Extreme statistics of non-intersecting Brownian motions

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- A. Comtet (LPTMS, Orsay)
- P. J. Forrester (Dept. of Math., Melbourne)
- S. N. Majumdar (LPTMS, Orsay)
- K. Liechty (Ann Arbor, U. of Michigan)
- J. Rambeau (Inst. Theor. Phys., Cologne)
- J. Randon-Furling (Univ. Paris 1, Paris)

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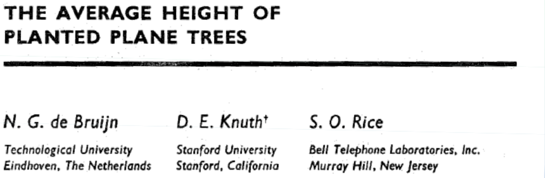
References:

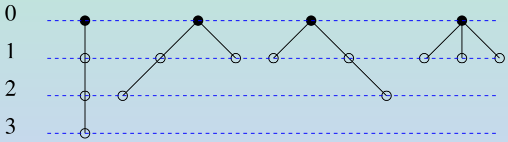
- G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling, Phys. Rev. Lett. **101**, 150601 (2008)
- J. Rambeau, G. S., Europhys. Lett. **91**, 60006 (2010); Phys. Rev. E **83**, 061146 (2011)
- P. J. Forrester, S. N. Majumdar, G. S., Nucl. Phys. B **844**, 500 (2011)
- J. Baik, K. Liechty, G. S., J. Math. Phys. **53**, 083303 (2012)

Height of rooted planar tree


THE AVERAGE HEIGHT OF
PLANTED PLANE TREES

N. G. de Bruijn — Technological University, Eindhoven, The Netherlands
D. E. Knuth† — Stanford University, Stanford, California
S. O. Rice — Bell Telephone Laboratories, Inc., Murray Hill, New Jersey

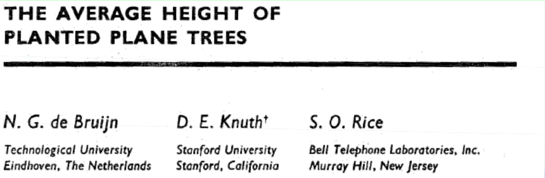
- Height of rooted plane trees $H_{1,n}$ with $n+1$ nodes



$H_{1,n=3} = 3$ $\quad$ $H_{1,n=3} = 2$ $\quad$ $H_{1,n=3} = 2$ $\quad$ $H_{1,n=3} = 1$

Q : $\langle H_{1,n} \rangle$ for large n ?

Height of rooted planar tree



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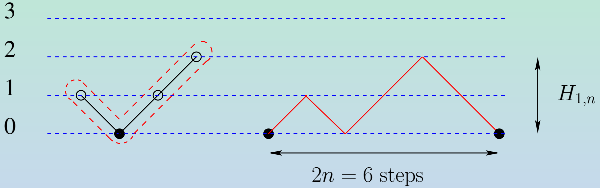
- Mapping between rooted plane trees and Dyck paths



$$\lim_{n\to\infty} \frac{\langle H_{1,n}\rangle}{\sqrt{2n}} = \langle H_1\rangle = \sqrt{\frac{\pi}{2}}$$

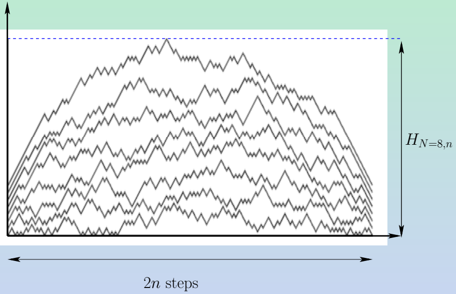
Generalization to N non-intersecting Dyck paths

