Next-to-leading order in the large N expansion of the multi-orientable random tensor model

Matti Raasakka

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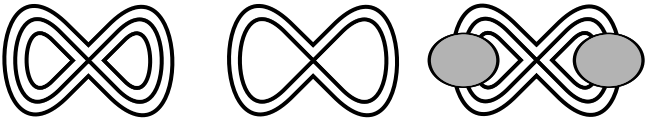

Journée Cartes
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Outline of the talk

1. A very brief history and motivation of tensor models
2. Tensor models and the large N expansion
3. Multi-orientable random tensor model
4. Large N expansion for the multi-orientable model
5. Classification of the next-to-leading order graphs
6. Critical behavior of the next-to-leading order series
7. Summary and outlook

A bit of history & motivation

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  - An important technique is the **large N expansion**, which is controlled by the topology of the surfaces dual to Feynman graphs: in particular, the leading order contribution to the partition function is given by the planar graphs.
  - Thus, the large N expansion allowed to study in detail the **planar sector** of the models and use them, for example, for enumeration of planar maps (and much more).
  - By simultaneous scaling of N and the coupling constant, the **double-scaling limit** allowed to define a continuum limit, where all topologies contribute, which was then possible to connect to 2d gravity.
- However, only recently a similarly powerful control over random tensor model partition functions has been achieved by restricting the class of graphs that arise from the perturbative series. . .

Tensor models and the large N expansion

- A tensor model is specified by its partition function

$$\mathcal{Z}(\lambda_t) = \int \left[\prod_c \mathrm{d}\phi_c \mathrm{d}\overline{\phi}_c\right] e^{-\mathcal{S}(\phi_c,\overline{\phi}_c;\lambda_t)} ,$$

where ϕ_c are rank-d complex tensors of size N,

$$\mathcal{S}(\phi_c,\overline{\phi}_c;\lambda_t) = \sum_c \overline{\phi}_c \cdot \phi_c - \sum_t \lambda_t V_t(\phi_c,\overline{\phi}_c) ,$$

and $\overline{\phi}_c \cdot \phi_c := \sum_{i_1,\dots,i_d} \overline{(\phi_c)_{i_1,\dots,i_d}} (\phi_c)_{i_1,\dots,i_d}$, $i_k = 1,\dots,N$ for all k.

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- Feynman graphs arise as graphical representations of summands in the perturbative expansion of the partition function

$$\mathcal{Z}(\lambda_t) = \sum_{k=0}^{\infty} \int \left[\prod_c \mathrm{d}\phi_c \mathrm{d}\overline{\phi}_c\right] \left(\sum_t \lambda_t V_t(\phi_c,\overline{\phi}_c)\right)^k e^{-\sum_c \overline{\phi}_c \cdot \phi_c} ,$$

where $\exp(-\sum_c \overline{\phi}_c \cdot \phi_c)$ is just a product of Gaussian measures for the tensor components of all tensors.

- For any order k in the perturbative series, the contributing terms are represented by Feynman graphs with k vertices, connected by oriented edges labelled by the index c. The edges encode the Isserlis-Wick pairings in calculating the expectation values of monomials for the Gaussian measure: $E[x_1 x_2 \cdots x_{2p}] = \sum \prod E[x_i x_j]$.
- The free energy $\mathcal{F} := N^{-d} \ln \mathcal{Z}$ may be expressed as a sum over the connected vacuum Feynman graphs Γ as

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$$\mathcal{F}(\lambda_t) = \sum_{\Gamma} \mathcal{A}(\Gamma)\,,$$

  where $\mathcal{A}(\Gamma)$ is called the amplitude of the graph.
- The large N expansion is facilitated by the fact that we have

$$\mathcal{A}(\Gamma) \propto N^{-\omega(\Gamma)}\,.$$

  Thus, the expansion in $1/N$ is controlled by the **degree** ω.
- Different tensor models may incorporate different classes of graphs and possibly also have different expressions for the degree ω.

- **Colored** tensor models were introduced by Razvan Gurau in 2009: Feynman graphs for d dimensions may be represented as bipartite $(d+1)$-edge-colored regular graphs of vertex-degree $d+1$.

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- The edge-coloring allows for an improved control over the perturbative expansion and the graph combinatorics. In particular, the degree ω has a simple expression in terms of topological data of certain 2d subgraphs of the colored graphs called **jackets**.

