A canonical tree decomposition for chirotopes

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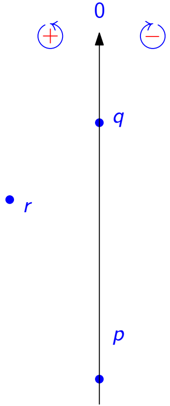
Orientation, chirotope of labelled points



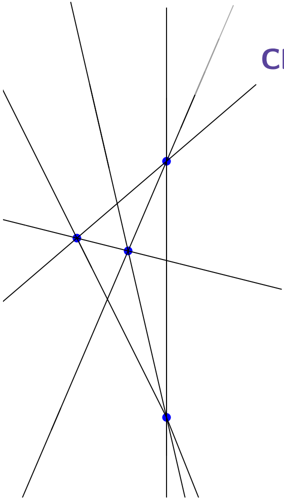
Orientation of three labelled points

$$\chi(p,q,r) = \begin{cases} +1 & \text{if } \mathfrak{p}_p, \mathfrak{p}_q, \mathfrak{p}_r \text{ oriented CCW,} \\ -1 & \text{if } \mathfrak{p}_p, \mathfrak{p}_q, \mathfrak{p}_r \text{ oriented CW,} \\ 0 & \text{if } \mathfrak{p}_p, \mathfrak{p}_q, \mathfrak{p}_r \text{ aligned.} \end{cases}$$

$$\chi(p,q,r) = sign \begin{vmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ 1 & 1 & 1 \end{vmatrix}$$

Remark: $\chi(p,q,r) = \chi(q,r,p) = \chi(r,p,q)$

Orientation, chirotope of labelled points



Chirotope of labelled points set

- Point set $\mathcal{P} = \{\mathfrak{p}_\ell\}_{\ell \in X}$ labelled by X
- Points in general position (no three aligned, no parallel lines)
- $\chi_{\mathcal{P}} : (X)_3 \to \{-1, +1\}$

$$\chi_{\mathcal{P}}(x, y, z) = \begin{cases} +1 & \text{if } \mathfrak{p}_x, \mathfrak{p}_y, \mathfrak{p}_z \text{ oriented CCW,} \\ -1 & \text{if } \mathfrak{p}_x, \mathfrak{p}_y, \mathfrak{p}_z \text{ oriented CW.} \end{cases}$$

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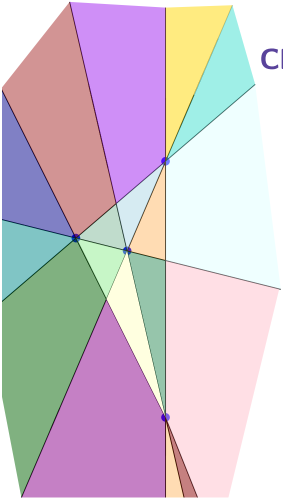

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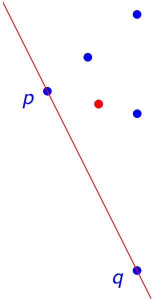
- Chirotopes allow to abstract from coordinates

Chirotopes encode many useful properties



Example of properties

- $[p, q] \in Conv(\mathcal{P})$
  $\Leftrightarrow \forall x \in X \setminus \{p, q\}, \chi(p, q, x) = cst$

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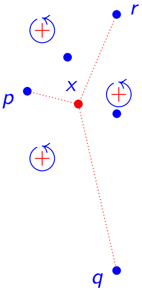

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- $x \in \Delta(p,q,r) \Leftrightarrow$
  $\chi_{\mathcal{P}}(p,q,x) = \chi_{\mathcal{P}}(q,r,x) = \chi_{\mathcal{P}}(r,p,x)$

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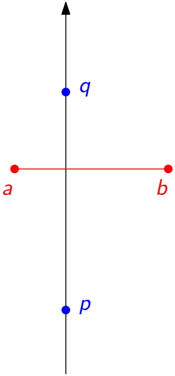

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  $\chi_{\mathcal{P}}(p, q, x) = \chi_{\mathcal{P}}(q, r, x) = \chi_{\mathcal{P}}(r, p, x)$
- (p, q) separates $[a, b]$
  $\Leftrightarrow \chi(p, q, a) = -\chi(p, q, b)$
- $[p, q]$ and $[a, b]$ are intersecting $\Leftrightarrow \ldots$

Chirotopes

Chirotopes are a useful combinatorial and geometric object...

- Finite number t_n of chirotopes on n elements

- Useful for exact algorithms, not depending on coordinates
- Can be used for benchmarking algorithms

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...but also quite complex to understand!

- Decide whether $f : (X)_3 \to \{-1, 1\}$ is realizable is NP-hard [Shor 91]
- Number of chirotopes t_n exactly known only for $n \leq 11$ (up to relabelling) [Aichholzer et al 2002]
- Best known asymptotics $t_n = n^{4n+\Theta(n/\log n)}$ [Goodman and Pollak 93]

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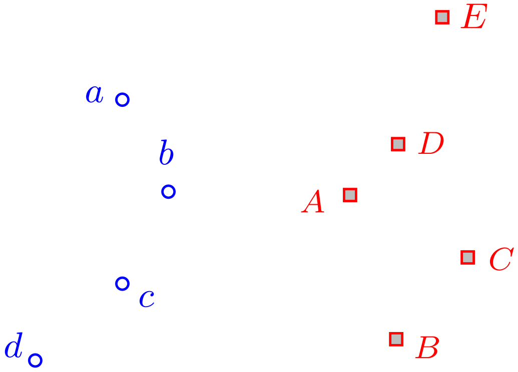
The space of chirotopes is hard to explore!

