Truncations of unitary matrices and Brownian bridges

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Plan

Motivation : Computational biology

Aim : measure of the similarity between two genomic (long) sequences.

Let $\mathfrak{S}_n$ be the set of permutations of $[n]$. If $\sigma, \tau \in \mathfrak{S}_n$, set

$$O_p(\sigma, \tau) = \#\{i \leqslant p : \sigma \circ \tau^{-1}(i) \leqslant p\}\,, p = 1, \cdots, n\,.$$

and compare them with the results of a random permutation.

G. Chapuy introduced the discrepancy process

$$T^{(n)}_{\lfloor ns \rfloor, \lfloor nt \rfloor}(\sigma) = \#\{i \leqslant \lfloor ns \rfloor : \sigma(i) \leqslant \lfloor nt \rfloor\}\,, s, t \in [0, 1]\,,$$

Theorem (G. Chapuy 2007)

The sequence

$$n^{-1/2}\left(T^{(n)}_{\lfloor ns \rfloor, \lfloor nt \rfloor}(\sigma) - stn\right)\ ,\ s, t \in [0, 1]$$

converges in distribution to the bivariate tied down Brownian bridge, of covariance $(s \wedge s' - ss')(t \wedge t' - tt')\,.$

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Matrix representation

If σ is represented by the matrix $U(\sigma)$, the integer $T^n_{p,q}(\sigma)$ is the sum of all elements of the upper-left $p \times q$ submatrix of $U(\sigma)$, i.e.

$$T^n_{p,q}(\sigma) = \mathrm{Tr}\left[D_p U(\sigma) D_q U(\sigma)^*\right]$$

where $D_k = \mathrm{diag}(1, \cdots, 1, 0, \cdots, 0)$ (k times 1) .

Instead of picking randomly σ in the group $\mathfrak{S}_n$, we propose to pick a random element U in the group $\mathbb{U}(n)$ (resp. $\mathbb{O}(n)$) and to study

The main statistic

$$T^n_{p,q} = \mathrm{Tr}(D_1 U D_2 U^*) = \sum_{i \leqslant p, j \leqslant q} |U_{ij}|^2 .$$

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Plan

Main result

Theorem (CDM+AR, *RMTA* 2011)

The process

$$W^{(n)} = \left\{ T^{(n)}_{\lfloor ns \rfloor, \lfloor nt \rfloor} - \mathbb{E} T^{(n)}_{\lfloor ns \rfloor, \lfloor nt \rfloor} \,,\ s, t \in [0,1] \right\}$$

converges in distribution in the Skorokhod space $\mathrm{D}([0,1]^2)$ *to the* *bivariate tied-down Brownian bridge* $\sqrt{\frac{2}{\beta}} W^{(\infty)}$ *where* $W^{(\infty)}$ *is a centered continuous Gaussian process on* $[0,1]^2$ *of covariance*

$$\mathbb{E}[W^{(\infty)}(s,t) W^{(\infty)}(s',t')] = (s \wedge s' - ss')(t \wedge t' - tt'),$$

$\beta = 2$ *in the unitary case and* $\beta = 1$ *in the orthogonal case.*

Normalizations

- No normalization here !
- If σ is Haar distributed in $\mathfrak{S}_n$, then U_{ij} is Bernoulli of parameter $1/n$ and
$$\mathrm{Var}(|U_{ij}|^2) \sim n^{-1}$$
- If U is Haar distributed in $\mathbb{U}(n)$, then the column vector $(U_{i,j})_{i=1}^n$ is uniform on the (complex) sphere of dim n, and $|U_{ij}|^2$ is Beta distributed with parameters $(1, n-1)$ and
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- If O is Haar distributed in $\mathbb{O}(n)$, then $|O_{ij}|^2$ is Beta distributed with parameters $(1/2, (n-1)/2)$ and
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Previous related results

- If q is fixed, Silverstein (1981) proved that the process

$$n^{1/2}\left(\sum_{i=1}^{\lfloor ns\rfloor}|U_{iq}|^2 - s\right)\ ,\ s\in[0,1]$$

converges in distribution to the (univariate) Brownian bridge, continuous gaussian process of covariance $s(1-s)$.

