# Algebraic Combinatorial Aspects of Nonlinear Differential Systems

Hoang Ngoc Minh
Equipe CALIN, LIPN - UMR 7030.

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# Summary

1. Introduction,
2. Nonlinear dynamical Systems,
3. Diagonal series,
4. Polylogarithms, multiple harmonic sums and polyzêtas,
5. Nonlinear differential equations.

# INTRODUCTION

# Particular cases : Fuchsian differential equations (FDE)

$$\dot{q}(z) = [M_0 u_0(z) + M_1 u_1(z)]\, q(z), \quad y(z) = \lambda q(z), \quad q(z_0) = \eta,$$

where $M_0, M_1 \in \mathcal{M}_{n,n}(\mathbb{C}), \lambda \in \mathcal{M}_{1,n}(\mathbb{C}), \eta \in \mathcal{M}_{n,1}(\mathbb{C})$ and $u_0(z), u_1(z) \in \mathcal{C}$.

Example (hypergeometric equation)

$$z(1-z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0 t_1 y(z) = 0.$$

Let $q_1(z) = y(z)$ and $q_2(z) = z(1-z)\dot{y}(z)$. One has

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \left[ \begin{pmatrix} 0 & 0 \\ -t_0 t_1 & -t_2 \end{pmatrix} \frac{1}{z} - \begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix} \frac{1}{1-z} \right] \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Here,

$\lambda = \begin{pmatrix} 1 & 0 \end{pmatrix}, M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix}, M_1 = -\begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix},$

$\eta = \begin{pmatrix} q_1(z_0) \\ q_2(z_0) \end{pmatrix}.$

# Examples of Nonlinear Dynamical Systems

## Example (harmonic oscillator)

$\dot{y}(z) + k_1 y(z) + k_2 y^2(z) = u_1(t).$

$$
\begin{aligned}
\dot{q}(z) &= A_0(q)u_0(z) + A_1(q)u_1(z) \quad \text{with } u_0(z) \equiv 1, \\
A_0 &= -(k_1 q + k_2 q^2)\frac{\partial}{\partial q}, \\
A_1 &= \frac{\partial}{\partial q}, \\
y(z) &= q(z).
\end{aligned}
$$

## Example (Duffing's equation)

$\ddot{y}(z) + a\dot{y}(z) + by(z) + cy^3(z) = u_1(z).$

$$
\begin{aligned}
\dot{q}(z) &= A_0(q)u_0(z) + A_1(q)u_1(z) \quad \text{with } u_0(z) \equiv 1, \\
A_0 &= -(aq_2 + b^2 q_1 + cq_1^3)\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q_1}, \\
A_1 &= \frac{\partial}{\partial q_2}, \\
y(z) &= q_1(z).
\end{aligned}
$$

# Previous work

For (FDE), one can base on the R. Jungen thesis[1] *"Sur les séries de Taylor n'ayant que des singularités algébrico-logarithmiques sur leur cercle de convergence"* (1931).

But for nonlinear differential equations ?
One can appoximate the nonlinear differential systems by linear ones, and then one can base one self on the Jungen's thesis.

[1]This thesis influence quitely the works of

- M.P. Schützenberger, *"On a theorem of R. Jungen"* (1962),
- M. Fliess, *"Sur divers produits de séries formelles"* (1974),
- Ph. Flajolet & A. Odlyzko, *"The Average Height of Binary Trees and Other Simple Trees"* (1982).

# NONLINEAR DYNAMICAL SYSTEMS

# Nonlinear Dynamical Systems

Let $(\mathcal{D}, d)$ be a $k$-commutative associative differential algebra with unit ($\mathrm{ch}(k) = 0$) and $\mathcal{C}$ be a differential subfield of $\mathcal{D}$.

$y(z) = \sum_{n \geq 0} y_n z^n$ is the output of :

$$(NLS) \quad \begin{cases} y(z) &= f(q(z)), \\ \dot{q}(z) &= A_0(q)u_0(z) + A_1(q)u_1(z), \\ q(z_0) &= q_0, \end{cases}$$

where :

- $u_0(z), u_1(z) \in \mathcal{C}$,
- the state $q = (q_1, \ldots, q_N)$ belongs the complex analytic manifold $Q$ of dimension $N$ and $q_0$ is the initial state,
- the observation $f \in \mathcal{O}$, with $\mathcal{O}$ is the ring of holomorphic functions over $Q$,
- For $i = 0..1$, $A_i = \sum_{j=1}^{N} A_i^j(q) \frac{\partial}{\partial q_j}$ is an analytic vector field[2] over $Q$, with $A_i^j(q) \in \mathcal{O}$, for $j = 1, \ldots, N$.

[2] A vector field $A_i$ is said to be linear if the $A_i^j(q), j = 1..N$, are constants.