Watermelon uniform random generation with applications

Nicolas Bonichon*, Mohamed Mosbah

LaBRI-Université Bordeaux 1, 351 Cours de la Libération, 33405 Talence, France

Theoretical Computer Science 307 (2003) 241 – 256



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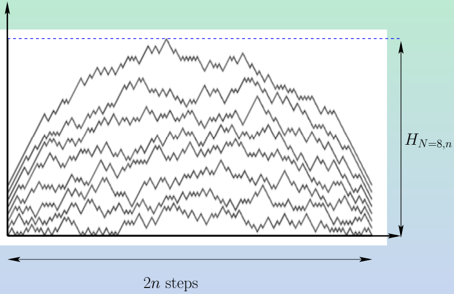

$\Longrightarrow$ numerical estimate for $\langle H_{N=8,n} \rangle$

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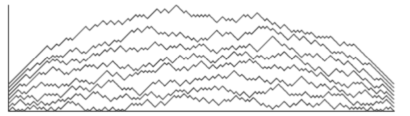

$$\frac{\langle H_{N,n}\rangle_{\rm num}}{\sqrt{2n}} = \langle H_N\rangle_{\rm num} \sim \sqrt{1.67\, N}$$

Generalization to N non-intersecting Dyck paths

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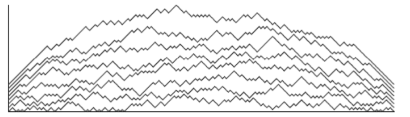

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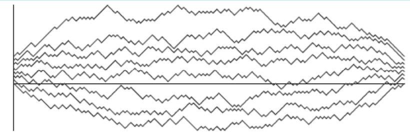

$$\frac{\langle H_{N,n}\rangle_{\rm num}}{\sqrt{2n}} = \langle H_N\rangle_{\rm num} \sim \sqrt{0.82\,N}$$

Generalization to N non-intersecting Dyck paths

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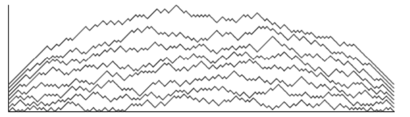

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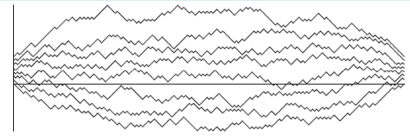


$$\frac{\langle H_{N,n}\rangle_{\rm num}}{\sqrt{2n}} = \langle H_N\rangle_{\rm num} \sim \sqrt{0.82\,N}$$

Q : can one compute $\langle H_N \rangle$?

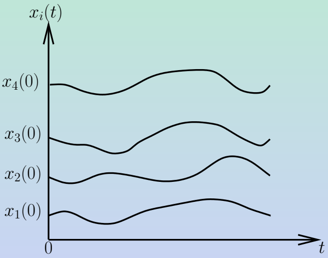
Non-intersecting Brownian motions in 1d

- N Brownian motions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t) \; , \; \langle \zeta_i(t) \zeta_j(t') \rangle = \delta_{i,j} \delta(t - t')$$
$$x_1(0) < x_2(0) < \ldots < x_N(0)$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < \ldots < x_N(t) \; , \quad \forall t \geq 0$$



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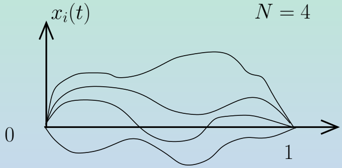

watermelons

Non-intersecting Brownian motions in 1d

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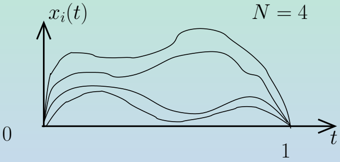


watermelons "with a wall"

Vicious walkers in physics

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 48, NUMBER 5 1 MARCH 1968

Soluble Model for Fibrous Structures with Steric Constraints

P.-G. DE GENNES

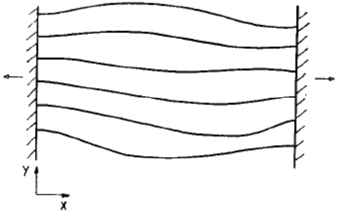
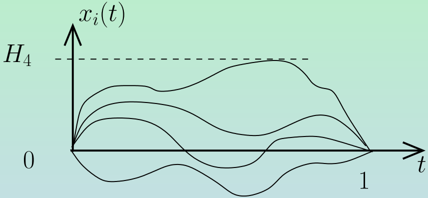


FIG. 1. Model for a two-dimensional fiber structure. The component chains are assumed to be attached to two plates I and F and placed under tension. The chains are bent by thermal fluctuations. Different chains cannot intersect each other.