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Asymptotic studies :

1) p, q fixed, $n \to \infty$ (large sample framework),
2) q fixed, $n, p \to \infty$ and $p/n \to s < 1$ fixed (high-dimensional framework, see Fujikoshi et al. 2008).
3) $p/n \to s$, $q/n \to t$ with s, t fixed. This case is considered in the Bai and Silverstein's book, and a CLT for $T_{p,q}$ was proved by Bai et al. (2009).

Plan

Asymptotics of the 1-marginals (i.e. s, t fixed)

Set $p = \lfloor ns \rfloor$, $q = \lfloor nt \rfloor$ and

$$A_{p,q} = D_p U D_q U^* = V_{p,q} V_{p,q}^*$$

where $V_{p,q} = D_p U D_q$ is the upper-left submatrix of U. As proved by Collins (2005) $A_{p,q}$ belongs to the Jacobi unitary ensemble (JUE) and

$$T_{p,q}^{(n)} = \operatorname{Tr} A_{p,q} = p \int x d\mu^{(p,q)}(x) \,,$$

where $\mu^{(p,q)}$ is the empirical spectral distribution

$$\mu^{(p,q)} = \frac{1}{p} \sum_{k=1}^{p} \delta_{\lambda_k^{(p)}} \,,$$

and the $\lambda_k^{(p)}$'s are the eigenvalues of $A_{p,q}$.

For the JUE, the equilibrium measure is the Kesten-McKay distribution. If $s \leqslant \min(t, 1-t)$ it has the density

$$\pi_{u_-,u_+}(x) := C_{u_-,u_+} \frac{\sqrt{(x-u_-)(u_+-x)}}{2\pi x(1-x)} \mathbf{1}_{(u_-,u_+)}(x) \tag{1}$$

where $0 \leqslant u_- < u_+ \leqslant 1$ ($u_\pm$ depending on s, t).

LLN

$$\lim_n \frac{1}{n} T^{(n)}_{\lfloor ns\rfloor,\lfloor nt\rfloor} = s \int x\pi_{u_-,u_+}(x)dx = st\,,$$

CLT

$$T^{(n)}_{\lfloor ns\rfloor,\lfloor nt\rfloor} - \mathbb{E}T^{(n)}_{\lfloor ns\rfloor,\lfloor nt\rfloor} \Rightarrow \mathcal{N}(0, s(1-s)t(1-t))$$

Plan

Fidi convergence and tightness

To prove the fidi convergence, it is enough to prove that for any $(a_i)_{i\leqslant k} \in \mathbb{R}$ and $(s_i, t_i)_{i\leqslant k} \in [0,1]^2$, $p_i = \lfloor ns_i \rfloor$, $q_i = \lfloor nt_i \rfloor$ the random variable

$$X^{(n)} = \sum_{i=1}^{k} a_i[\mathrm{Tr}(D_{p_i} U D_{q_i} U^\star) - \mathbb{E}(\mathrm{Tr}(D_{p_i} U D_{q_i} U^\star))]$$

where $D_{p_i} = I_{p_i}$, $D_{q_i} = I_{q_i}$, converges in distribution to the normal distribution with the good variance.

We use the method of cumulants.

To prove tightness, we will take benefit of the structure of $T^{(n)}_{\lfloor ns \rfloor, \lfloor nt \rfloor}$ as a sum with stationary increments. A sufficient condition (via the Bickel-Wichura criterion) is

$$\mathbb{E}\left(\mathrm{Tr}(D_p U D_q U^\star) - \mathbb{E}\,\mathrm{Tr}(D_p U D_q U^\star)\right)^4 = O(p^2 q^2 n^{-4})\,. \qquad (2)$$

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The first calculations give

$$\mathbb{E}|U_{ij}|^{2k} = \frac{(n-1)!k!}{(n-1+k)!}$$

$$\mathbb{E}\left(|U_{i,j}|^2|U_{i,k}|^2\right) = \frac{1}{n(n+1)}\,,\ \mathbb{E}\left(|U_{i,j}|^2|U_{k,\ell}|^2\right) = \frac{1}{n^2-1}\,.$$

but for the fidi and tightness, we need mixed moments of higher order. In fact, we gave a complete proof (fidi convergence + tightness) using a formula for the cumulants of variables of the form

$$X = \mathrm{Tr}(AUBU^*)$$

for deterministic matrices A, B of size n.