# Structural $\mathbb{C}$-automaton associated to $(NLS)$

Any $(NLS)$ can be associated to a *structural* $\mathbb{C}$-*automaton* characterizing the structure of the differential algebra defined by $\{A_i\}_{i=0,1}$.
For any $i = 1, .., N$, let $D_j$ denotes $\partial/\partial q_i$. Let $\mathbf{r}$ be a multi-index $(r_1, \ldots, r_N)$ and let $D^{\mathbf{r}}$ denotes the differential operator $D_1^{r_1} \ldots D_N^{r_N}$.
The *infinite* structural $\mathbb{C}$-automaton is the 5-uple $(X, \mathcal{F}, I, \tau, \lambda)$, where

- $X = \{x_0, x_1\}$,
- $\mathcal{F}$ is the $\mathbb{C}$-vector space generated by the operators $D^{\mathbf{r}}$,
- $I$ is the initial state,
- $\tau(x_i), i = 0, ..1$, is the linear endomorphism of $\mathcal{F}$ describing the right action[3] of $A_i$ on differential operator $D^{\mathbf{r}}$,
- $\lambda$ is the row vector whose $i^{\mathtt{th}}$ component is $D_i f$.

The *truncated* structural $\mathbb{C}$-automaton is obtained by choosing the states that are met along the successful path and of length less or equal to $m$.
This gives a $\mathbb{C}$-automaton recognizes a rational power series over $X$.

[3]This action is given by $D^{\mathbf{r}} A_i = \sum_{j=1}^{N} \sum_{\mathbf{s} \leq \mathbf{r}} \binom{\mathbf{r}}{\mathbf{s}} D^{\mathbf{r}-\mathbf{s}} A_i^j(q) D^{\mathbf{s}} D_j$, with $\mathbf{r} = (r_1, \ldots, r_N), \mathbf{s} = (s_1, \ldots, s_N)$ and $\mathbf{s} \leq \mathbf{r} \iff s_1 \leq r_1, \ldots, s_k \leq r_N$ and $\binom{\mathbf{r}}{\mathbf{s}} = \prod_{j=1}^{N} \binom{r_j}{s_j}$.

# Examples of structural $\mathbb{C}$-automaton

## Example (harmonic oscillator)

Putting $F := -(k_1 q + k_2 q^2)$, one has $A_0 = FD, A_1 = D$.
$X = \{x_0, x_1\}, \mathcal{F} = \mathrm{span}_{\mathbb{C}}\{D^i\}_{i \geq 0}, I = \{\mathrm{Id}\}, \lambda = \begin{pmatrix} q & 1 & 0 & \ldots & 0 \end{pmatrix}$.
The $\mathbb{C}$-automaton cell is given by

$$\begin{aligned} D^i A_1 &= D^{i+1}, \\ D^i A_0 &= FD^{i+1} + \binom{i}{1}[DF]D^{i-1} + \binom{i}{2}[D^2 F]D^{i-2}. \end{aligned}$$

## Example (Duffing's equation)

Putting $F := -(aq_2 + b^2 q_1 + cq_1^3)$, one has $A_0 = FD_1 + D_2, A_1 = D_2$.
$X = \{x_0, x_1\}, \mathcal{F} = \mathrm{span}_{\mathbb{C}}\{D_1^i D_2^j\}_{i \geq 0}^{j \geq 0}, I = \{\mathrm{Id}\}, \lambda = \begin{pmatrix} q_1 & 1 & 0 & \ldots & 0 \end{pmatrix}$.
The $\mathbb{C}$-automaton cell is given by

$$\begin{aligned} D_1^i D_2^j A_1 &= D_1^i D_2^{j+1}, \\ D_1^i D_2^j A_0 &= FD_1^i D_2^{j+1} \\ &+ \binom{i}{1}[D_1 F]D_1^{i-1} D_2^{j+1} + \binom{i}{2}[D_1^2 F]D_1^{i-2} D_2^{j+1} + \binom{i}{3}[D_1^3 F]D_1^{i-3} D_2^{j+1} \\ &- jaD_1^i D_2^j + q_2 D_1^{i+1} D_2^j + jD_1^{i+1} D_2^{j-1}. \end{aligned}$$

# Our works

Let $X = \{x_0, x_1\}$ with $x_0 < x_1$. For any $w = x_{i_1} \cdots x_{i_k} \in X^*$, let

$$\mathcal{A}(1_{X^*}) = \mathrm{Id}, \qquad \mathcal{A}(w) = A_{i_1} \circ \ldots \circ A_{i_k},$$

$$\alpha_{z_0}^{z}(1_{X^*}) = 1, \qquad \alpha_{z_0}^{z}(w) = \int_{z_0}^{z} \int_{z_0}^{z_1} \ldots \int_{z_0}^{z_{k-1}} u_{i_1}(z_1)dz_1 \ \cdots \ u_{i_k}(z_k)dz_k.$$