Mutually avoiding sets, modular decomposition



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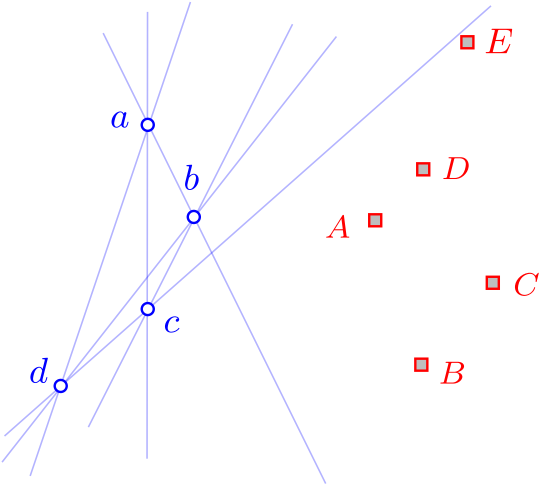

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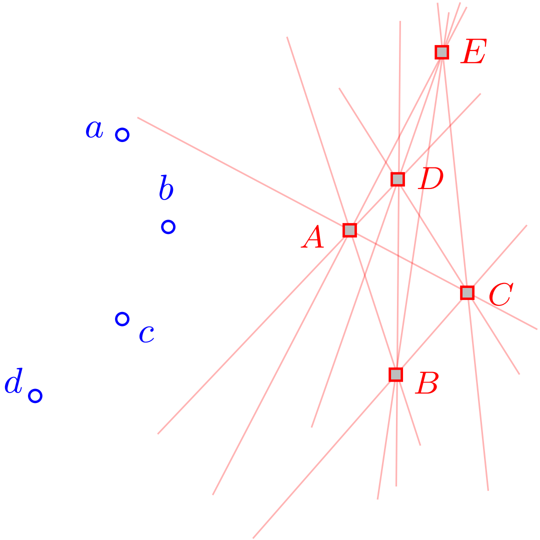

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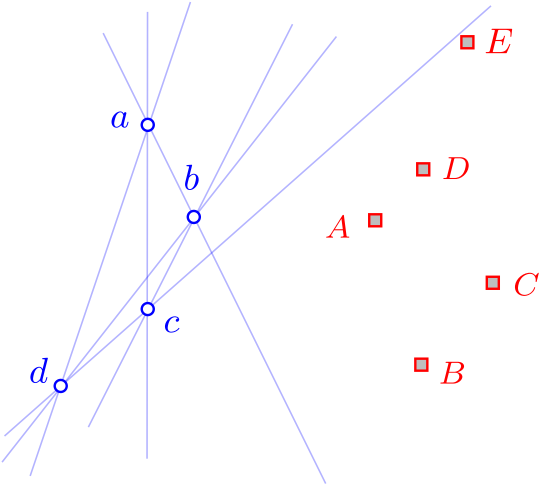

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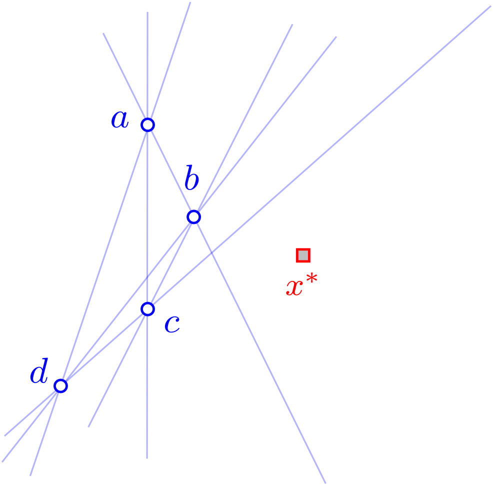

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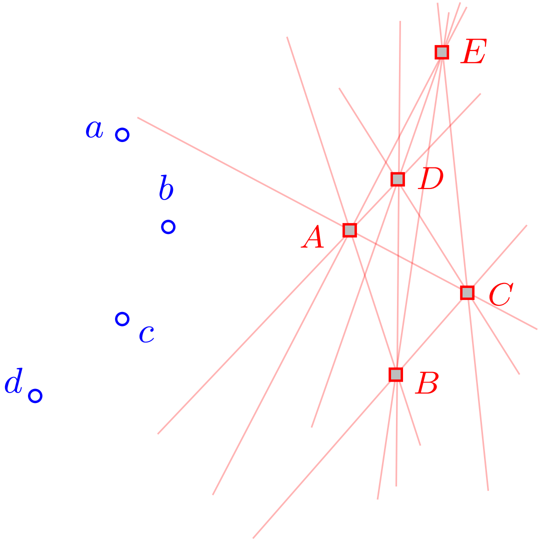