Vicious walkers in physics and maths

- P. G. de Gennes, *Soluble Models for fibrous structures with steric constraints* (1968)
- M. E. Fisher, *Walks, Walls, Wetting and Melting* (1984)
- D. J. Grabiner, *Brownian motion in a Weyl chamber, non-colliding particles, and random matrices* (1999)
- C. Krattenthaler, A. J. Guttmann, X. G. Viennot, *Vicious walkers, friendly walkers and Young tableaux* (2000)
- P. L. Ferrari, K. Johansson, N. O'Connell, M. Praehofer, H. Spohn, C. Tracy, H. Widom... *Stochastic growth models, directed polymers* (from 2000)
- ...

Extreme statistics of vicious walkers

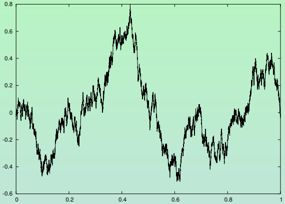


Maximal height of watermelons

$$x_1(t) < x_2(t) < ... < x_N(t)$$
$$H_N = \max_{\tau}[x_N(\tau), 0 \leq \tau \leq 1]$$
$$\langle H_N \rangle \ = ?$$

Extreme statistics of Brownian motion

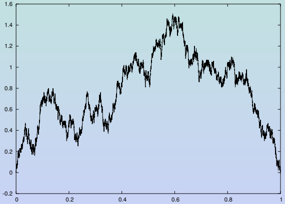
- Brownian bridge



$$H_1 = \max_{\tau}[x(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_1 \rangle = \sqrt{\frac{\pi}{8}}$$

- Brownian excursion



$$H_1 = \max_{\tau}[x(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_1 \rangle = \sqrt{\frac{\pi}{2}}$$

de Bruijn, Knuth, Rice '72
Flajolet, Odlyzko '90

Numerical computation of $\langle H_N \rangle$

N. Bonichon, M. Mosbah (2003)

Watermelon uniform random generation with applications

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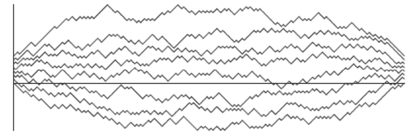
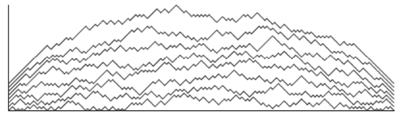
Theoretical Computer Science 307 (2003) 241 – 256

$$H_N = \max_\tau [x_N(\tau), 0 \le \tau \le 1]$$



$$\langle H_N \rangle_{\rm num} \sim \sqrt{0.82\, N}$$

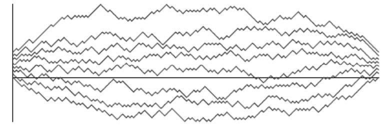


$$\langle H_N \rangle_{\rm num} \sim \sqrt{1.67\, N}$$

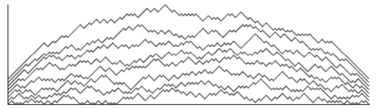
Main results

1. Connection between watermelons and random matrices

$\Longrightarrow$ exact asymptotic results for $\langle H_N \rangle$ for $N \gg 1$