**Theorem (Deneufchâtel,Duchamp,HNM, 2010)**

*Let $S = \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\, w \in \mathcal{D}\langle\langle X \rangle\rangle$. The following conditions are equivalent :*

- i) *The family $(\alpha_{z_0}^{z}(w))_{w \in X^*}$ of coefficients of $S$ is free over $\mathcal{C}$.*
- ii) *The family of coefficients $(\alpha_{z_0}^{z}(x))_{x \in X \cup \{1_{X^*}\}}$ is free over $\mathcal{C}$.*

Therefore, by successive Picard iterations, one get

$$y(z) = \sum_{w \in X^*} \mathcal{A}(w) \circ f(q_0)\, \alpha_{z_0}^{z}(w) = [(\mathcal{A} \otimes \alpha_{z_0}^{z})\mathcal{D}](f(q_0)),$$

where, $\mathcal{D} = \sum_{w \in X^*} w \otimes w$.

# Chen-Fliess generating series

- Chen series

$$S_{z_0 \rightsquigarrow z} = \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\; w.$$

Any Chen generating series $S_{z_0 \rightsquigarrow z}$ is group-like, for $\Delta_{\sqcup\!\sqcup}$, and it depends only on the homotopy class of $z_0 \rightsquigarrow z$ (**Ree**).

The product of two Chen generating series $S_{z_1 \rightsquigarrow z_2}$ and $S_{z_0 \rightsquigarrow z_1}$ is the Chen generating series $S_{z_0 \rightsquigarrow z_2} = S_{z_1 \rightsquigarrow z_2} S_{z_0 \rightsquigarrow z_1}$ (**Chen**).

- The generating series of the polysystem $\{A_i\}_{i=0,1}$ and of the observation $f \in \mathcal{O}$ is given by

$$\sigma f \quad := \quad \sum_{w \in X^*} \mathcal{A}(w) \circ f \;\; w \quad \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_N]\!]\langle\!\langle X \rangle\!\rangle.$$

$$\sigma f_{|_q} \quad := \quad \sum_{w \in X^*} \mathcal{A}(w) \circ f_{|q} \;\; w \quad \in \mathbb{C}\langle\!\langle X \rangle\!\rangle.$$

The last is called Fliess generating series of $\{A_i\}_{i=0,1}$ and of $f$ at $q$.

For any $f, g \in \mathcal{O}$ anf for any $\lambda, \mu \in \mathbb{C}$, one has (**Fliess**)

$$\sigma(\nu f + \mu g) = \sigma(\nu f) + \sigma(\mu g) \quad \text{and} \quad \sigma(fg) = \sigma f \sqcup\!\sqcup \sigma g.$$

# DIAGONAL SERIES

# Lyndon words

- A word is a Lyndon word if it is less than each of its right factors (for the lexicographical ordering).

### Example

$\{x_0, x_1\}, x_0 < x_1$. The Lyndon words of length $\leq 5$ are $x_0, x_0^4 x_1$, $x_0^3 x_1, x_0^3 x_1^2, x_0^2 x_1, x_0^2 x_1 x_0 x_1, x_0^2 x_1^2, x_0^2 x_1^3, x_0 x_1, x_0 x_1 x_0 x_1^2, x_0 x_1^2, x_0 x_1^3, x_0 x_1^4, x_1$.

- For any $w \in X^*, w = l_1^{i_1} \ldots l_k^{i_k}, \quad l_1 > \ldots > l_k$ (**Širšov**).

### Example

$x_1 x_0 x_1^2 x_0 x_1^2 x_0^2 x_1 = x_1 . x_0 x_1^2 . x_0 x_1^2 . x_0^2 x_1 = x_1 (x_0 x_1^2)^2 x_0^2 x_1$.

- $\mathcal{L}yn(X)$ : the set of Lyndon words over $X$ and forms a transcendence basis for the shuffle algebra (**Radford**).

### Example

$x_0 x_1 x_0^2 x_1 = x_0 x_1 ⧢ x_0^2 x_1 - 3\ x_0^2 x_1 x_0 x_1 - 6\ x_0^3 x_1^2$,
$x_0^3 x_1 x_0^4 x_1 = x_0^3 x_1 ⧢ x_0^4 x_1 - 5 x_0^4 x_1 x_0^3 x_1 - 15 x_0^5 x_1 x_0^2 x_1 - 35 x_0^6 x_1 x_0 x_1 - 70 x_0^7 x_1^2$.

- Let $Y = \{y_i\}_{i \geq 1}$ with $y_1 > y_2 > \ldots$. Then $l \in \mathcal{L}yn X \setminus \{x_0\} \iff \Pi_Y l \in \mathcal{L}yn(Y)$ (**Perrin**).

# Standard factorization and PBW basis

- The standard factorization of $l \in \mathcal{L}ynX \setminus X$, noted by $\mathrm{st}(l)$, is $(u, v)$, where $u, v \in \mathcal{L}ynX$ s.t. $l = uv$ and $v$ is the proper longest right factor of $l$ verifying $u < uv < v$.