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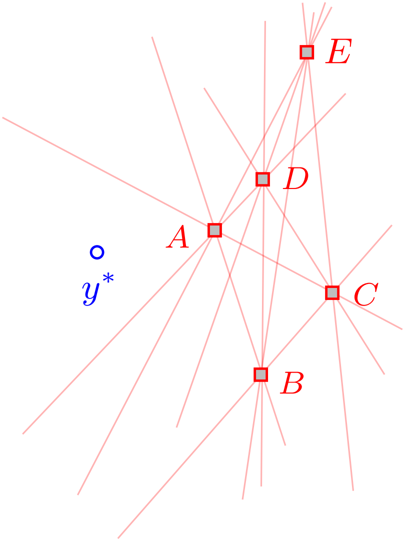

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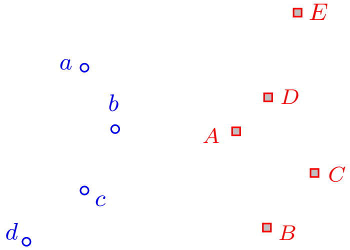

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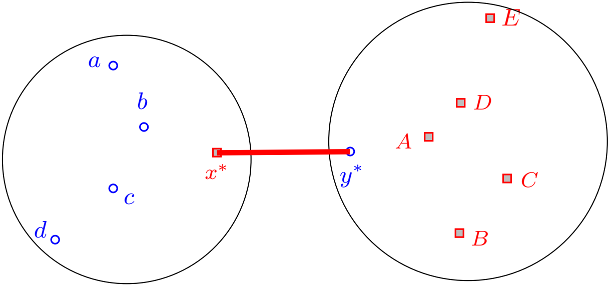
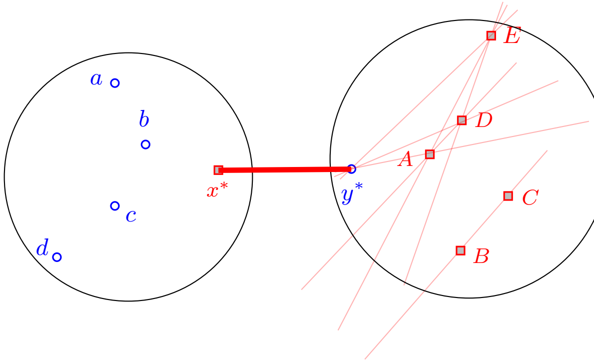


Chirotope is read following proxies!

$$\chi(a, d, B) = \chi(a, d, x^*)$$

Mutually avoiding sets, modular decomposition



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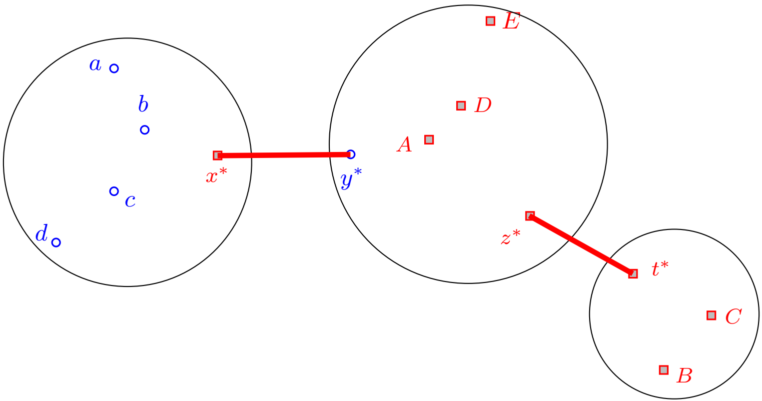
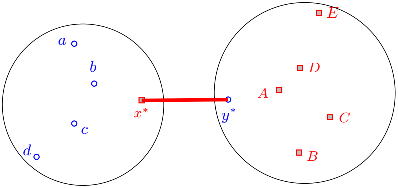


Chirotope is read following proxies!

$$\chi(a, d, B) = \chi(a, d, x^*) \qquad \chi(a, E, B) = \chi(y^*, E, z^*)$$

Converse operation: Bowtie operation



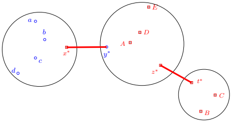
- χ sign function on $X \cup \{x^*\}$, and ξ on $Y \cup \{y^*\}$
- the bowtie $\kappa \stackrel{\text{def}}{=} \chi\,{}_{x^*}\!\bowtie_{y^*} \xi$ is defined on $X \cup Y$ by:

$$\begin{cases} \kappa(x_1, x_2, x_3) & = & \chi(x_1, x_2, x_3) & \text{if } x_1, x_2, x_3 \text{ are all in } X; \\ \kappa(x_1, x_2, y) & = & \chi(x_1, x_2, x^*) & \text{if } x_1, x_2 \text{ are in } X \text{ and } y \text{ is in } Y; \\ \kappa(x, y_2, y_3) & = & \xi(y^*, y_2, y_3) & \text{if } x \text{ is in } X \text{ and } y_2, y_3 \text{ are in } Y; \\ \kappa(y_1, y_2, y_3) & = & \xi(y_1, y_2, y_3) & \text{if } y_1, y_2, y_3 \text{ are all in } Y. \end{cases}$$

- $\chi\,{}_{x^*}\!\bowtie_{y^*} \xi$ is a realizable chirotope if and only if χ and ξ are realizable and x^* and y^* are extreme in χ and ξ. [Bouvel,Féray,Goaoc,K.]

