Asymptotic normality of pattern counts in conjugacy classes

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Definitions

Permutations

Conjugacy invariant permutations

Patterns

Results

Uniform case: (Hofer)

Partial results: (Féray), (Hamaker and Rhoades) and (Kammoun)

General case: (Dubach) and (Féray and Kammoun)

Proofs

Comparison techniques

Weighted dependency graphs

Universality (Aléa days)

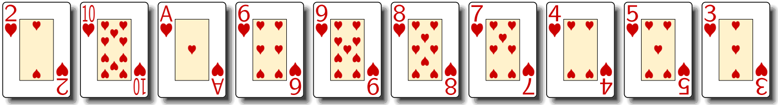
I.I.D.

Random matrices

Longest increasing (decreasing) subsequence

Conjugacy invariant permutations

Permutation



Word:

2 10 1 6 9 8 7 4 5 3

Descents

Peaks

Patterns

Longest increasing subsequence

RSK

Cycles:

$(1,2,10,3)(4,6,8)(5,9)(7)$

Total number of cycles

Number of cycles of length i

Conjugacy class

Matrix:

$$\begin{bmatrix} 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0 \end{bmatrix}$$

Question: we fix the value of a function, we study another.

Example in LIPN: Bassino et al.

- Condition: Separable i.e. 0 occurrence of the patterns 2413 and 3142
- Function to study: Longest increasing subsequence / proportion of other patterns.

Cycle Structure and Spectrum

- $\#$ total number of cycles
- $\#_i$ number of cycles of length i

If $0 \leq p < q$ and $\mathrm{GCD}(p,q) = 1$, then

$$\text{Multiplicity of eigenvalue } e^{\frac{p}{q}2\pi \mathrm{i}} \text{ is } \sum_{r\geq 1} \#_{rq}(\sigma)$$

In particular:

$$\#(\sigma) = \text{Multiplicity of eigenvalue } 1$$

$$\mathrm{Tr}(\sigma^k) = \sum_{i|k} i\#_i(\sigma) \quad \text{and} \quad k\#_k(\sigma) = \sum_{i|k} \mathrm{Tr}(\sigma^i)\mu(i)$$

Where $\mu(i)$ is the Möbius function defined as:

$$\mu(i) = \begin{cases} 0 & \text{if } i \text{ is divisible by the square of a prime number,} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct prime numbers.} \end{cases}$$

Conjugacy Classes

The conjugacy class of σ is $\{\pi\sigma\pi^{-1}, \pi \in \mathfrak{S}_n\}$.

Theorem

Let σ, ρ be two permutations.
There is equivalence between:

- *σ and ρ are in the same conjugacy class*
- *σ and ρ have the same cycle structure, i.e., $\forall i \geq 1, \#_i(\sigma) = \#_i(\rho)$.*
- *σ and ρ have the same spectrum (considering multiplicities)*
- $\forall i \geq 1, \mathrm{Tr}(\sigma^i) = \mathrm{Tr}(\rho^i)$.

Conjugacy invariant

- Definition: σ_n is conjugacy invariant if for all ρ,

$$\rho\sigma_n\rho^{-1} \stackrel{d}{=} \sigma_n.$$

- σ_n is conjugacy invariant if and only if $\mathbb{P}(\sigma_n = \sigma)$ is a function of the cycle structure of σ.

Conjugacy invariant

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- σ_n is conjugacy invariant if and only if $\mathbb{P}(\sigma_n = \sigma)$ is a function of the cycle structure of σ.
- Example 1: Ewens

$$\mathbb{P}(\sigma_n = \sigma) = \frac{\theta^{\#\sigma}}{C_{n,\theta}}.$$

- **Example 2: Uniform permutation within a conjugacy class.**
- Example 3: Uniform Involutions / Derangements.

Morally: Conditioned on the cycle structure, the permutation is chosen uniformly.

Descents

We denote by $D(\sigma) = \{i : \sigma(i+1) < \sigma(i)\}$.
We assume that $(\sigma_n)_{n\geq 1}$ is a sequence of random permutations such that for all n, σ_n is conjugacy invariant of size n.
Furthermore, we suppose that $\frac{\#_1\sigma_n}{n} \to \alpha$

Theorem (Kim and Lee 2020)

$$\frac{\mathrm{card}(D(\sigma_n)) - \frac{(1-\alpha^2)n}{2}}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{1-4\alpha^3+3\alpha^4}{12}\right).$$

Goal: prove similar results for other functions.