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Recall : the multivariate cumulants are defined by

$$\kappa_r(a_1, \cdots, a_r) := (-i)^r \frac{\partial^r}{\partial \xi_1 \cdots \partial \xi_r} \log \mathbb{E} \exp i \sum \xi_k a_k \,.$$

They are related with moments by

$$\kappa_r(a_1, \cdots, a_r) = \sum_{C \in \mathcal{P}(r)} \text{Möb}(C, 1_r) \mathbb{E}_C(a_1, \cdots, a_r)$$

where

- $\mathcal{P}(r)$ is the set of partitions of $[r]$
- If $C = \{C_1, \cdots, C_k\}$ is the decomposition of C in blocks, then

$$\text{Möb}(C, 1_r) = (-1)^{k-1}(k-1)! \;\;,\;\; \mathbb{E}_C(a_1, \dots, a_r) = \prod_{i=1}^{k} \mathbb{E}(\prod_{j \in C_i} a_j).$$

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Proposition (Particular case of Mingo, Sniady, Speicher)

Let U *be Haar distributed on* $\mathbb{U}(n)$*. Let* $D = (D_1, \dots D_k)$ *and* $\bar{D} = (D_{\bar{1}}, \dots D_{\bar{k}})$ *be two families of deterministic matrices of size* n*. We set, for* $1 \leqslant i \leqslant r$*,* $X_i = \mathrm{Tr}(D_i U D_{\bar{i}} U^\star)$ *. Then,*

$$\kappa_r(X_1, \dots, X_r) = \sum_{\alpha, \beta \in \mathcal{S}_r} \sum_{A} C_{\beta\alpha^{-1}, A} \, \mathrm{Tr}_\alpha(\bar{D}) \, \mathrm{Tr}_{\beta^{-1}}(D) \tag{3}$$

where in the second sum $A \in \mathcal{P}(r)$ *is such that* $\beta\alpha^{-1} \leqslant A$ *and* $A \vee \beta \vee \alpha = 1_r$*, and* $C_{\sigma, A}$ *are the "relative cumulants" of the unitary Weingarten function . Moreover, if the sequence* $\{D, \bar{D}\}_n$ *has a limit distribution, then for* $r \geqslant 3$*,*

$$\lim_{n \to \infty} \kappa_r(X_1, \dots, X_r) = 0.$$

The needed formula relies on the notion of second order freeness introduced by Mingo, Sniady and Speicher (06-07).

Roughly speaking, whereas the freeness, introduced by Voiculescu, provides the asymptotic behavior of expectation of traces of random matrices, the second order freeness describes the leading order of the fluctuations of these traces.

To reach these cumulants, we use the Möbius formula and estimate the moments.

Finally, moments, which are expectations of products of entries of $\mathbb{U}$ can be described by the Weingarten function defined as follows (Collins Sniady).

The needed formula relies on the notion of second order freeness introduced by Mingo, Sniady and Speicher (06-07).
Roughly speaking, whereas the freeness, introduced by Voiculescu, provides the asymptotic behavior of expectation of traces of random matrices, the second order freeness describes the leading order of the fluctuations of these traces.
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Plan

Combinatorics of the unitary and orthogonal groups

- Let $\mathcal{M}_{2k}$ be the set of pairings of $[2k]$, i.e. of partitions where each block consists of exactly two elements. It is then convenient to encode the set $[2k]$ by
$$[2k] \cong \{1, \ldots, k, \bar{1}, \ldots, \bar{k}\}.$$
Given two pairings p_1, p_2, we define the graph $\Gamma(p_1, p_2)$ as follows. The vertex set is $[2k]$ and the edge set consists of the pairs of p_1 and p_2. Let $\mathrm{loop}(p_1, p_2)$ the number of connected components of $\Gamma(p_1, p_2)$.
- Let $\mathcal{M}_{2k}^{\mathcal{U}}$ denote the set of pairings of $[2k]$, pairing each element of $[k]$ with an element of $[\bar{k}]$.