First properties of decomposition

- κ is indecomposable if there is no nontrivial decomposition $\kappa = \chi \,_{x^*}\!\bowtie_{y^*} \xi$
- Every chirotope admits a decomposition built from indecomposable chirotopes
- Every decomposition can be represented by a (indecomposable) chirotope tree



- It allows a nice description of a realizable chirotope while avoiding providing a full realization

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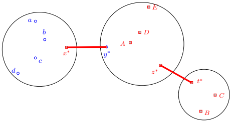
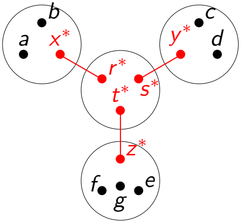
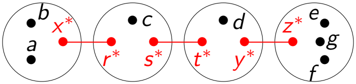


- It allows a nice description of a realizable chirotope while avoiding providing a full realization
- Is the chirotope tree canonical (unique)?

Convex issue

Two trees with the same associated chirotope:





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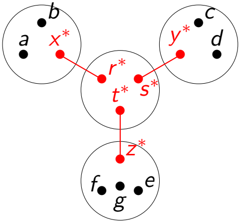
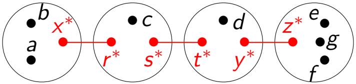
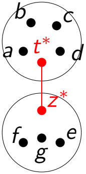




But if we merge / recompose adjacent convex nodes, same tree:



Unicity of decomposition

A chirotope tree is canonical:

- If every node is either convex or indecomposable
- There is no edge between two convex nodes

Proposition: Every chirotope admits a unique canonical tree (up to relabelling the proxies) [BFGK.]

Proof sketch

Define two transformations on chirotope trees:

- $\xrightarrow{\lozenge}$ that merges two adjacent convex nodes
- $\xrightarrow{\bowtie}$ that decomposes a nonconvex node in two

Proposition: **[BFGK.]** The transformation $\Rightarrow := \xrightarrow{\lozenge} \cup \xrightarrow{\bowtie}$ terminates and is locally confluent: if $T \Rightarrow T_1$ and $T \Rightarrow T_2$ then there exists T_3 such that $T_1 \Rightarrow^* T_3$ and $T_2 \Rightarrow^* T_3$.

The main difficulty resides in the case where a node can be decomposed in two manners.

Good/Bad news

Good news:

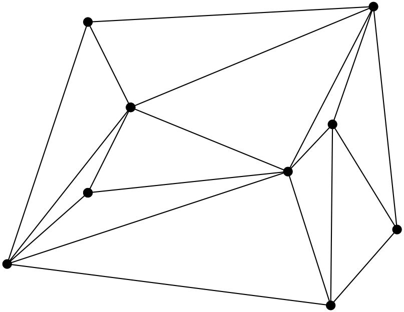
- Chirotope trees provide a nice way to build chirotopes from smaller ones
- Unicity gives a simple way to prove that two chirotopes built recursively are different

Bad news: For n large enough we have [BFGK.]

$$d_n/t_n = \mathcal{O}(n^{-3})$$

Triangulations

A triangulation of a point set $\mathcal{P}$ is a maximal crossing-free set of edges between elements of $\mathcal{P}$.



The set of triangulations of $\mathcal{P}$ only depends on its chirotope.

We write $\mathcal{T}_\kappa$ the set of triangulations of a chirotope κ.

Triangulations

Many questions are still open:

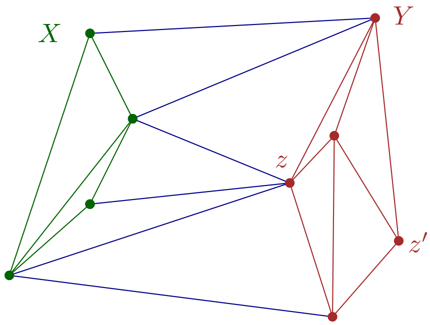
- For every κ on n elements,

  $|\mathcal{T}_\kappa| = \mathcal{O}(30^n)$ [Sharir and Sheffer, 2011]
  $|\mathcal{T}_\kappa| = \Omega(2.63^n)$ [Aichholzer et al, 2016]
- It is conjectured that the minimal is $\mathcal{O}(3.47^n)$ [Hurtado and Noy 97]
- Max known: Koch chains has $\approx 9.08^n$ triangulations [Rutschmann and Wettstein 2022]

Algorithmically: Compute the number of triangulations of a given point set:

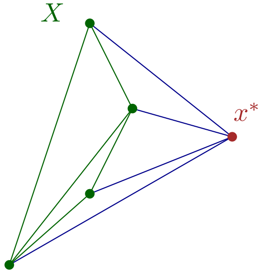
- $\mathcal{O}(n^2 2^n)$ [Alvarez and Seidel 2013]
- $\mathcal{O}(n^{(11+o(1))\sqrt{n}})$ [Marx and Miltzow 2016]