Example
$\mathrm{st}(x_0^2x_1x_0x_1) = (x_0^2x_1, x_0x_1)$.

- $\mathcal{L}ie_{\mathbb{C}}\langle X\rangle$ (resp. $\mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$) : set of Lie polynomials (resp. power series) over $X$ and of coefficients in $\mathbb{C}$.
- $\{S_l;\ l \in \mathcal{L}yn(X)\}$ is a basis of $\mathcal{L}ie_{\mathbb{C}}\langle X\rangle$, where the *bracket form* $S_l$ of Lyndon word $l$ is defined by $S_x = x$ if $x \in X$ and $S_l = [S_u, S_v]$ if $(u, v) = \mathrm{st}(l)$.
- The PBW basis $\mathcal{B} = \{S_w;\ w \in X^*\}$ is obtained by putting

$$S_w = S_{l_1}^{i_1} S_{l_2}^{i_2} \ldots S_{l_k}^{i_k} \quad \text{for} \quad w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k,$$

- The dual basis $\check{\mathcal{B}} = \{\check{S}_w;\ w \in X^*\}$ is obtained by putting $\check{S}_{1_{X^*}} = 1_{X^*}, \check{S}_l = x\check{S}_u$ for $l = xu \in \mathcal{L}ynX$ and

$$\check{S}_w = \frac{\check{S}_{l_1}^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \check{S}_{l_k}^{\sqcup\!\sqcup\, i_k}}{i_1! \ldots i_k!} \quad \text{for} \quad w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k.$$

# Diagonal series and Lie elements

- $\mathcal{D} = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{l \otimes \hat{l}} = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\check{S}_l \otimes S_l}$ (**Schützenberger**).

- Let $S \in \mathbb{C}\langle\langle X \rangle\rangle$. $S$ is called group-like if $\Delta_{\sqcup\!\sqcup} S = S \otimes S$.
- $S$ is said to be primitive if $\Delta_{\sqcup\!\sqcup} S = 1 \otimes S + S \otimes 1$.
- $S$ satisfies Friedrichs' (multiplicative) criterion $\langle S | u \sqcup\!\sqcup v \rangle = \langle S | u \rangle \langle S | v \rangle$.

- The following assertions are equivalent (**Ree**)
  i) $S \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X \rangle\rangle$.
  ii) $e^S$ verifies Friedrichs' (multiplicative) criterion.
  iii) $S$ is primitive.
  iv) $e^S$ is group-like.

One has similar results over $Y = \{y_i\}_{i \geq 1}$ with $y_1 > y_2 > \ldots$.