Classical Pattern

Let π be a permutation of size k. An *occurrence* of the (classical) pattern π in a permutation σ is a vector $(i_1, \cdots, i_k)$ with $i_1 < \cdots < i_k$ such that $\sigma(i_1) \ldots \sigma(i_k)$ has the same relative order as the elements of π.
Examples:

- For the permutation $\sigma = 2173456$,
  the vector $(i_1, i_2, i_3) = (2, 3, 7)$ is an occurrence of the pattern $\pi = 132$
  (176 has the same relative order as $\pi = 132$.)
- An occurrence of 21 is an inversion.
- An occurrence of $123 \cdots k$ is an increasing subsequence of length k.

Vincular Pattern

Definition

Let π be a permutation of size k and A be a subset of $[k-1]$. An *occurrence* of the vincular pattern (π, A) in a permutation σ is a vector $(i_1, \cdots, i_k)$ with $i_1 < \cdots < i_k$ satisfying:

- $(i_1, \cdots, i_k)$ is an occurrence of the classical pattern π in σ.
- For every s in A, $i_{s+1} = i_s + 1$.

Examples:

- $(\pi, \emptyset)$: is the classical pattern π
- An occurrence of $(21, \{1\})$: is a descent
- For the permutation $\sigma = 2173456$, the vector $(i_1, i_2, i_3) = (2, 3, 7)$
  - is an occurrence of the pattern $(\pi = 132, A = \{1\})$
  - not an occurrence of $(\pi = 132, A = \{1, 2\})$

Notation: $\mathfrak{N}^{\pi,A}(\sigma)$: pattern counts (number of occurrences of the patterns).

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- General case: (Dubach) and (Féray and Kammoun)

Proofs

- Comparison techniques
- Weighted dependency graphs

Universality (Aléa days)

- I.I.D.
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Uniform case

Fix $\Pi = (\pi, A)$, and let k be the size of π.

Theorem (Hofer (2018))

We assume that σ_n uniform of size n

$$\frac{\mathfrak{N}^{\Pi}(\sigma_n) - \mathbb{E}(\mathfrak{N}^{\Pi}(\sigma_n))}{n^{k-\frac{1}{2}-\mathrm{card}(A)}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \sigma_{\Pi}^2).$$

With

- $\sigma_{\Pi}^2 > 0$.

Generalises:

- $k = 2$: Fulman (2004)
- Consecutive: Goldstein (2005)
- Monotone: Bonà (2010)
- Classical: Janson et al. (2015)
- Without positivity: Féray (2013)

Ewens

Recall: Ewens distribution.

$$\mathbb{P}(\sigma_n = \sigma) = \frac{\theta^{\#\sigma}}{C_{n,\theta}}.$$

Fix $\Pi = (\pi, A)$, and $\theta \geq 0$. Let k be the size of π.

Theorem (Féray (2013))

We assume that σ_n follows the Ewens distribution with parameter θ. Then,

$$\frac{\mathfrak{N}^{\Pi}(\sigma_n) - \mathbb{E}(\mathfrak{N}^{\Pi}(\sigma_n))}{n^{k-\frac{1}{2}-\mathrm{card}(A)}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \sigma_{\Pi}^2).$$

Few cycles

Let σ_n is conjugacy invariant of size n

Theorem (Kammoun 2020)

We assume that $\frac{\#(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} 0.$

Then, $\frac{\mathfrak{N}^{\Pi}(\sigma_n)-\mathbb{E}(\mathfrak{N}^{\Pi}(\sigma_n))}{n^{k-\frac{1}{2}-\mathrm{card}(A)}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma_{\Pi}^2).$

Theorem (Hamaker and Rhoades (2022))

We assume that: for all i $\#_i(\sigma_n) \xrightarrow[n\to\infty]{d} 0.$

Then, $\frac{\mathfrak{N}^{\Pi}(\sigma_n)-\mathbb{E}(\mathfrak{N}^{\Pi}(\sigma_n))}{n^{k-\frac{1}{2}-\mathrm{card}(A)}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma_{\Pi}^2)$

If we combine both techniques.