Triangulations of bowties



$T \in \mathcal{T}_\kappa$

Triangulations of bowties



$$T' = \pi_{Y \to x^*}(T)$$

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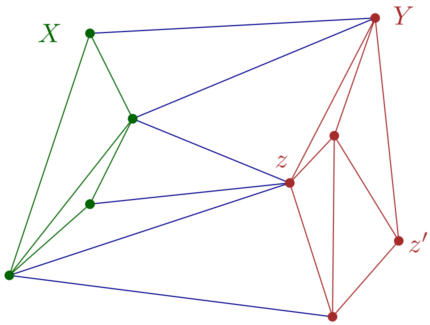

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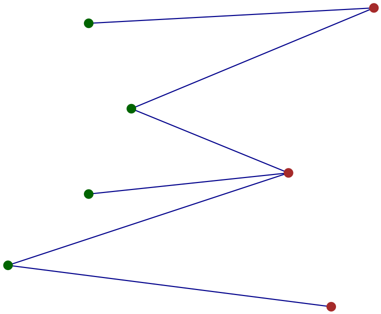
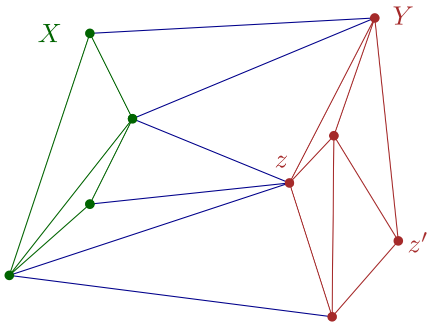


T_{XY}

Triangulations of bowties



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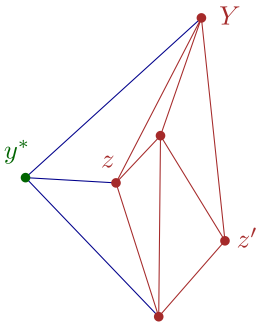
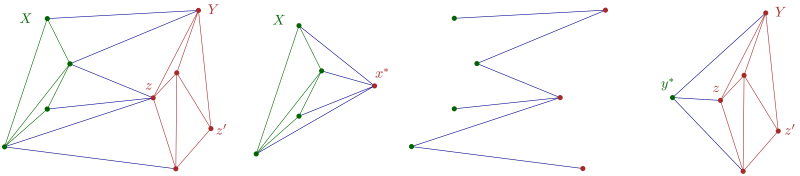
Triangulations of bowties



$$T'' = \pi_{X \to y^*}(T)$$

Triangulations of bowties



$T \in \mathcal{T}_\kappa$ $\quad$ $T' = \pi_{Y \to x^*}(T)$ $\quad$ T_{XY} $\quad$ $T'' = \pi_{X \to y^*}(T)$

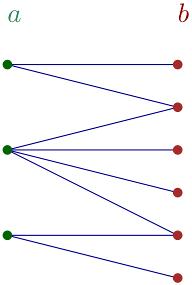
Bijection between:

- Triangulations of $\kappa = \chi \,_{x^*}\!\bowtie_{y^*} \xi$
- Triplets of triangulations of χ, ξ, and maximal crossing-free families of edged between the neighbors of x^* and y^*

$$|\mathcal{T}_\kappa| = \sum_{a,b \geq 2} \binom{a+b-2}{a-1} [x^a] P_{\chi,x^*}(x) \; [y^b] P_{\xi,y^*}(y).$$

Triangulations of bowties

Two ingredients:



$|T_{XY}|$ only depends on a and b:

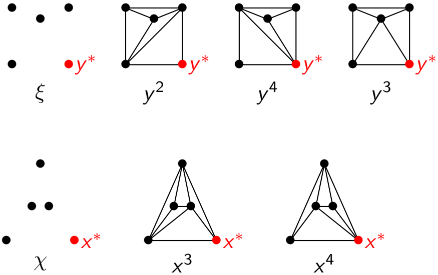
$$|T_{XY}| = \left(\!\!\binom{b}{a-1}\!\!\right) = \binom{a+b-2}{a-1}$$

Triangulations of bowties

Two ingredients: Triangulation polynomial

$$P_{\chi,x^*}(x) = \sum_{T \in \mathcal{T}_\chi} x^{\deg_T(x^*)} \qquad P_{\chi,x^*}(1) = |\mathcal{T}_\chi|$$

Example: $P_{\chi,x^*}(x) = x^3(x+1)$ and $P_{\xi,y^*}(y) = y^2(1+y+y^2)$.