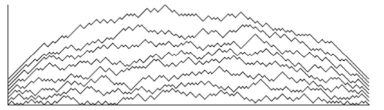


$$\langle H_N \rangle \sim \sqrt{N}$$



$$\langle H_N \rangle \sim \sqrt{2\,N}$$

2 Exact results for the full distribution of H_N



Cumulative distribution

$$F_N(M) = \text{Prob.}[H_N \leq M]$$

$$F_N(M) \to \mathcal{F}_1\left(2^{11/6} N^{1/6} \left| M - \sqrt{2N} \right| \right) \; , \; N \to \infty$$

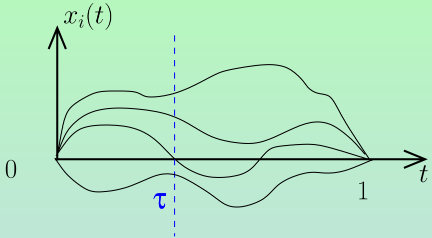
$\mathcal{F}_1$ is the Tracy-Widom distribution for GOE random matrices

Outline

Outline

1. Vicious walkers and random matrices
2. Connection with stochastic growth models
3. Exact computation using Feynman-Kac formula
4. Large N limit for watermelons with a wall
5. Conclusion

Non intersecting Brownian motions and RMT



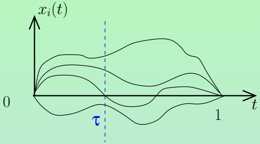
- Joint probability of $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ at fixed time τ

$$P_{\rm joint}(x_1, x_2, \cdots, x_N, \tau) \propto \prod_{i<j=1}^{N} (x_i - x_j)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^N x_i^2}$$

$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of the Gaussian Unitary Ensemble (GUE, $\beta = 2$)

Non intersecting Brownian motions and RMT



- $H \equiv H(t)$, $N \times N$ Hermitian matrices from GUE:

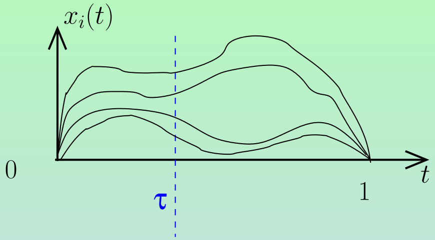
$$H_{mn}(t) = \begin{cases} \frac{1}{\sqrt{2}}\left(B_{mn}(t) + i\,\tilde{B}_{mn}(t)\right) , & m < n , \\ B_{mm}(t) , & m = n \\ \frac{1}{\sqrt{2}}\left(B_{nm}(t) - i\,\tilde{B}_{nm}(t)\right) , & m > n , \end{cases}$$

  where $B_{m,n}, \tilde{B}_{m,n}$ independent Brownian bridges

- Eigenvalues of $H(t)$

  $\{\lambda_1(t) < \lambda_2(t) < \cdots < \lambda_N(t)\} \stackrel{d}{=} \{x_1(t) < x_2(t) < \cdots < x_N(t)\}$

Non intersecting Brownian motions and RMT



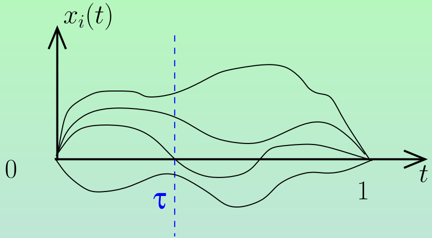
- Joint probability of $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ at fixed time τ

$$P_{\rm joint}(\mathbf{x}, \tau) \propto \prod_{i=1}^{N} x_i^2 \prod_{1 \leq i < j \leq N} (x_i^2 - x_j^2)^2 e^{-\frac{\mathbf{x}^2}{\sigma^2(\tau)}}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of the Bogoliubov-de Gennes type (class C)

A. Atland, M. R. Zirnbauer '96

Non intersecting Brownian motions and RMT



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$$P_{\text{joint}}(x_1, x_2, \cdots, x_N, \tau) \propto \prod_{i<j=1}^{N} (x_i - x_j)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^{N} x_i^2}$$