# Computational examples

| $I$ | $\Pi_Y(I)$ | $S_I$ | $\check{S}_I$ | $\Pi_Y(\check{S}_I)$ |
|---|---|---|---|---|
| $x_0$ | | $x_0$ | $x_0$ | |
| $x_1$ | $y_1$ | $x_1$ | $x_1$ | $y_1$ |
| $x_0x_1$ | $y_2$ | $[x_0, x_1]$ | $x_0x_1$ | $y_2$ |
| $x_0^2x_1$ | $y_3$ | $[x_0, [x_0, x_1]]$ | $x_0^2x_1$ | $y_3$ |
| $x_0x_1^2$ | $y_2y_1$ | $[[x_0, x_1], x_1]$ | $x_0x_1^2$ | $y_2y_1$ |
| $x_0^3x_1$ | $y_4$ | $[x_0, [x_0, [x_0, x_1]]]$ | $x_0^3x_1$ | $y_4$ |
| $x_0^2x_1^2$ | $y_3y_1$ | $[x_0, [[x_0, x_1], x_1]]$ | $x_0^2x_1^2$ | $y_3y_1$ |
| $x_0x_1^3$ | $y_2y_1^2$ | $[[[x_0, x_1], x_1], x_1]$ | $x_0x_1^3$ | $y_2y_1^2$ |
| $x_0^4x_1$ | $y_5$ | $[x_0, [x_0, [x_0, [x_0, x_1]]]]$ | $x_0^4x_1$ | $y_5$ |
| $x_0^3x_1^2$ | $y_4y_1$ | $[x_0, [x_0, [[x_0, x_1], x_1]]]$ | $x_0^3x_1^2$ | $y_4y_1$ |
| $x_0^2x_1x_0x_1$ | $y_3y_2$ | $[[x_0, [x_0, x_1]], [x_0, x_1]]$ | $2x_0^3x_1^2 + x_0^2x_1x_0x_1$ | $2y_4y_1^2 + y_3y_2$ |
| $x_0^2x_1^3$ | $y_3y_1^2$ | $[x_0, [[[x_0, x_1], x_1], x_1]]$ | $x_0^2x_1^3$ | $y_3y_1^2$ |
| $x_0x_1x_0x_1^2$ | $y_2^2y_1$ | $[[x_0, x_1], [[x_0, x_1], x_1]]$ | $3x_0^2x_1^3 + x_0x_1x_0x_1^2$ | $3y_3y_1^2 + y_2^2y_1$ |
| $x_0x_1^4$ | $y_2y_1^3$ | $[[[[x_0, x_1], x_1], x_1], x_1]$ | $x_0x_1^4$ | $y_2y_1^3$ |
| $x_0^5x_1$ | $y_6$ | $[x_0, [x_0, [x_0, [x_0, [x_0, x_1]]]]]$ | $x_0^5x_1$ | $y_6$ |
| $x_0^4x_1^2$ | $y_5y_1$ | $[x_0, [x_0, [x_0, [[x_0, x_1], x_1]]]]$ | $x_0^4x_1^2$ | $y_5y_1$ |
| $x_0^3x_1x_0x_1$ | $y_4y_2$ | $[x_0, [[x_0, [x_0, x_1]], [x_0, x_1]]]$ | $2x_0^4x_1^2 + x_0^3x_1x_0x_1$ | $2y_5y_1 + y_4y_2$ |
| $x_0^3x_1^3$ | $y_4y_1^2$ | $[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$ | $x_0^3x_1^3$ | $y_4y_1^2$ |
| $x_0^2x_1x_0x_1^2$ | $y_3y_2y_1$ | $[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$ | $3x_0^3x_1^3 + x_0^2x_1x_0x_1^2$ | $3y_4y_1^2 + y_3y_2y_1$ |
| $x_0^2x_1^2x_0x_1$ | $y_3y_1y_2$ | $[[x_0, [[x_0, x_1], x_1]], [x_0, x_1]]$ | $6x_0^3x_1^3 + 3x_0^2x_1x_0x_1^2 + x_0^2x_1^2x_0x_1$ | $6y_4y_1^2 + 3y_3y_2y_1 + y_3y_1y_2$ |
| $x_0^2x_1^4$ | $y_3y_1^3$ | $[x_0, [[[[x_0, x_1], x_1], x_1], x_1]]$ | $x_0^2x_1^4$ | $y_3y_1^3$ |
| $x_0x_1x_0x_1^3$ | $y_2^2y_1^2$ | $[[x_0, x_1], [[[x_0, x_1], x_1], x_1]]$ | $4x_0^2x_1^4 + x_0x_1x_0x_1^3$ | $4y_3y_1^3 + y_2^2y_1^2$ |
| $x_0x_1^5$ | $y_2y_1^4$ | $[[[[[x_0, x_1], x_1], x_1], x_1], x_1]$ | $x_0x_1^5$ | $y_2y_1^4$ |

# POLYLOGARITHM-HARMONIC SUM-POLYZETA

# Chen series and generating series of polylogarithms

Let $u_0(z) = \dfrac{1}{z}, u_1(z) = \dfrac{1}{1-z}$ and $\omega_0(z) = u_0(z)dz, \omega_1(z) = u_1(z)dz$.

$$\begin{aligned} \forall w \in X^*x_1, \quad \alpha_0^z(w) &= \mathrm{Li}_w(z), \\ \mathrm{P}_w(z) &:= (1-z)^{-1}\,\mathrm{Li}_w(z) = \sum_{n\geq 1} \mathrm{H}_w(n)\, z^n, \\ \mathrm{Li}_{x_0}(z) &:= \log z, \\ \mathrm{L}(z) &:= \sum_{w\in X^*} \mathrm{Li}_w(z)\, w, \\ \mathrm{P}(z) &:= (1-z)^{-1}\mathrm{L}(z). \end{aligned}$$

Let

$$(DE) \quad dG(z) = [x_0\, \omega_0(z) + x_1\, \omega_1(z)]G(z).$$

## Proposition

- $S_{z_0 \rightsquigarrow z}$ *satisfies* $(DE)$ *with* $S_{z_0 \rightsquigarrow z_0} = 1$,
- $\mathrm{L}(z)$ *satisfies* $(DE)$ *with* $\mathrm{L}(z) \widetilde{_{z\to 0}} \exp(x_0 \log z)$.

*Hence,* $S_{z_0 \rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}$*, or equivalently,* $\mathrm{L}(z) = S_{z_0 \rightsquigarrow z}\mathrm{L}(z_0)$.