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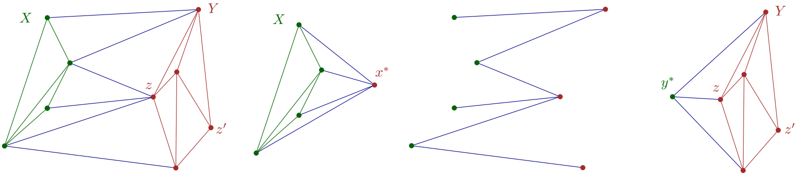

$T \in \mathcal{T}_\kappa$ $T' = \pi_{Y \to x^*}(T)$ T_{XY} $T'' = \pi_{X \to y^*}(T)$

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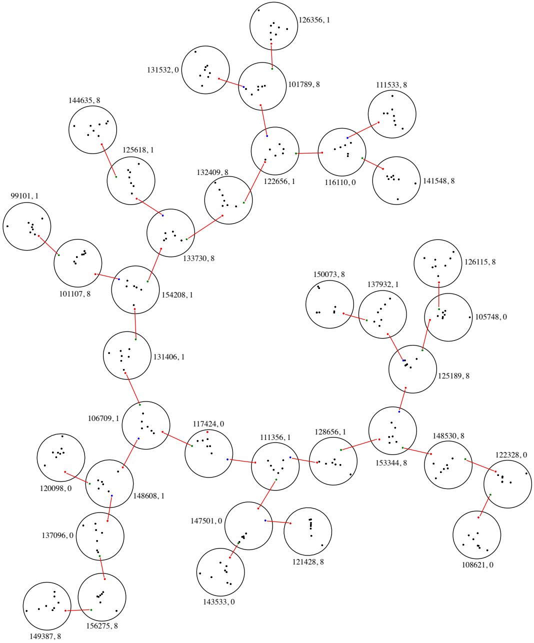
Triangulations of chirotope trees



36 nodes, each decorated with a chirotope of size 9, adding up to 254 elements.

Number of triangulations?

Triangulations of chirotope trees

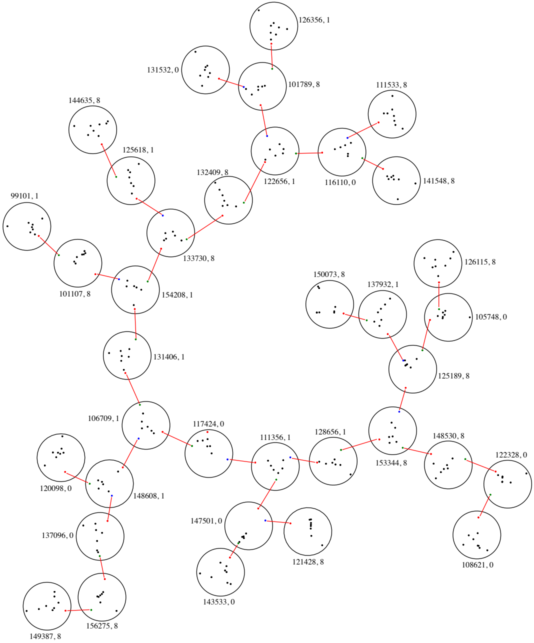
- Same ideas with the same bijection.
- However the full triangulation polynomial of every node is needed:

$$Q_{\xi,\{x_i^*\}}(x_1,\ldots,x_k,\{y_{i,j}\}_{1\le i<j\le k}) = \sum_{T\in\mathcal{T}_\xi}\prod_{i=1}^{k} x_i^{\deg_T(x_i^*)}\cdot \prod_{\substack{x_i^* x_j^*\in T\\ i<j}} y_{i,j}\,.$$

- and the formula are more complex, as we compute multivariate polynomials instead of just a number.

Conclusion: **[BFGK.]** the number of triangulation of a chirotope tree can be computed in polynomial time from the full triangulation polynomials of its nodes.

Triangulations of chirotope trees



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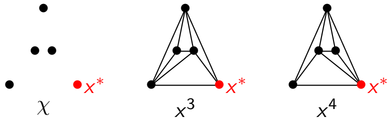
Number of triangulations:

$$|T_\kappa| \approx 5.92966751.10^{180}$$

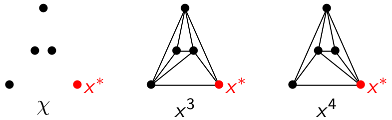
computed exactly in a few seconds using sage.

Computing triangulation polynomials

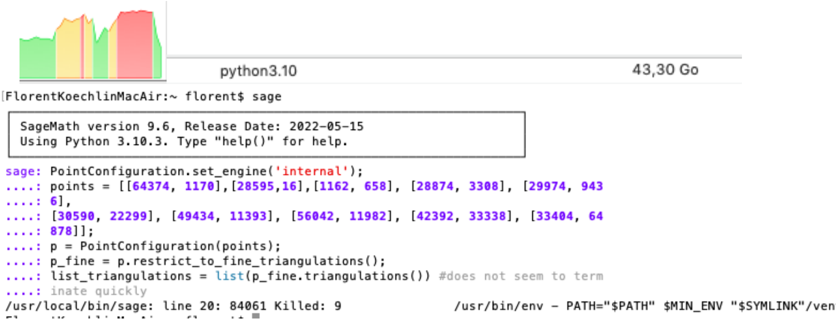


- First idea: enumerate all triangulations with Sage and deduce the polynomial

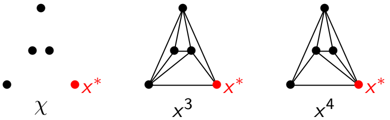
Computing triangulation polynomials



◦ First idea: enumerate all triangulations with Sage and deduce the polynomial → bug in Sage!