Theorem (Not written anywhere)

We assume that: for all i $\frac{\#_i(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} 0.$

Then, $\frac{\mathfrak{N}^{\Pi}(\sigma_n)-\mathbb{E}(\mathfrak{N}^{\Pi}(\sigma_n))}{n^{k-\frac{1}{2}-\mathrm{card}(A)}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma_{\Pi}^2)$

Our result

Fix $\Pi = (\pi, A)$,

Theorem (Féray and Kammoun (2023))

We assume that σ_n is conjugacy invariant of size n and that $\frac{\#_1(\sigma_n)}{n} \xrightarrow[n\to\infty]{d} \alpha$, $\frac{\#_2(\sigma_n)}{n} \xrightarrow[n\to\infty]{d} \beta$. *Then*

$$\frac{\mathfrak{N}^{\Pi}(\sigma_n) - \mathbb{E}(\mathfrak{N}^{\Pi}(\sigma_n))}{n^{k-\frac{1}{2}-\mathrm{card}(A)}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \sigma^2_{\Pi,\alpha\beta}).$$

Moreover, if $A = \emptyset$, then $\sigma^2_{\Pi,\alpha,\beta} = 0$ if and only if $(\alpha, \beta) = (1, 0)$.

Remarks:

- Hofer (2018) implies that $\sigma^2_{\Pi,0,0} > 0$ for any Π.
- It is easy to see that $\sigma^2_{\Pi,1,0} = 0$ for any Π. (Identity)
- $\sigma^2_{\Pi,\alpha\beta}$ is a polynomial in (α & β). (Hamaker and Rhoades (2022))
- Dubach (2024) proved the same result for classical patterns ($A = \emptyset$) + speed of convergence.

Conjecture: for any Π, $\sigma^2_{\Pi,\alpha,\beta} = 0$ if and only if $(\alpha, \beta) = (1, 0)$.
Questions: for which patterns, $\sigma^2_{\Pi,\alpha,\beta}$ does not depend on β ? (consecutive)?

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Comparison techniques

- Initially for the longest increasing subsequence / RSK (Kammoun 2018).
- Works for other combinatorial structures (coloured permutations, k-arrangements, etc.)

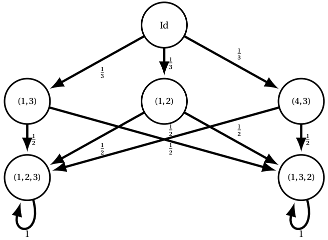
We give the proof of

Theorem (Kammoun 2020)

We assume that $\frac{\#(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} 0.$

Then, $\frac{\mathfrak{N}^{\Pi}(\sigma_n) - \mathbb{E}(\mathfrak{N}^{\Pi}(\sigma_n))}{n^{k-\frac{1}{2}-\mathrm{card}(A)}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \sigma_{\Pi}^2).$

Simple random walk a directed version of the Cayley graph of $\mathfrak{S}_n$.



- If we start from any conjugacy invariant measure, the stationary measure is Ewens with parameter 0.
- In each step, $\mathfrak{N}^{\Pi}$ varies at most by $\frac{2}{k!} n^{k-\mathrm{card}(A)-1}$.

$$\begin{aligned} \left|\mathfrak{N}^{\Pi}(\sigma_n) - \mathfrak{N}^{\Pi}(\sigma_n^{unif})\right| &\leq \left|\mathfrak{N}^{\Pi}(\sigma_n) - \mathfrak{N}^{\Pi}(\sigma_{0,n}^{Ew})\right| + \left|\mathfrak{N}^{\Pi}(\sigma_{0,n}^{Ew}) - \mathfrak{N}^{\Pi}(\sigma_n^{unif})\right| \\ &\leq \frac{2}{k!} n^{k-\mathrm{card}(A)-1} (\#\sigma_n + \underbrace{\#\sigma_n^{unif}}_{\approx \log(n)}) \end{aligned}$$

We want that $\left|\mathfrak{N}^{\Pi}(\sigma_n) - \mathfrak{N}^{\Pi}(\sigma_n^{unif})\right| = o(n^{k-\mathrm{card}(A)-\frac{1}{2}})$.
It is sufficient that $\#\sigma_n = o(\sqrt{n})$.