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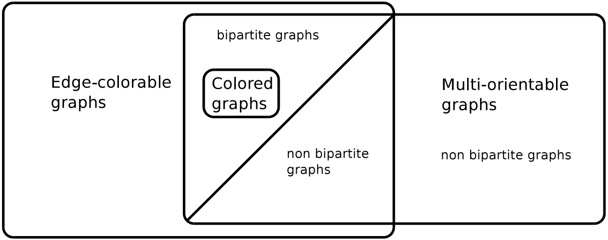
- **Colored** tensor models were introduced by Razvan Gurau in 2009: Feynman graphs for d dimensions may be represented as bipartite $(d+1)$-edge-colored regular graphs of vertex-degree $d+1$.
- The edge-coloring allows for an improved control over the perturbative expansion and the graph combinatorics. In particular, the degree ω has a simple expression in terms of topological data of certain 2d subgraphs of the colored graphs called **jackets**.
- Recently, the large N expansion of colored tensor models has been under intensive investigation. Some very important recent advances:
  - The first derivation of the large N expansion for colored tensor models. [Gurau (2011)]
  - The leading order ($\omega = 0$) sector is given by the so-called melonic graphs, which correspond to a subclass of triangulations of a d-sphere. [Bonzom, Gurau, Riello, Rivasseau (2011)]
  - The next-to-leading order ($\omega = 1$) sector was classified and summed over. [Kaminski, Oriti, Ryan (2013)]
  - All orders in ω were classified and enumerated, and the existence of a double-scaling limit established. [Gurau, Schaeffer (2013)] and [Dartois, Gurau, Rivasseau (2013)]

Multi-orientable tensor model

- The multi-orientable tensor model introduced by Adrian Tanasa in 2011: Incorporates a strictly larger set of graphs than the corresponding rank-4 colored tensor model.



[Dartois, Rivasseau, Tanasa (2013)]

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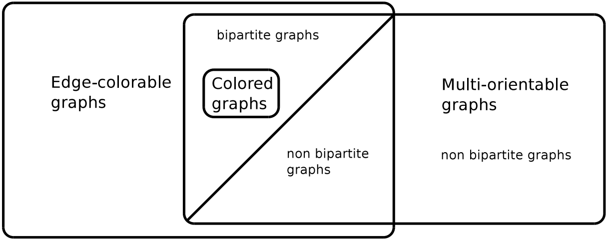


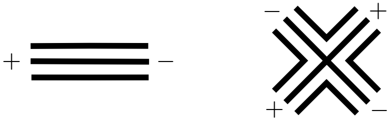
[Dartois, Rivasseau, Tanasa (2013)]

Question:

How much of the large N scaling properties of colored tensor models generalize to this larger family of tensor graphs?

Multi-orientable tensor model

- The edges and vertices of the multi-orientable tensor model can be represented as



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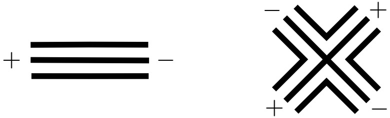
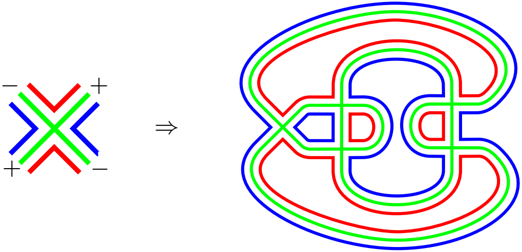


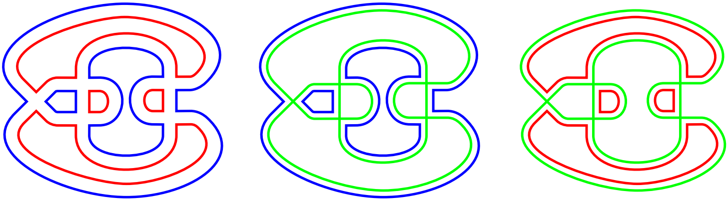
- The strands, representing the tensor indices, can be classified into three types [Dartois, Rivasseau, Tanasa (2013)]



- The strands running inside the vertices (green) are called **inner** strands while the others (blue and red) are called **outer** strands.