# Noncommutative generating series of convergent polyzêtas

Let $X = \{x_0, x_1\}$ (resp. $Y = \{y_i\}_{i \geq 1}$) with $x_0 < x_1$ (resp. $y_1 > y_2 > \ldots$). Let $\mathcal{L}ynX$ (resp. $\mathcal{L}ynX$) be the transcendence basis of $(\mathbb{C}\langle X\rangle, \shuffle)$ (resp. $(\mathbb{C}\langle Y\rangle, \stuffle)$) and let $\{\hat{l}\}_{l\in\mathcal{L}ynX}$ (resp. $\{\hat{l}\}_{l\in\mathcal{L}ynY}$) be its dual basis. Then

## Theorem (HNM, 2009)

*We have* $\Delta_{\shuffle} \mathrm{L} = \mathrm{L} \otimes \mathrm{L}$ *and* $\Delta_{\stuffle} \mathrm{H} = \mathrm{H} \otimes \mathrm{H}$.

$$\text{Moreover, let } \mathrm{L}_{\mathrm{reg}}(z) := \prod_{\substack{l\in\mathcal{L}ynX \\ l\neq x_0, x_1}}^{\searrow} e^{\mathrm{Li}_l(z)\,\hat{l}} \text{ and } \mathrm{H}_{\mathrm{reg}}(N) := \prod_{\substack{l\in\mathcal{L}ynY \\ l\neq y_1}}^{\searrow} e^{\mathrm{H}_l(N)\,\hat{l}}.$$

*Then* $\mathrm{L}(z) = e^{x_1 \log\frac{1}{1-z}} \mathrm{L}_{\mathrm{reg}}(z) e^{x_0 \log z}$ *and* $\mathrm{H}(N) = e^{y_1 \mathrm{H}_1(N)} \mathrm{H}_{\mathrm{reg}}(N)$.
*We put* $Z_{\shuffle} := \mathrm{L}_{\mathrm{reg}}(1)$ *and* $Z_{\stuffle} := \mathrm{H}_{\mathrm{reg}}(\infty)$.

## Theorem (*à la* Abel theorem, HNM, 2005)

*Let* $\Pi_Y \mathrm{L}$ *and* $\Pi_Y Z_{\shuffle}$ *be the projections of* $\mathrm{L}$ *and* $Z_{\shuffle}$ *over* $Y$. *Then*

$$\lim_{z\to 1} e^{y_1 \log\frac{1}{1-z}} \Pi_Y \mathrm{L}(z) = \lim_{N\to\infty} \exp\left[-\sum_{k\geq 1} \mathrm{H}_{y_k}(N) \frac{(-y_1)^k}{k}\right] \mathrm{H}(N) = \Pi_Y Z_{\shuffle}.$$

## Corollary

$Z_{\shuffle}$ *and* $Z_{\stuffle}$ *are group-likes and* $Z_{\stuffle} = e^{-\gamma\, y_1} \Gamma(1 + y_1) \Pi_Y Z_{\shuffle}$.

## Successive derivations of $\mathrm{L}$

For any $w = x_{i_1} \ldots x_{i_k} \in X^*$ and for any derivation multi-index $\mathbf{r} = (r_1, \ldots, r_k)$ of degree $\deg \mathbf{r} = |w| = k$ and of weight $\mathrm{wgt}\, \mathbf{r} = k + r_1 + \ldots + r_k$, let us define the monomial $\tau_{\mathbf{r}}(w)$ by

$$\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1}) \ldots \tau_{r_k}(x_{i_k}) = [u_{i_1}^{(r_1)}(z) \ldots u_{i_k}^{(r_k)}(z)]\, x_{i_1} \ldots x_{i_k}.$$

In particular, for any integer $r$

$$\tau_r(x_0) = u_0^{(r)}(z)\, x_0 = \frac{-r!x_0}{(-z)^{r+1}},$$

$$\text{and} \qquad \tau_r(x_1) = u_1^{(r)}(z)\, x_1 = \frac{r!x_1}{(1-z)^{r+1}}.$$

### Theorem (HNM, 2003)

*For any $n \in \mathbb{N}$, we have, $\mathrm{L}^{(n)}(z) = P_n(z)\mathrm{L}(z)$, where*

$$P_n(z) = \sum_{wgt\, \mathbf{r}=n} \sum_{w \in X^n} \prod_{i=1}^{\deg \mathbf{r}} \binom{\sum_{j=1}^{i} r_j + j - 1}{r_i} \tau(w) \in \mathcal{D}\langle X \rangle.$$

# Operations on $\mathrm{P}_w(z) = (1-z)^{-1}\,\mathrm{Li}_w(z)$

For $f(z) = \sum_{n\geq 0} a_n z^n$, since multiplying or dividing by $z$ acts simply on $[z^n]f(z)$, then let us study the effect of multiplying or dividing by $1-z$.