Weighted dependency graphs

Initially developed by Féray (2018).
Works for other combinatorial structures.
We give a proof of

Theorem (Féray and Kammoun (2023))

We assume that σ_n is conjugacy invariant of size n and that $\frac{\#_1(\sigma_n)}{n} \xrightarrow[n\to\infty]{d} \alpha$, $\frac{\#_2(\sigma_n)}{n} \xrightarrow[n\to\infty]{d} \beta$. *Then*

$$\frac{\mathfrak{N}^{\Pi}(\sigma_n) - \mathbb{E}(\mathfrak{N}^{\Pi}(\sigma_n))}{n^{k-\frac{1}{2}-\mathrm{card}(A)}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \sigma^2_{\Pi,\alpha\beta}).$$

Cumulants

Definition

$$\kappa_r(X_1,\dots,X_r) = [t_1 t_2 \cdots t_r] \log(\mathrm{E}(e^{\sum_{j=1}^n t_j X_j}))$$

For simplicity, we write $\kappa_r(X) := \kappa_r(X,\cdots,X)$.

- $X \sim \mathcal{N}(m,\sigma^2)$ if and only if for all $r \geq 3$, $\kappa_r(X) = 0$
- If X_1 and X_2 are independent, then $\kappa_r(X_1+X_2) = \kappa_r(X_1) + \kappa_r(X_2)$
- $\kappa_r(X+C) = \kappa_r(X)$ if $r \geq 2$
- $\kappa_r(\alpha X) = \alpha^r \kappa_r(X)$
- If $\{X_1,\dots X_i\}$ and $\{Y_{i+1},\dots Y_r\}$ are independent (and non-empty), then $\kappa_r(X_1,\dots,X_i,Y_{i+1},\dots,Y_r) = 0$

Proof of the CLT For $r \geq 3$

$$K_r\left(\frac{\sum_{i=1}^n X_i - n\mathbb{E}(X_1)}{\sqrt{n}}\right) = K_r\left(\frac{\sum_{i=1}^n X_i}{\sqrt{n}}\right) = \frac{1}{n^{\frac{r}{2}}}\sum_{i=1}^n \kappa_r(X_i) = \frac{n}{n^{\frac{r}{2}}}\kappa_r(X_1) = o(1)$$

Weak dependency

- If $\{X_1, \dots, X_r\}$ are "weakly dependent", then $\kappa_r(X_1, \dots, X_r) \approx \mathbf{0}$.
- Dependency graphs: a graph with weights on the edges. Vertices are indexed by random variables, and weights measure the "dependency".
- If the weights are sufficiently "small", we have a CLT for the sum of the variables.

Uniform Permutation

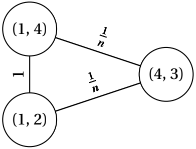
- Example: σ_n is uniform and $A_{i,j} = 1[\sigma_n(i) = j]$.
- If $i \neq j$ and $k \neq m$, then

$$\mathbb{E}(A_{i,k}A_{j,m}) = \frac{1}{n(n-1)} \approx \frac{1}{n^2} = \mathbb{E}(A_{i,k})\mathbb{E}(A_{j,m}).$$

- if $k \neq m$, then $\mathbb{E}(A_{i,k}A_{i,m}) = 0$ and $\mathbb{E}(A_{i,k})\mathbb{E}(A_{j,m}) = \frac{1}{n^2}$.