Multi-orientable tensor model

- By removing any one of the three types of strands, we end up with a ribbon graph, called a **jacket** of the original tensor graph:



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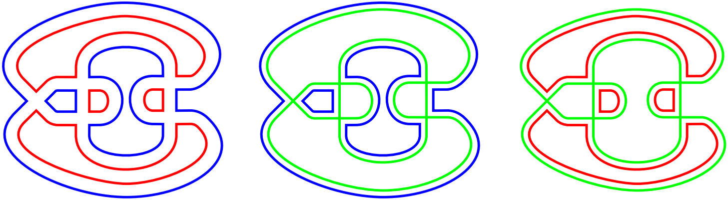


- The **degree** of a MO graph is the sum over the genera of the jackets

$$\omega = \sum_J g_J \, .$$

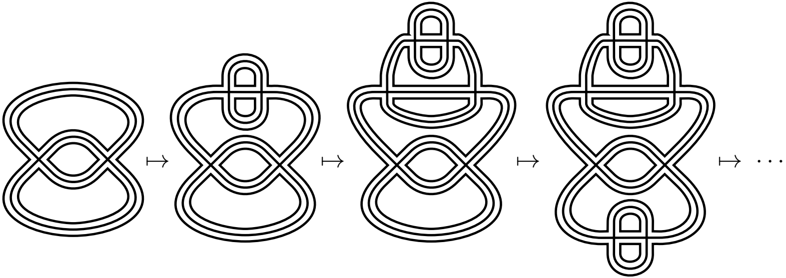
The genus g_J is obtained through the Euler characteristic formula

$$g_J = 1 - \frac{1}{2}(F_J - L_J + V_J) \in \mathbb{N}/2 \; (!!!) \, ,$$

where $F_J, L_J, V_J = \#$ of faces, lines, vertices of J.

Leading order in $1/N$ for the multi-orientable tensor model

- The leading order $\omega = 0$ is still given by the melonic sector obtained by sequential melonic insertions to the elementary melon. [Dartois, Rivasseau, Tanasa (2013)]



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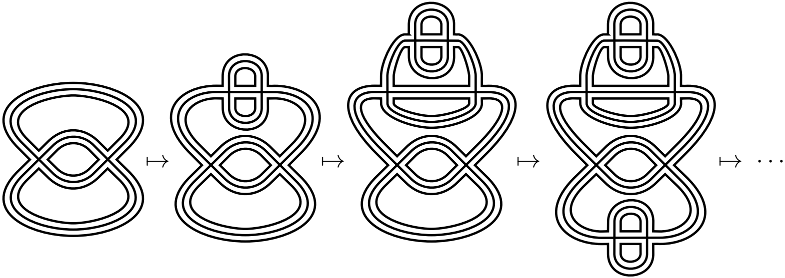


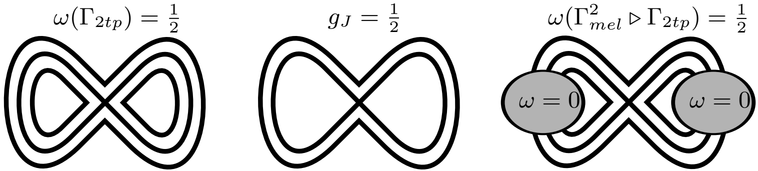
- The melonic graphs can be mapped to trees, and thus counted exactly. The leading order series has the same behavior as the colored model:

$$\mathcal{F}_{\mathrm{LO}} \propto \text{const.} + \left(1 - \frac{\lambda^2}{\lambda_c^2}\right)^{2-\gamma_{\mathrm{LO}}}, \qquad \gamma_{\mathrm{LO}} = \frac{1}{2}.$$

[Bonzom, Gurau, Riello, Rivasseau (2011)]

Next-to-leading order in $1/N$ for the multi-orientable tensor model

- The multi-orientable next-to-leading order sector is given by $\omega = 1/2$, because of non-orientable jackets, not $\omega = 1$ as for colored models.
- Simplest NLO graph is the double-tadpole:



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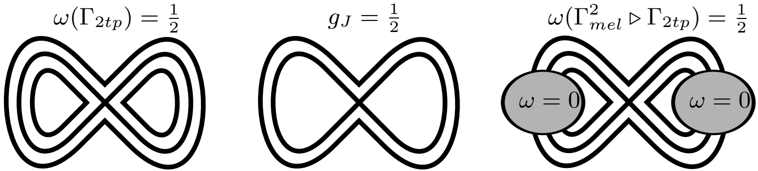


- Any insertion of a melonic 2-point subgraph conserves the degree.

