)-a\Psi(\lambda).\end{aligned}$$

Proposition ($\Psi=\Psi_{\phi,\mu}\in\mathcal{B}_\theta,\ \theta\geq 1$)

1) $\Delta_a\Psi\in\mathcal{B}_{\theta-1},\ \forall\, a>0.$
2) $\Omega_b\Psi\in\mathcal{B}_\theta$, *for some* $b>0\Longrightarrow\Psi\in\mathcal{B}_\theta.$
3) *If* $\theta\geq 2$, *then*

$$\Omega_b\Psi\in\mathcal{B}_\theta,\ \forall\, 0<b<1\Longleftrightarrow\lambda\Psi'(\lambda)\in\mathcal{B}_\theta\Longleftrightarrow x\frac{\mu(dx)}{dx}\searrow,$$

and

$$\chi_a\Psi(\lambda)\Longleftrightarrow x^2\frac{\mu(dx)}{dx}\nearrow.$$

Prospects on $\mathcal{B}_\theta$

- The operator

$$\Xi_a \Psi(\lambda) = \Psi(a\lambda) - a^\theta \Psi(\lambda)$$

is of interest: for all $a > 0$, the Barnes function satifies

$$e^{-b_a \lambda} \, a^{\lambda^2} \, \frac{\big(G(1+\lambda)\big)^{a^2}}{G(1+a\lambda)} = e^{\Psi_a(\lambda)} = \mathbb{E}[X_a^\lambda],$$

where b_a is explicit, $\Psi_a \in \mathcal{LE} \subset \mathcal{B}_2, \;\; X_a \sim \mathbf{EGGC} \cap \mathbf{SN}$.

- We have seen that the Barnes-type function intervene in Mod-Gaussian convergence. Is a “generalized mod-Gaussian” convergence feasible ?, i.e

$$e^{\Psi_N} \, \mathbb{E}[e^{-\lambda Z_N}] \longrightarrow \Phi(\lambda), \;\; \Psi_N \in \mathcal{B}_\theta ?$$

Thank you for your attention !

Sergei N. Bernstein, 1880–1968

Mark Yor, 1949–2014

Some References

1. Bertoin, J.; Roynette, B.; Yor, M.: Some Connections Between (Sub)Critical Branching Mechanisms and Bernstein Functions.arXiv:math/0412322v1 [math.PR] 16 Dec 2004.
2. Bridaa, S.; Sonia Fourati, S.; Jedidi, W., Some new classes and techniques in the theory of Bernstein functions. Chapter book in "A Lifetime of Excursions Through Random Walks and Lévy Processes, A Volume in Honour of Ron Doneys 80th Birthday", *Progress in Probability*, Springer Series 78, 139–155, 2021.
3. Jacod, J.; Kowalski, E.; Nikeghbali, A.: Mod-Gaussian convergence: New limit theorems in probability and number theory. Forum Math. 23 (2011), 835-873.
4. Jedidi, W.; Simon, T.; Wang M.: Density solutions to a class of integro-differential equations. Journal of Mathematical Analysis and Applications, J. Math.Anal.Appl. 458, 134152 (2018).
5. Nikeghbali, A., Yor, M.: The Barnes G function and its relations with sums and products of generalized gamma convolution variables. Elect. Comm. in Probab. 14, 396411 (2009).