For any $U = (i_\ell, j_\ell)_{1 \le \ell \le r}$, let $G(U)$, be the complete graph with vertices U and the weight of $((i,j),(k,l))$ is $\begin{cases} 1 & \text{if } i = k \text{ or } j = l \\ \frac{1}{n} & \text{otherwise.} \end{cases}$

For example, if $U = ((1,4),(1,2),(4,3),(1,2))$, $G(U)$



Uniform Permutation

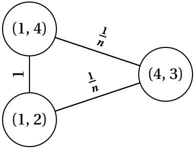
Theorem (Féray 2018)

For all $r \geq 1$, there exists C_r such that: For all integers n, for all $U = (i_\ell, j_\ell)_{1 \leq \ell \leq r}$

$$\kappa_r(A_{i_1,j_1}, \ldots, A_{i_r,j_r}) \leq C_r \mathrm{M(U)} n^{-\mathrm{card}(U)}$$

where

- *$M(U)$ is the maximum weight of a spanning tree of $G(U)$.*
- *$\mathrm{card}(U)$ is the number of distinct elements in U.*

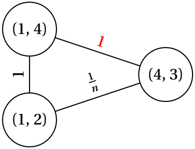
For example, if $U = ((1,4),(1,2),(4,3),(1,2))$, $G(U) =$ 

For all n, $\kappa_r(A_{1,4}, A_{1,2}, A_{4,3}, A_{1,2})) \leq C_4 \frac{1}{n} n^{-3} = C_4 n^{-4}$

New graphs

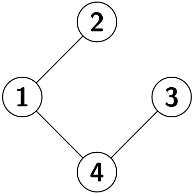
- $G^1(U)$, the complete graph with vertices U and the weight of $((i,j),(k,l))$ is 1 if $i=k$ or $j=l$ or $i=j$ or $k=l$, and $\frac{1}{n}$ otherwise.

For example, if $U=((1,4),(1,2),(4,3),(1,2))$, $G^1(U)=$



- $G^2(U):=([n],E=U)$

For example, if $U=((1,4),(1,2),(4,3),(1,2))$, $G^2(U)=$



Uniform Permutation within a Conjugacy Class

σ_n^λ is uniform within the conjugacy class λ and $A_{i,j} = 1[\sigma_n^\lambda(i) = j]$.

Theorem (Féray and Kammoun 2023)

For all $r \geq 1$, there exists C_r such that: For all integers n, for all $U = (i_\ell, j_\ell)_{1 \leq \ell \leq r}$

$$\kappa_r(A_{i_1,j_1}, \ldots, A_{i_r,j_r}) \leq C_r \mathrm{M(U)} n^{CC(U) - \mathrm{card}(U)}$$

where

- *$M(U)$ is the maximum weight of a spanning tree of $G^1(U)$, the complete graph with vertices U and the weight of $((i,j),(k,l))$ is 1 if $i = k$ or $j = l$ or $i = j$ or $k = l$, and $\frac{1}{n}$ otherwise.*
- card(U) *is the number of distinct elements in U.*
- *$CC(U)$ the number of nontrivial connected components in the graph $G^2(U) = ([n], E = U)$*

Application: Patterns

If we denote by $X^{(\pi,A)}$ the number of occurrences of the pattern (π, A), we have

$$X^{(\pi,A)}(\sigma_n^\lambda) = \sum_{\substack{i_1<\dots<i_k \\ i_{s+1}=i_s+1 \text{ for } s\in A}} \sum_{\substack{j_1,\dots,j_k \\ j_{\pi^{-1}(1)}<\dots<j_{\pi^{-1}(k)}}} A_{i_1,j_1}\cdots A_{i_k,j_k}.$$

To conclude: The magic of weighted dependency graphs: We can "easily" move from controlling mixed cumulants of $\left\{A_{i,j} : (i,j) \in [n]^2\right\}$ to controlling mixed cumulants of $\left\{A_{i_1,i_2}\cdots A_{i_r,j_r} : (i_1,j_1,\dots,i_r,j_r) \in [n]^{2r}\right\}$.
We obtain

$$\kappa_r(X^{(\pi,A)}(\sigma_n^\lambda)) \le C_{k,r} n^{r(k-\mathrm{card}(A)-1)+1},$$

and thus

$$\kappa_r\left(\frac{X^{(\pi,A)}(\sigma_n^\lambda) - \mathbb{E}(X^{(\pi,A)}(\sigma_n^\lambda))}{n^{k-\mathrm{card}(A)-\frac{1}{2}}}\right) \le C_{k,r} n^{1-\frac{r}{2}}$$

Motivation: universality

Central Limit Theorem
Let $X_1, X_2, \dots, X_n$ be i.i.d with $\mathrm{Var}(X_i) = \sigma^2 < +\infty$. Then,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X_1)\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

.