Question:

But are the graphs so obtained all the possible NLO ($\omega = 1/2$) graphs?

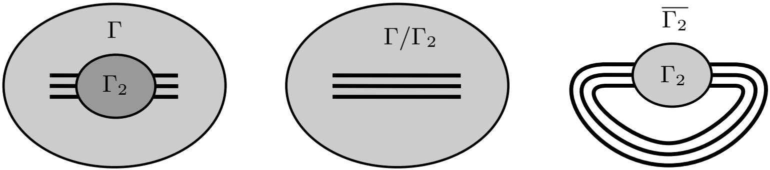
Next-to-leading order graphs for the MO tensor model

Lemma 1:

Let Γ be an MO vacuum Feynman graph, and Γ_2 an MO 2-point subgraph of Γ. Let us denote by Γ/Γ_2 the graph obtained by replacing Γ_2 inside Γ with a propagator. We then have the relation

$$\omega(\Gamma) = \omega(\Gamma/\Gamma_2) + \omega(\overline{\Gamma_2})\,,$$

where $\overline{\Gamma_2}$ denotes the vacuum graph obtained by gluing the external legs of Γ_2 to each other.



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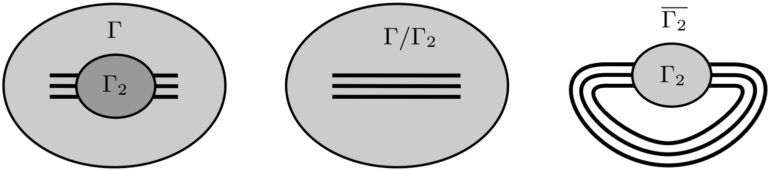

Easy to prove using the helpful expression $\omega(\Gamma) = 3 + \frac{3}{2}V_\Gamma - F_\Gamma$, which can be derived from $V_J = V_\Gamma$, $L_J = L_\Gamma = 2V_\Gamma$ and $\sum_J F_J = 2F_\Gamma$.

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Definition:

A **NLO melon-free graph** is a graph with $\omega = \frac{1}{2}$ and no melonic 2-point subgraphs.

- The double-tadpole is an NLO melon-free graph, since it does not contain melonic 2-point subgraphs.
- All NLO graphs can be obtained by melonic insertions into the melon-free NLO graphs.

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$\Rightarrow$ The melon-free graphs classify the NLO graphs into families related through insertions and contractions of melonic 2-point subgraphs.

It is then sufficient to focus on the melon-free NLO multi-orientable graphs.

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Definition:

A graph is **2-particle-irreducible (2PI)** if it does not contain any proper non-trivial 2-point subgraphs.

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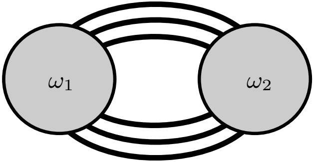
Lemma 2:

A NLO melon-free graph of the MO model is 2-particle-irreducible.

$$\omega_1 + \omega_2 = 1/2$$



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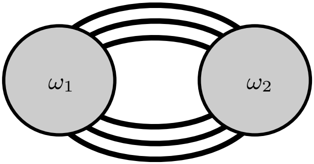

(i) By Lemma 1, either $\omega_1 = 0$ and $\omega_2 = 1/2$, or vice versa. (ω_i is the degree of the corresponding vacuum graph.)

(ii) $\omega = 0$ only for the propagator and melonic 2-point graphs.

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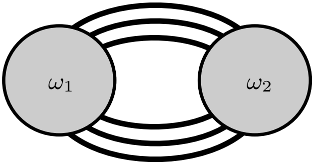

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(ii) $\omega = 0$ only for the propagator and melonic 2-point graphs.

$\Rightarrow$ The part with $\omega = 0$ must be a propagator for a melon-free graph.

$\Rightarrow$ No non-trivial 2-point subgraphs in a NLO melon-free graph.

Main theorem:

The only NLO melon-free graph of the MO model is the double-tadpole.

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Assume Γ is a NLO melon-free graph.

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$\Rightarrow$ $F_{\Gamma,o} = V_\Gamma + 2$ for the number of faces formed by the outer strands.