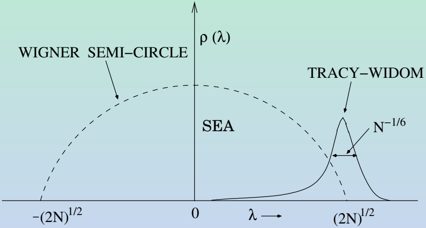
$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of the Gaussian Unitary Ensemble (GUE, $\beta = 2$)

Non intersecting Brownian motions and RMT

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE, $\beta = 2$)
- Mean density $\rho(\lambda)$ of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$ for GUE

$$\rho(\lambda) = \frac{1}{N} \sum_{\alpha=1}^{N} \langle \delta(\lambda - \lambda_\alpha) \rangle$$



Non intersecting Brownian motions and RMT

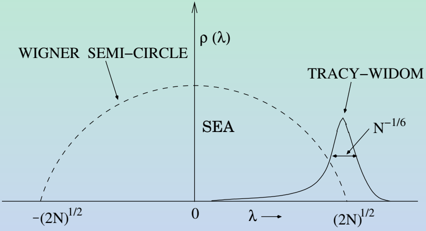
- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE, $\beta = 2$)
- Largest eigenvalue of random matrices from GUE

$$\lambda_{\max} = \max_{1 \leq i \leq N} \lambda_i$$

$$= \sqrt{2N} + \frac{N^{-\frac{1}{6}}}{\sqrt{2}} \chi_2$$

$$\text{Prob.}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi)$$

Tracy – Widom distribution



Non intersecting Brownian motions and RMT

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE, $\beta = 2$)
- Largest eigenvalue of random matrices from GUE

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$$= \sqrt{2N} + \frac{N^{-\frac{1}{6}}}{\sqrt{2}} \chi_2$$

$$\text{Prob.}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi)$$

Tracy – Widom distribution

$$\mathcal{F}_2(\xi) = \exp\left[-\int_{\xi}^{\infty} (s-\xi) q^2(s)\, ds\right]$$

where $q(s)$ satisfies Painlevé II

$$q''(s) = s\, q(s) + 2q^3(s)$$

$$q(s) \sim \text{Ai}(s)\ ,\ s \to \infty$$

C. Tracy, H. Widom '94

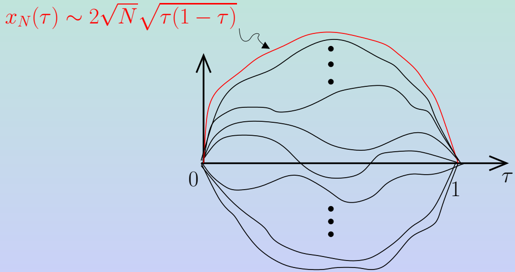
Watermelons in the limit of large N

- Consequences for watermelons without wall for large N

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim \sqrt{2N} + \frac{N^{-1/6}}{\sqrt{2}}\chi_2$$

$$\text{Prob.}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi), \text{ Tracy-Widom distribution for } \beta = 2$$

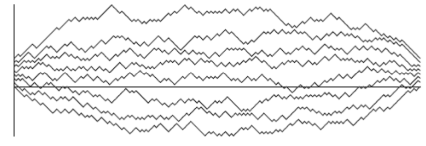
- When $N \to \infty$, $x_N(\tau)$ reaches a deterministic elliptic shape

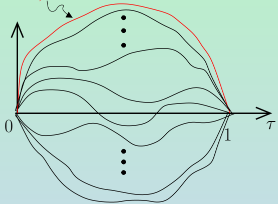


Asymptotic behavior of $\langle H_N \rangle$: without wall

- Consequences for watermelons without wall for large N

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim \sqrt{2N} + \frac{N^{-1/6}}{\sqrt{2}}\chi_2$$