Let $G^{\mathbb{U}(n)}$ be the Gram matrix

$$G^{\mathbb{U}(n)} = \left(G^{\mathbb{U}(n)}(p_1, p_2)\right)_{p_1,p_2\in\mathcal{M}_{2k}^{\mathcal{U}}} := \left(n^{\mathsf{loop}(p_1,p_2)}\right)_{p_1,p_2\in\mathcal{M}_{2k}^{\mathcal{U}}} .$$

The unitary Weingarten matrix $\mathrm{Wg}^{\mathbb{U}(n)}$ is defined as the pseudo-inverse of $G^{\mathbb{U}(n)}$, i.e. such that $GWG = W$ and $WGW = G$.
Let $G^{\mathbb{O}(n)}$ be the Gram matrix

$$G^{\mathbb{O}(n)} = \left(G^{\mathbb{O}(n)}(p_1, p_2)\right)_{p_1,p_2\in\mathcal{M}_{2k}} := \left(n^{\mathsf{loop}(p_1,p_2)}\right)_{p_1,p_2\in\mathcal{M}_{2k}} .$$

The unitary Weingarten matrix $\mathrm{Wg}^{\mathbb{O}(n)}$ is defined as the-pseudo inverse of $G^{\mathbb{O}(n)}$.
Owing to some isomorphisms, these functions of two arguments may be reduced to functions of one argument only. In particular in the unitary case, each p_i is associated with a permutation α_i and

$$G^{\mathbb{U}(n)}(p_1, p_2) =: G(\alpha_2^{-1}\alpha_1) = n^{\#\text{cycles of } \alpha_2^{-1}\alpha_1} .$$

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Proposition

For every choice of indices $\mathbf{i} = (i_1, \dots, i_k, i_{\bar{1}}, \dots, i_{\bar{k}})$ *and* $\mathbf{j} = (j_1, \dots, j_k, j_{\bar{1}}, \dots, j_{\bar{k}})$,

$$\mathbb{E}\left(U_{i_1 j_1} \dots U_{i_k j_k} \bar{U}_{i_{\bar{1}} j_{\bar{1}}} \dots \bar{U}_{i_{\bar{k}}, j_{\bar{k}}}\right) = \sum_{p_1, p_2 \in \mathcal{M}_{2k}^{U}} \delta_{\mathbf{i}}^{p_1} \delta_{\mathbf{j}}^{p_2} \, \mathsf{Wg}^{\mathbb{U}(n)}(p_1, p_2)$$

$$\mathbb{E}\left(O_{i_1 j_1} \dots O_{i_k j_k} \bar{O}_{i_{\bar{1}} j_{\bar{1}}} \dots \bar{O}_{i_{\bar{k}}, j_{\bar{k}}}\right) = \sum_{p_1, p_2 \in \mathcal{M}_{2k}} \delta_{\mathbf{i}}^{p_1} \delta_{\mathbf{j}}^{p_2} \, \mathsf{Wg}^{\mathbb{O}(n)}(p_1, p_2)$$

where $\delta_{\mathbf{i}}^{p_1}$ *(resp.* $\delta_{\mathbf{j}}^{p_2}$) *is equal to* 1 *or* 0 *if* $\mathbf{i}$ *(resp.* $\mathbf{j}$*) is constant on each pair of* p_1 *(resp.* p_2*) or not.*

Plan

Random truncation

B. Farrell (2011) studied truncated unitary matrices, either deterministic (Discrete Fourier Transform)

$$\mathrm{DFT}_{jk}^{(n)} = \frac{1}{\sqrt{n}} e^{-2i\pi(j-1)(k-1)/n}$$

or Haar distributed, when each row is chosen independently with probability s and each column is chosen independently with probability t. He proved that the ESD converges to the Kesten-McKay distribution.

Plan

Main result

We can embed this model in a two parameter framework,

$$\mathcal{T}^{(n)}(s,t) = \sum_{1 \leqslant i,j \leqslant n} |U_{ij}|^2 1_{R_i \leqslant s} 1_{C_j \leqslant t}$$

Theorem (CDM+AR+VB 2013)

If U is Haar in $\mathbb{U}(n)$ or $\mathbb{O}(n)$, or if U is the DFT matrix, then

$$n^{-1/2}\left(\mathcal{T}^{(n)} - \mathbb{E}\mathcal{T}^{(n)}\right) \xrightarrow{\text{law}} \mathcal{W}^\infty$$

$$\mathcal{W}^\infty(s,t) = sB_0^{(2)}(t) + tB_0^{(1)}(s)\ ,\ \ s,t \in [0,1]\,,$$

with $B_0^{(1)}$ and $B_0^{(2)}$ two independent Brownian bridges.