$$
\begin{aligned}
[z^n](1-z)\mathrm{P}_w(z) &= \mathrm{H}_w(n) - \mathrm{H}_w(n-1).\\
[z^n]\frac{\mathrm{P}_w(z)}{1-z} &= \sum_{k=0}^{n} \mathrm{H}_w(k)\\
&= \begin{cases} (n+1)\mathrm{H}_w(n) - \mathrm{H}_{y_{s-1}w'}(n) \text{ if } w = y_s w', s \neq 1.\\ (n+1)\mathrm{H}_w(n) - \displaystyle\sum_{j=1}^{n} \mathrm{H}_{w'}(j-1) \text{ if } w = y_1 w', \end{cases}
\end{aligned}
$$

and, more generally,

$$
\begin{aligned}
[z^n](1-z)^k\mathrm{P}_w(z) &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j \mathrm{H}_w(n-j),\\
[z^n]\frac{\mathrm{P}_w(z)}{(1-z)^k} &= \sum_{n\geq j_1\geq\cdots\geq j_k\geq 0} \mathrm{H}_w(j_k).
\end{aligned}
$$

# NONLINEAR DIFFERENTIAL EQUATIONS

# Nonlinear differential equations with three singularities

$y(z) = \sum_{n \geq 0} y_n z^n$ is the output of :

$$(NS) \begin{cases} y(z) &= f(q(z)), \\ \dot{q}(z) &= \dfrac{A_0(q)}{z} + \dfrac{A_1(q)}{1-z}, \\ q(z_0) &= q_0, \end{cases}$$

$(\rho, \check{\rho}, C_f)$ and $(\rho, \check{\rho}, C_i)$, for $i = 0, .., m$, are convergence modules of $f$ and $\{A_i^j\}_{j=1,..,n}$ respectively at $q \in \mathrm{CV}(f) \Cap_{i=0,..,m}^{j=1,..,n} \mathrm{CV}(A_i^j)$.

$\sigma f_{|_{q_0}} = \sum_{w \in X^*} \mathcal{A}(w)(f(q_0))\, w$ satisfies the $\chi-$growth condition.

The duality between $\sigma f_{|_{q_0}}$ and $S_{z_0 \rightsquigarrow z}$ consists on the convergence (precisely speaking, the convergence of a duality pairing) of the Fliess' fundamental formula which is extended as follows

## Theorem (HNM, 2007)

$$y(z) = \langle \sigma f_{|_{q_0}} \| S_{z_0 \rightsquigarrow z} \rangle = \sum_{w \in X^*} \langle \mathcal{A}(w)(f(q_0)) | w \rangle \langle S_{z_0 \rightsquigarrow z} | w \rangle.$$

## Corollary

*The output $y$ of nonlinear differential equation with three singularities admits then the following expansions*

$$\begin{aligned} y(z) &= \sum_{w\in X^*} g_w(z)\,\mathcal{A}(w)(f(q_0)),\\ &= \sum_{k\geq 0}\sum_{n_1,\ldots,n_k\geq 0} g_{x_0^{n_1}x_1\ldots x_0^{n_k}x_1}(z)\ \mathrm{ad}_{A_0}^{n_1} A_1\ldots\mathrm{ad}_{A_0}^{n_k} A_1 e^{\log z A_0}(f(q_0)),\\ &= \exp\Big(\sum_{w\in X^*} g_w(z)\,\mathcal{A}(\pi_1(w))(f(q_0))\Big),\\ &= \prod_{l\in\mathcal{L}ynX} \exp\Big(g_l(z)\,\mathcal{A}(\hat{l})(f(q_0))\Big), \end{aligned}$$

*where, for any $w\in X^*$, $g_w\in\mathrm{LI}_{\mathcal{C}}$ and*

$$\pi_1(w)=\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\sum_{v_1,\cdots,v_k\in X^*\setminus\{1_{X^*}\}}\langle w|v_1 ⧢ \cdots ⧢ v_k\rangle\, v_1\cdots v_k.$$

# Asymptotics of the output

The output $y$ of nonlinear differential equation with three singularities is then combination of the elements belonging the $\mathrm{LI}_{\mathcal{C}}$.

For $z_0 = \varepsilon \to 0^+$, the asymptotic behaviour of the output $y$ at $z = 1$ is given by

$$y(1) \underset{\varepsilon\to 0^+}{\widetilde{}} \langle \sigma f_{|_{q_0}} \| S_{\varepsilon \rightsquigarrow 1-\varepsilon} \rangle = \sum_{w \in X^*} \langle \mathcal{A}(w) \circ f_{|_{q_0}} | w \rangle \langle S_{\varepsilon \rightsquigarrow 1-\varepsilon} | w \rangle,$$

with $S_{\varepsilon \rightsquigarrow 1-\varepsilon} \underset{\varepsilon\to 0^+}{\widetilde{}} e^{-x_1 \log \varepsilon} Z_{\sqcup\!\sqcup}\ e^{-x_0 \log \varepsilon}$.