Computing triangulation polynomials



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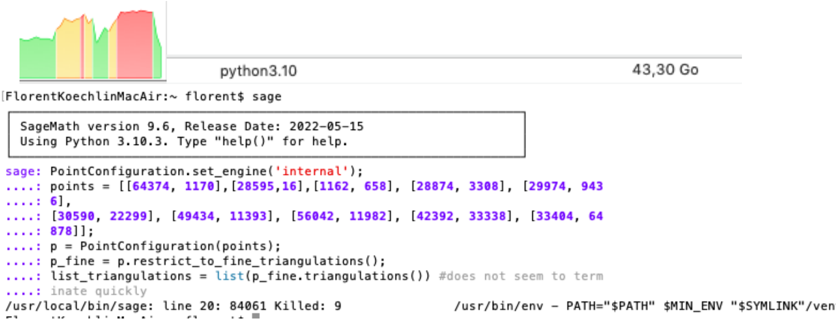

- Better idea: adapt the $\mathcal{O}(n^2 2^n)$ algorithm of [Alvarez and Seidel 2013] to compute the polynomial

Chain Triangulations: an analyzable case

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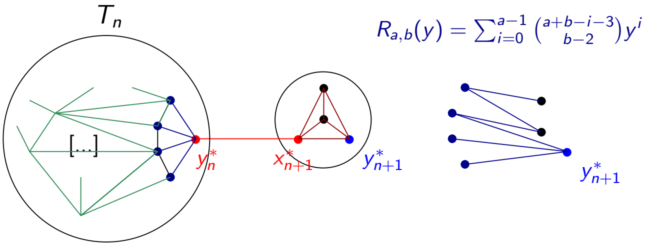

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- Let us introduce $F(y,u) \stackrel{\text{def}}{=} \sum_{k \geq 1} P_k(y) u^k$

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- Kernel method: find $y(u)$ analytic canceling the kernel.
- We obtain $F(1,u) = \sum_{n\geq 1} |\mathcal{T}_{\chi_n}| u^n$ (A066357 !) and

$$|\mathcal{T}_{\chi_n}| \sim_{n\to\infty} \frac{3-2\sqrt{2}}{\sqrt{2\pi}} \frac{16^n}{n^{3/2}}.$$

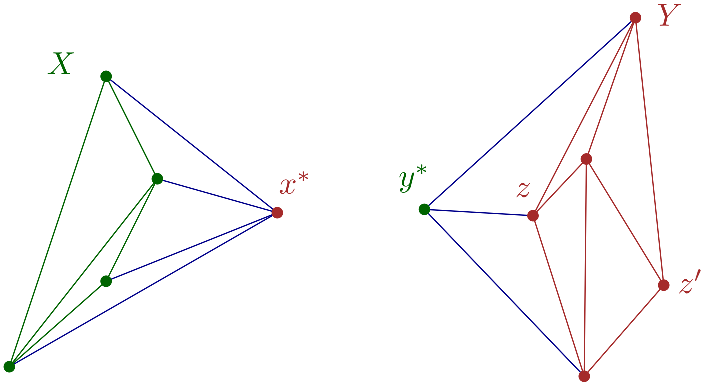
Conclusion

- We have seen a canonical chirotope decomposition
- Natural sens of "factorizing" chirotope
- The decomposition can be used to compute the number of triangulations of complex chirotopes

Many open questions:

- What is the complexity of computing the canonical decomposition?
- Can more complex configurations be analyzed analytically?
- Can we unify it with other classical constructions?

Further work: faster computation



$$|\mathcal{T}_\kappa| = \sum_{a,b\geq 2} \binom{a+b-2}{a-1} [x^a] P_{\chi,x^*}(x)\ [y^b] P_{\xi,y^*}(y)$$

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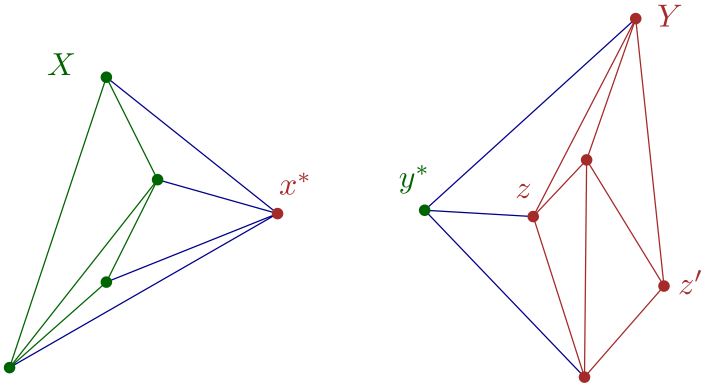

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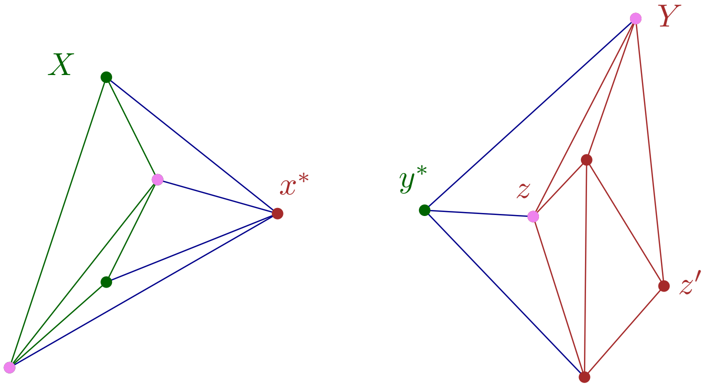