- The maximal height is reached for $\tau = 1/2$

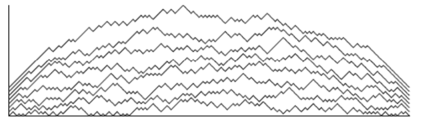
$$H_N = \max_{\tau}[x_N(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_N \rangle = \langle x_N(\tau = \frac{1}{2}) \rangle \sim \sqrt{N} \quad vs. \quad \langle H_N \rangle_{\rm num} \sim \sqrt{0.82\, N}$$

Asymptotic behavior of $\langle H_N \rangle$: with a wall

- Consequences for watermelons without wall for large N

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim 2\sqrt{N} + \frac{N^{-1/6}}{2^{2/3}}\chi_2$$



$$\text{Prob.}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi),$$

Tracy-Widom distribution for $\beta = 2$

- The maximal height is reached for $\tau = 1/2$

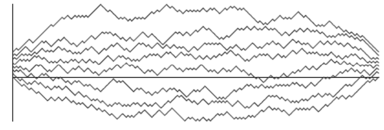
$$H_N = \max_\tau [x_N(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_N \rangle = \langle x_N(\tau = \frac{1}{2}) \rangle \sim \sqrt{2N} \quad \textit{vs.} \quad \langle H_N \rangle_{\text{num}} \sim \sqrt{1.67\, N}$$

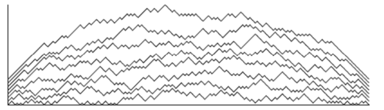
Summary I

- Connection between watermelons and random matrices

$\Longrightarrow$ exact asymptotic results for $\langle H_N \rangle$ for $N \gg 1$



$$\langle H_N \rangle \sim \sqrt{N}$$

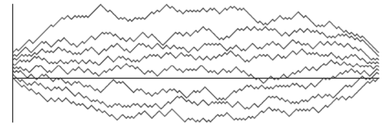


$$\langle H_N \rangle \sim \sqrt{2N}$$

Summary I

- Connection between watermelons and random matrices

$$\Longrightarrow \text{exact asymptotic results for } \langle H_N \rangle \text{ for } N \gg 1$$



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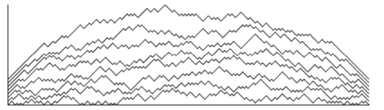


$$\langle H_N \rangle \sim \sqrt{2\,N}$$

What about the fluctuations of H_N ?

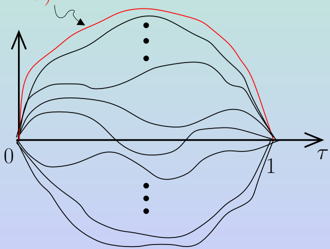
Fluctuations of H_N in the limit of large N

- Consequences for watermelons without wall for large N

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim \sqrt{2N} + \frac{N^{-1/6}}{\sqrt{2}}\chi_2$$

$$\text{Prob.}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi),$$ Tracy-Widom distribution for $\beta = 2$