If $y(z) = \sum_{n \geq 0} y_n z^n$ then, the coefficients of its ordinary Taylor expansion belong the harmonic algebra and there exist algorithmically computable coefficients $a_i \in \mathbb{Z}$, $b_i \in \mathbb{N}$ and $c_i$ belong a completion of the $\mathbb{C}$-algrebra generated by $\mathcal{Z}$ and by the Euler's $\gamma$ constant, such that

$$y_n \underset{n\to\infty}{\widetilde{}} \sum_{i \geq 0} c_i n^{a_i} \log^{b_i} n.$$

# Finite parts of the output

## Definition

For any $f \in \mathcal{O}$ such that

$$\langle \sigma f_{|_{q_0}} \| S_{z_0 \rightsquigarrow z} \rangle = \sum_{n \geq 0} y_n z^n$$

and for $z_0 = \varepsilon \to 0^+$, let

$$\phi(f_{|_{q_0}}) \underset{z \to 1}{\widetilde{\quad}} \text{f.p. } y(z) \quad \text{in the scale} \quad \{(1-z)^a \log(1-z)^b\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$$

$$\psi(f_{|_{q_0}}) \underset{n \to \infty}{\widetilde{\quad}} \text{f.p. } y_n \quad \text{in the scale} \quad \{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}.$$

## Proposition

*For any $f, g \in \mathcal{O}$ anf for any $\lambda, \mu \in \mathbb{C}$, one has*

$$\phi((\nu f + \mu g)_{|_{q_0}}) = \phi(\nu f_{|_{q_0}}) + \phi(\mu g_{|_{q_0}}) \quad \textit{and} \quad \phi(fg_{|_{q_0}}) = \phi(f_{|_{q_0}})\phi(g_{|_{q_0}}),$$

$$\psi((\nu f + \mu g)_{|_{q_0}}) = \psi(\nu f_{|_{q_0}}) + \psi(\mu g_{|_{q_0}}) \quad \textit{and} \quad \psi(fg_{|_{q_0}}) = \psi(f_{|_{q_0}})\psi(g_{|_{q_0}}).$$

# Successive derivations of the output

Let $n \in \mathbb{N}$,

$$\begin{aligned} y^{(n)}(z) &= \langle \sigma f_{|_{q_0}} \| \frac{d^n}{dz^n} S_{z_0 \rightsquigarrow z} \rangle \\ &= \langle \sigma f_{|_{q_0}} \| \mathrm{L}^{(n)}(z) \mathrm{L}(z_0)^{-1} \rangle \\ &= \langle \sigma f_{|_{q_0}} \| P_n(z) \mathrm{L}(z) \mathrm{L}(z_0)^{-1} \rangle \\ &= \langle P_n(z) \triangleleft \sigma f_{|_{q_0}} \| \mathrm{L}(z) \mathrm{L}(z_0)^{-1} \rangle \\ &= \langle P_n(z) \triangleleft \sigma f_{|_{q_0}} \| S_{z_0 \rightsquigarrow z} \rangle, \end{aligned}$$

where the polynomial $P_n(z) \in \mathcal{D}\langle X \rangle$ is defined as follows

$$P_n(z) = \sum_{\mathrm{wgt}\, \mathbf{r} = n} \sum_{w \in X^n} \prod_{i=1}^{\deg \mathbf{r}} \binom{\sum_{j=1}^{i} r_j + j - 1}{r_i} \tau(w).$$

Therefore, $P_n(z) \triangleleft \sigma f_{|_{q_0}} \in \mathcal{D}\langle\langle X \rangle\rangle$ is the non commutative generating series of $y^{(n)}$.

# Asymptotics of the successive derivation of the output

Let $k \in \mathbb{N}$, the successive derivation $y^{(k)}$ of the output of nonlinear differential equation with three singularities is then combination of the elements $g$ belonging the polylogarithm algebra.

For $z_0 = \varepsilon \to 0^+$, the asymptotic behaviour of the output $y$ at $z = 1$ is given by

$$\begin{aligned} y^{(k)}(1) \quad &\underset{\varepsilon \to 0^+}{\widetilde{\quad}} \quad \langle \sigma f_{|_{q_0}} \| P_k(\varepsilon) S_{\varepsilon \rightsquigarrow 1-\varepsilon} \rangle \\ &= \quad \sum_{w \in X^*} \langle \mathcal{A}(w) \circ f_{|_{q_0}} | w \rangle \langle P_k(\varepsilon) S_{\varepsilon \rightsquigarrow 1-\varepsilon} | w \rangle. \end{aligned}$$

If $y^{(k)}(z) = \sum_{n \geq 0} y_n^{(k)} z^n$ then, the coefficients of its ordinary Taylor expansion belong the harmonic algebra and there exist algorithmically computable coefficients $a_i \in \mathbb{Z}$, $b_i \in \mathbb{N}$ and $c_i$ belong a completion of the $\mathbb{C}$-algrebra generated by $\mathcal{Z}$ and by the Euler's $\gamma$ constant, such that

$$y_n^{(k)} \underset{n \to \infty}{\widetilde{\quad}} \sum_{i \geq 0} c_i n^{a_i} \log^{b_i} n.$$

THANK YOU FOR YOUR ATTENTION