Plan

Subordination

Let $S_n^{(1)}(s) = \sum_{i=1}^n \mathbf{1}_{R_i \leqslant s}$ and $S_n^{(2)}(t) = \sum_{j=1}^n \mathbf{1}_{C_j \leqslant t}$ and $\widetilde{U}_{ij} = |U_{ij}|^2$.

Proposition

If $\widetilde{U}$ is a random doubly stochastic matrix $n \times n$ with a distribution invariant by permutation of rows and columns, then

$$\mathcal{T}^{(n)} \overset{\text{law}}{=} \left(\mathsf{T}^{(n)}{}_{S_n^{(1)}(s), S_n^{(2)}(t)}, s, t \in [0,1] \right) . \tag{4}$$

We can then treat $\omega = (R_1, R_2, \cdots ; C_1, C_2, \cdots)$ as an environment.

Proposition (Quenched)

$$\mathcal{T}^{(n)} - n^{-1} S_n^{(1)} \otimes S_n^{(2)} \xrightarrow{\text{law}} \sqrt{\frac{2}{\beta}} W^{(\infty)} \quad \textit{for a.e. } \omega \, .$$

Proposition (Skorokhod embedding)

Let $A^{(n)}$ *be* $D([0,1]^2)$*-valued such that* $A^{(n)} \xrightarrow{\text{law}} A$*. Let* $S_n^{(1)}$ *and* $S_n^{(2)}$ *be two independent processes as above, independent upon* $A^{(n)}$*. Set* $\widetilde{S_n^{(1)}} = \left(n^{-1/2}(S_n^{(1)}(s) - ns)\ ,\ s \in [0,1]\right)$ *and idem for* $\widetilde{S_n^{(2)}}$ *and* $\mathcal{A}^{(n)} = \left(A^{(n)}_{\left(n^{-1}S_n^{(1)}(s), n^{-1}S_n^{(2)}(t)\right)}\ ,\ s,t \in [0,1]\right)$*. Then*

$$\left(\mathcal{A}^{(n)}, \widetilde{S}_n^{(1)}, \widetilde{S}_n^{(2)}\right) \xrightarrow{\text{law}} (A, B_0^{1)}, B_0^{(2)})$$

where $A, B_0^{(1)}, B_0^{(2)})$ *are independent and* $B_0^{(1)}$ *and* $B_0^{(2)}$ *are two BB.*

Lemma

$$\left(n^{-1/2}\left(n^{-1}S_n^{(1)}(s)S_n^{(2)}(t) - nst\right)\ s,t \in [0,1]\right) \xrightarrow{\text{law}} \mathcal{W}^{(\infty)}$$

Now,

$$\mathcal{T}^{(n)} - \mathbb{E}\mathcal{T}^{(n)} = \left(\mathcal{T}^{(n)} - \mathbb{E}^{\omega}\mathcal{T}^{(n)}\right) + \left(\mathbb{E}^{\omega}\mathcal{T}^{(n)} - \mathbb{E}\mathcal{T}^{(n)}\right) .$$

Proposition (annealed)

$$n^{-1/2}\left(\mathcal{T}^{(n)} - \mathbb{E}\mathcal{T}^{(n)}\right) \xrightarrow{\text{law}} \mathcal{W}^{\infty} .$$

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Plan

Open problem

Quantum groups, in particular quantum permutation group.
Haar, Weingarten

$$\int u_{i_1 j_1} \cdots u_{i_k j_k} = \sum_{p_1, p_2} \delta_{p_1, i} \delta_{p_2, j} W_{k,n}(p_1, p_2)$$

where p_1, p_2 are non-crossing partitions of $[k]$ and

$$W_{k,n} = G_{k,n}^{-1} \, , \; G_{k,n}(p_1, p_2) = n^{|p_1 \vee p_2|} \, .$$

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THANK YOU FOR YOUR ATTENTION!