$\Rightarrow$ $\omega(\Gamma) = 3 + \frac{3}{2}V - (F_{\Gamma,o} + F_{\Gamma,i}) = 1 + \frac{1}{2}V_\Gamma - F_{\Gamma,i}$, where $F_{\Gamma,i}$ is the number of faces in Γ formed by the inner strands.

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- Since the outer jacket is planar, the graph may be drawn on a plane so that the inner faces intersect only at vertices.
- For a connected graph with $F_{\Gamma,i} > 1$ any inner face must always intersect another inner face.

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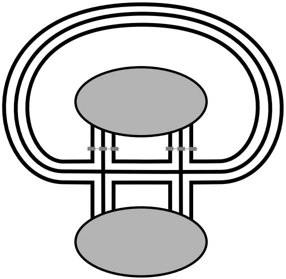
- Since the outer jacket is planar, the graph may be drawn on a plane so that the inner faces intersect only at vertices.
- For a connected graph with $F_{\Gamma,i} > 1$ any inner face must always intersect another inner face.

We may then concentrate on the properties of inner faces.

Next-to-leading order graphs for the MO tensor model



Now, assume that Γ is a NLO melon-free graph with $F_{\Gamma,i} > 1$, and there is an inner face f in Γ, which intersects the other inner faces of Γ only twice.

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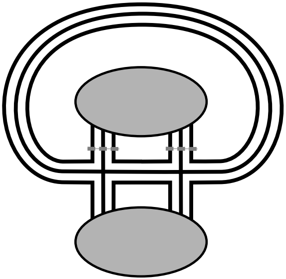


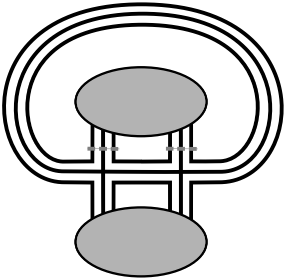
Now, assume that Γ is a NLO melon-free graph with $F_{\Gamma,i} > 1$, and there is an inner face f in Γ, which intersects the other inner faces of Γ only twice.

(i) f must intersect the same face twice, since the number of intersections between any pair of faces is even.

(ii) f cannot intersect itself, because this would correspond to a non-trivial 2-point subgraph in Γ, but Γ is 2PI, since it is NLO melon-free.

(iii) There are no further intersections between f and other faces of Γ.

Next-to-leading order graphs for the MO tensor model



Now, assume that Γ is a NLO melon-free graph with $F_{\Gamma,i} > 1$, and there is an inner face f in Γ, which intersects the other inner faces of Γ only twice.

(i) f must intersect the same face twice, since the number of intersections between any pair of faces is even.

(ii) f cannot intersect itself, because this would correspond to a non-trivial 2-point subgraph in Γ, but Γ is 2PI, since it is NLO melon-free.

(iii) There are no further intersections between f and other faces of Γ.

$\Rightarrow$ f divides the plane on which Γ is drawn into two separate regions.

$\Rightarrow$ The part of Γ inside f is a connected 2-point subgraph of Γ, so it must be trivial. But then the 2-point subgraph of Γ obtained by cutting just outside f is a melonic subgraph.

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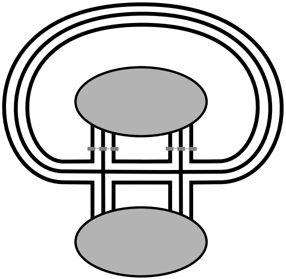

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$\Rightarrow$ Each inner face must intersect the other inner faces at least four times in a NLO melon-free graph Γ with $F_{\Gamma,i} > 1$.

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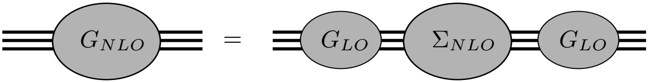
Corollary:

All graphs contributing to the next-to-leading order of the MO model arise from insertions of melonic 2-point subgraphs into the double-tadpole.