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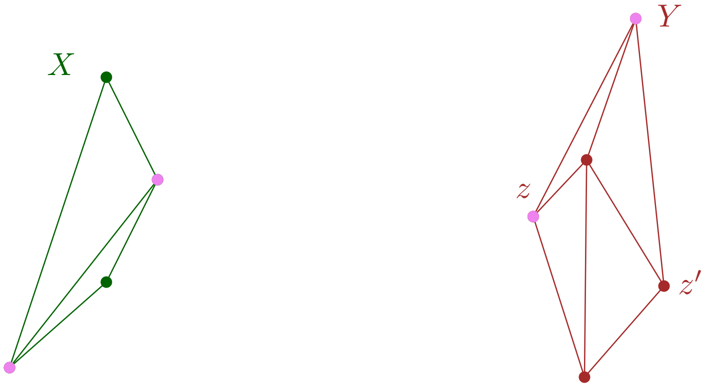

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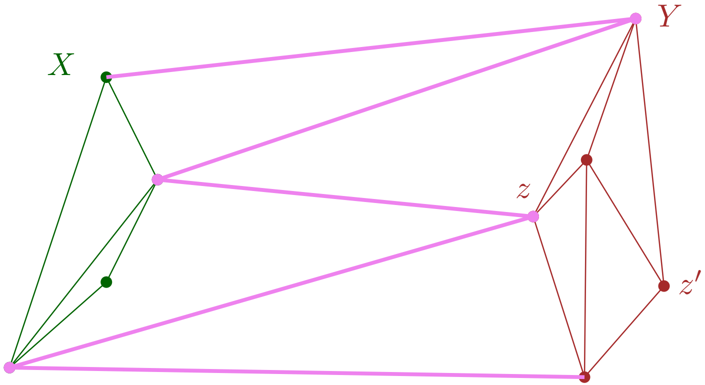

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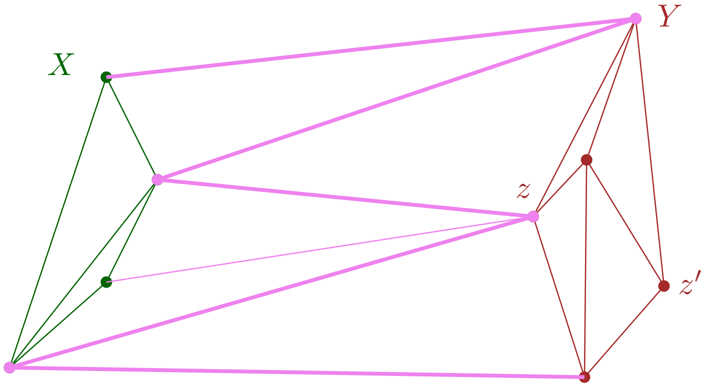

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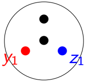
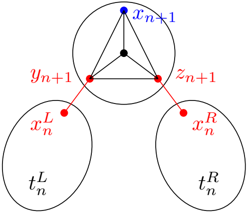
Further work: Binary chain

- Start with $\chi_1 =$ 

- Recursively build t_n:

$t_{n+1} =$

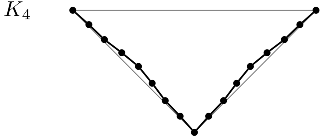


- Then if $Q_n(x) = (x+1)P_n(x+1)$:

$$Q_n(x) = \sum_{i=0}^{s} a_i x^i \Rightarrow Q_{n+1}(x) = (x+1)^4 \sum_{k=2}^{s} \left(\sum_{i=k}^{s} a_i x^{i-k} \right)^2$$

- Not analyzable for now, but greatly improves the possible number of iterations

Further work: Koch Chains [Rutschmann and Wettstein 2022]



- **Chain:** x-increasing sequence of points $x_1, \ldots, x_n$ such that $[x_i x_{i+1}]$ is forced in every triangulation
- every known configuration with many triangulations is a chain
- Every chain can be decomposed with only two operators $\wedge$ and $\vee$, and the basic chain of two points [Rutschmann and Wettstein 2022]

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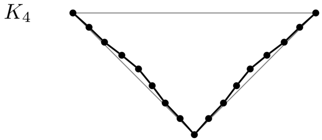

- **Koch chain**: best known configuration with maximal number of triangulations $\approx 9.08^n$
- Construction similar to ours (with "phantom" proxies)
- Similar techniques for counting triangulations
- We managed to generalize their construction beyond chains, but it seems that only chains reach the best number of triangulations

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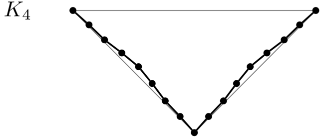

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Conlusion: still many things to look at!

Thank you!