- When $N \to \infty$, $x_N(\tau)$ reaches a deterministic elliptic shape



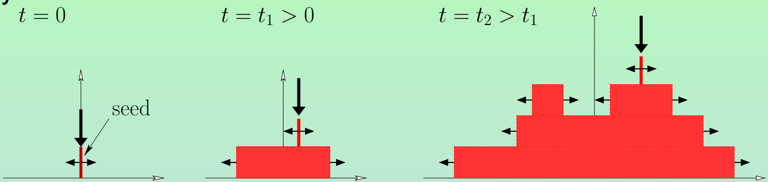
Fluctuations

$$x_N(\tau = 1/2) - \sqrt{N} \sim N^{-\frac{1}{6}}$$

Outline

Curved growing interface : the PNG droplet

- Polynuclear Growth Model



Curved growing interface : the PNG droplet

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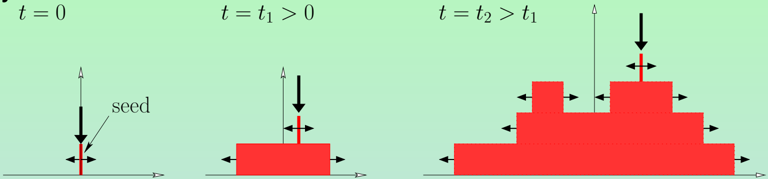
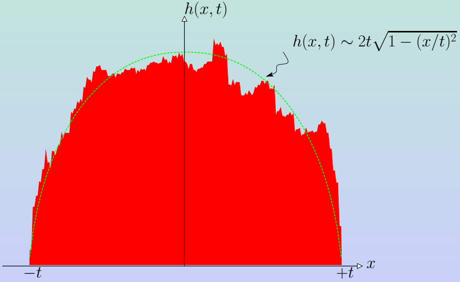


- At large time t the profile becomes droplet-like



Curved growing interface : the PNG droplet

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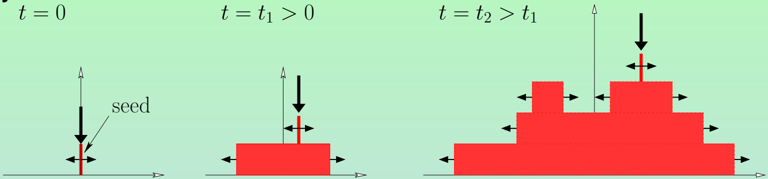

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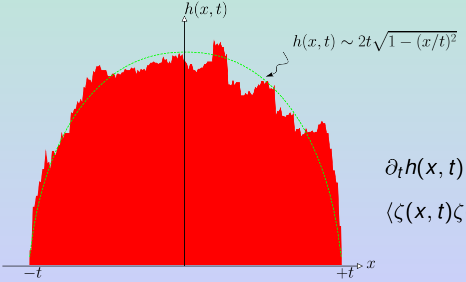


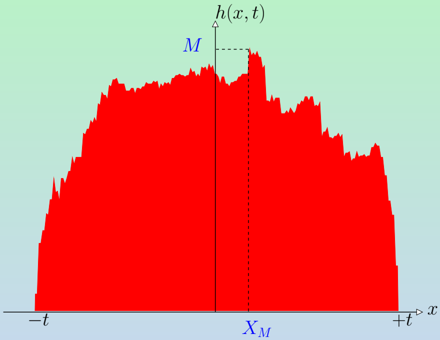
Fluctuations : KPZ equation

$$\partial_t h(x,t) = \nu \nabla^2 h(x,t) + \frac{\lambda}{2}(\nabla h(x,t))^2 + \zeta(x,t)$$

$$\langle \zeta(x,t)\zeta(x',t')\rangle = D\delta(x-x')\delta(t-t')$$

Curved growing interface : the PNG droplet

- Fluctuations : focus on extreme statistics



Curved growing interface : the PNG droplet

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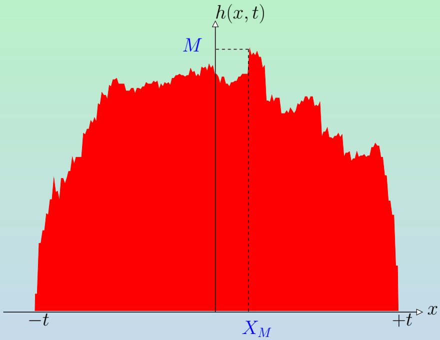
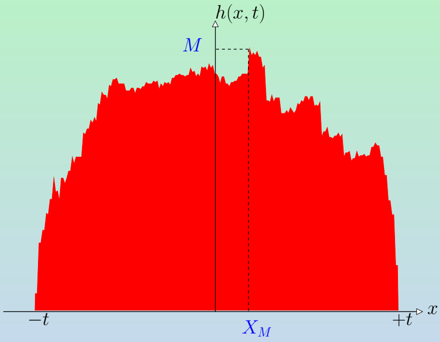