Next-to-leading order series for the MO tensor model

- Following the classification of NLO graphs, one may determine the sum over the NLO amplitudes of the model by relating it to the LO series.
- Consider the connected and the 1PI 2-point functions G and Σ.
- We have $G_{NLO} = G_{LO}^2 \Sigma_{NLO}$ from the following graphical relation:



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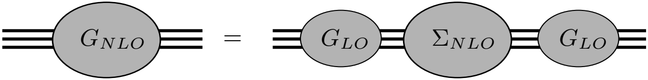
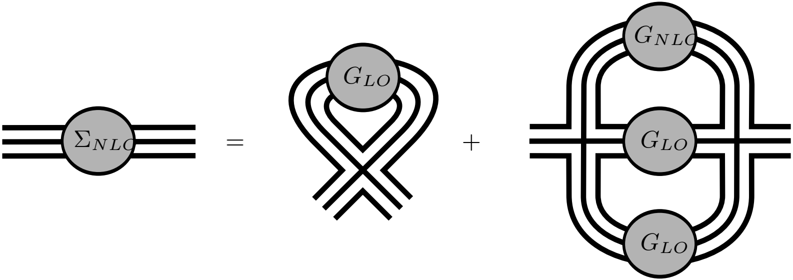


- On the other hand, $\Sigma_{NLO} = \lambda G_{LO} + 3\lambda^2 G_{LO}^2 G_{NLO}$ follows from:



Next-to-leading order series for the MO tensor model

- Substituting one to the other, we may solve for

$$G_{NLO} = \frac{\lambda G_{LO}^3}{1 - 3\lambda^2 G_{LO}^4} .$$

- Differentiating the LO two-point function relation $G_{LO} = 1 + \lambda^2 G_{LO}^4$ we get

$$\frac{\partial}{\partial \lambda} G_{LO} = \frac{2\lambda G_{LO}^4}{1 - 4\lambda^2 G_{LO}^3} = \frac{2\lambda G_{LO}^5}{1 - 3\lambda^2 G_{LO}^4} ,$$

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- Thus, we get the expression

$$G_{NLO} = \frac{\lambda}{G_{LO}^2} \frac{\partial}{\partial\lambda^2} G_{LO} \, ,$$

which implies, together with $G_{LO} \propto \text{const.} + (1 - (\lambda^2/\lambda_c^2))^{1/2}$,

$$G_{NLO} \propto \left(1 - \frac{\lambda^2}{\lambda_c^2}\right)^{-1/2} \, .$$

Next-to-leading order series for the MO tensor model

- We have from a Schwinger-Dyson equation the relation

$$G_{\text{NLO}} = 1 - 4\lambda^2 \frac{\partial}{\partial \lambda^2} \mathcal{F}_{\text{NLO}}$$

for the connected two-point function G_{NLO} and the free energy $\mathcal{F}_{\text{NLO}}$.

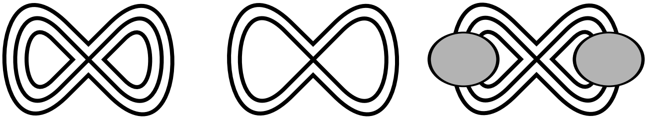
Critical behavior of the NLO free energy:

$$\mathcal{F}_{\text{NLO}} \propto \left(1 - \frac{\lambda^2}{\lambda_c^2}\right)^{2-\gamma_{\text{NLO}}}, \quad \text{where} \quad \gamma_{\text{NLO}} = 3/2\,.$$

- Thus we find the same critical value of the coupling constant for the NLO series as for the LO series. Nevertheless, one has a distinct value for the NLO susceptibility exponent.
- Such behavior is indicative of the existence of a **double-scaling limit** also for the multi-orientable tensor model.

Summary & outlook

- All next-to-leading order vacuum graphs of the multi-orientable random tensor model arise from insertions of melonic 2-point subgraphs into the double-tadpole graph.
- The next-to-leading order free energy has the same critical coupling constant as the leading order free energy, and a critical exponent $3/2$.



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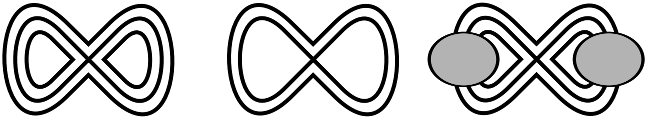

- This indicates a double-scaling limit also for the multi-orientable model. What about higher orders?
- In higher orders deviations from the colored model are enhanced. How to classify generic multi-orientable graphs without the convenience of color labels?
- Can one further loosen up the restrictions on the tensor graphs and retain control over the large N expansion? Is there a motivation?