The limit is universal (does not depend on the distribution of X_i).

Symmetry/independence + control = universality

Fisher-Tippett-Gnedenko Theorem

Let $X_1, X_2, \dots, X_n$ be i.i.d and $M_n = \max(X_1, X_2, \dots, X_n)$.
Suppose there exist constants $a_n > 0$ and b_n such that, for every real x,

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) \to G(x)$$

where $G(x)$ is a non-degenerate cumulative distribution function. Then, G is the cumulative distribution function of a Gumbel, Fréchet, or Weibull variable.
The limit fluctuations depend on the tail of the distribution of X_1.

Symmetry/Independence + Control = Universality

Wigner Matrices

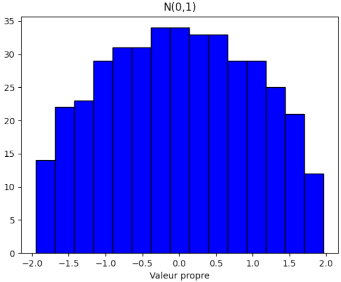
Let's define the symmetric matrix M as

$$M = \frac{1}{\sqrt{n}} \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & \dots & a_{1,n} \\ a_{1,2} & a_{2,2} & \dots & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \dots & \dots & a_{n,n} \end{bmatrix}$$

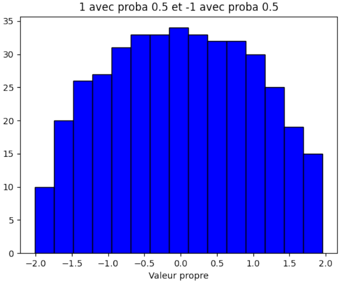
The entries $\{a_{i,j}\}_{1 \le i \le j \le n}$ are i.i.d. such that $\mathbb{E}(a_{1,1}) = 0$ and $\mathbb{E}(a_{1,1}^2) = 1$.

Let $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$ be the eigenvalues of M.

Histogram of Eigenvalues



Gaussian entries



Entries 1 or -1

Wigner's theorem

"The histogram of eigenvalues is not far from a semi-circle"

Theorem

The empirical spectral measure of the eigenvalues of M

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i},$$

converges weakly to the semi-circular law of Wigner as n tends to infinity.

But also[*],

- The largest eigenvalue converges to 2
- The fluctuations of the largest eigenvalue are of Tracy-Widom type
- Large deviations of the largest eigenvalues are universal
- The joint limit fluctuations of the first k eigenvalues are universal
- The local limit laws are universal
- The fluctuations of the number of points in [a,b] are universal

And for random permutations?

[*] Some conditions apply on the moments / the tail of the distribution

Longest Decreasing Subsequence

- $(\sigma(i_1), \dots, \sigma(i_k))$ is a decreasing subsequence of σ if $i_1 < i_2 < \cdots < i_k$ and $\sigma(i_1) > \cdots > \sigma(i_k)$.

Longest Decreasing Subsequence

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- LDS(σ): The length of the longest decreasing subsequence of σ.

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- LDS(σ): The length of the longest decreasing subsequence of σ.
- Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & \color{red}{8} & \color{red}{7} & \color{red}{5} & 2 & \color{red}{4} & \color{red}{3} \end{pmatrix}$$

LDS(σ) = 5.

Longest Decreasing Subsequence: Universality

We assume that σ_n is conjugation invariant and $\frac{\#_1(\sigma_n)}{n} \to \alpha$

Theorem (Dubach (2024+))

$$\frac{\mathrm{LDS}(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} 2\sqrt{1-\alpha}$$

Theorem (Kammoun 2018)

If $n^{\frac{-1}{6}} \min_{1\le i\le n} \left(\left(\sum_{j=1}^{i} \#_j(\sigma_n) \right) + \frac{\sqrt{n}}{i} \sum_{j=i+1}^{n} \#_j(\sigma_n) \right) \xrightarrow[n\to\infty]{\mathbb{P}} 0$, *then,* $\frac{\mathrm{LDS}(\sigma_n)-2\sqrt{n}}{\sqrt[6]{n}} \xrightarrow[n\to\infty]{d}$ *Tracy Widom*

Theorem (Guionnet, Kammoun 2023)

If σ_n is conjugacy invariant and $\#(\sigma_n) = o(\sqrt{n})$. Then, $\frac{\mathrm{LDS}(\sigma_n)}{\sqrt{n}}$ satisfies a LD principle

- *with speed $\sqrt{n}$ and rate function $J_{LDS,\frac{1}{2}}$.*
- *with speed n and rate function $J_{LDS,1}$*

With,

$$J_{LDS,\frac{1}{2}}(x) = \begin{cases} 2x\cosh^{-1}\frac{x}{2} & \text{if } x > 2 \\ +\infty & \text{if } x \leq 2 \end{cases}.$$

$$J_{LDS,1}(x) = \begin{cases} -1 + \frac{x^2}{4} + 2\ln\left(\frac{x}{2}\right) - \left(2 + \frac{x^2}{2}\right)\ln\left(\frac{2x^2}{4+x^2}\right) & \text{if } 0 < x \leq 2 \\ 0 & \text{if } x > 2 \\ +\infty & \text{if } x \leq 0 \end{cases}.$$

In other words: if σ_n is conjugation invariant and $\#(\sigma)$ "is low" then

$$-\log\left(\mathbb{P}\left(\frac{\mathrm{LDS}(\sigma_n)}{\sqrt{n}} \approx x\right)\right) \approx \begin{cases} \left(-1+\frac{x^2}{4}+2\ln\left(\frac{x}{2}\right)-\left(2+\frac{x^2}{2}\right)\ln\left(\frac{2x^2}{4+x^2}\right)\right) n & \text{if } x \in]0,2[] \\ 2x\cosh^{-1}\left(\frac{x}{2}\right) \sqrt{n} & \text{if } x > 2 \\ +\infty & \text{if } x \leq 0 \\ 0 & \text{if } x = 2 \end{cases}.$$

The same phenomenon appears for λ_1 (Wigner Matrices).

What we know

Type 1: Local events

- $\mathbb{P}(S \subset D(\sigma))$
- $\mathbb{P}(\sigma(10) > 10)$

Type 2: LLN / first order / global convergence

- $\frac{\mathfrak{N}^{\Pi}}{n^{k-\mathrm{card}(A)}}$
- $\frac{\mathrm{LDS}}{\sqrt{n}}$

The limit depends only on $\frac{\#_1}{n}$

Type 3: fluctuations (Poisson / Normal)

- $\mathrm{Tr}((\sigma_n \rho_n \pi_n \sigma_n^{-1} \rho_n^{-1} \pi_n)^{2024})$
- $\frac{\mathfrak{N}^{\Pi} - \mathbb{E}(\mathfrak{N}^{\Pi})}{n^{k-\mathrm{card}(A)-\frac{1}{2}}}$

The limit depends on $\frac{\#_1}{n}$ and $\frac{\#_2}{n^{\alpha}}$ for some α

Type 4: others

- $\frac{\mathrm{LDS}-2\sqrt{n}}{n^{\frac{1}{6}}}$
- Large deviations.

Universality if # is low.
There is still much work to be done.

	Exact calculation	Representations	Method of moments (and its variants)		Comparison	Geometric
			Random matrices	Dependency graphs		
Universality Permutations	Fulman, Kim, Lee	Hamaker and Rhoades	Kammoun and Maïda	Féray and Kammoun	Guionnet and Kammoun	Dubach
Functions	Descents Valleys	Descents Inversions Partterns (classic, (bi)-vincular, LAS	Trace of words	Descents Inversions Partterns (classic, vincular) Long.Altern.Sub	Descents Inversions Partterns (classic, vincular) Long.Altern.Sub LDS Long.Com.Sub. RSK Bord RSK shape Gran dev	Inversion Partterns classic RSK Shape LDS(order 1)
Limits	Normal	Constant	Poisson Mixtures	Normal	Normal Tracy Widom Airy, VKLS	Normal
Arxiv	2018, 2018 2019	2022	2019, 2022	2023	2018, 2020 2023	2024+

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‘Merci de votre attention