KPZ scaling

$$M - 2t \sim t^{1/3}$$
$$X_M \sim t^{2/3}$$

Curved growing interface : the PNG droplet

- Fluctuations : focus on extreme statistics

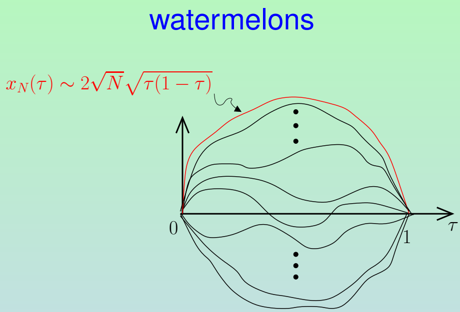


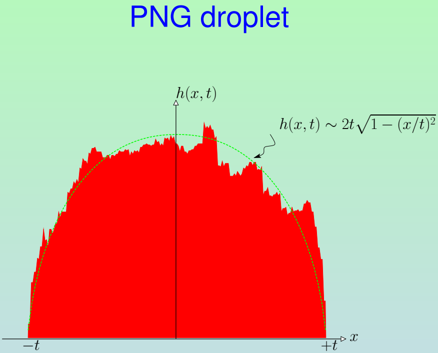
KPZ scaling

$$M - 2t \sim t^{1/3}$$
$$X_M \sim t^{2/3}$$

What is the joint distribution of M, X_M ?

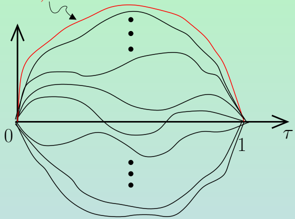
Vicious walkers and PNG droplet





$$x_N \iff h$$
$$\tau \iff x$$
$$N \iff t$$

Vicious walkers and PNG droplet



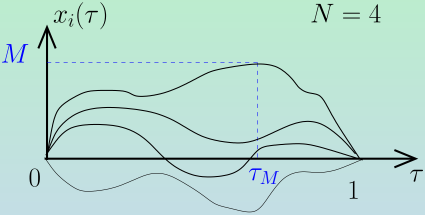
$$\frac{h(u t^{\frac{2}{3}}, t) - 2t}{t^{\frac{1}{3}}} \stackrel{d}{=} \frac{2\left[x_N\left(\frac{1}{2} + \frac{u}{2} N^{-\frac{1}{3}}\right) - \sqrt{N}\right]}{N^{-\frac{1}{6}}} \stackrel{d}{=} \mathcal{A}_2(u) - u^2$$

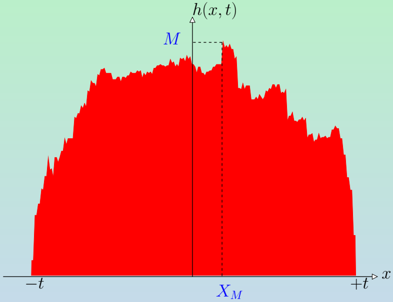
Prähofer & Spohn '00

$\mathcal{A}_2(u) \equiv$ Airy$_2$ process

Vicious walkers and PNG droplet

- Use this correspondence to study extreme statistics of PNG



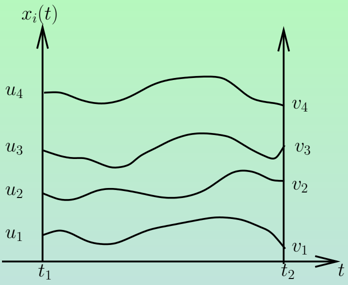


HERE: • exact computation of the distribution $P_N(M)$

• $P_N(M)$ in the $N \to \infty$ limit

Outline

Transition probability



$$\mathcal{P}_N(v_1, v_2, \cdots, v_N, t_2 | u_1, u_2, \cdots, u_N, t_1) = ?$$

Karlin-Mc Gregor (1959), Lindström (1973), Gessel-Viennot (1985)

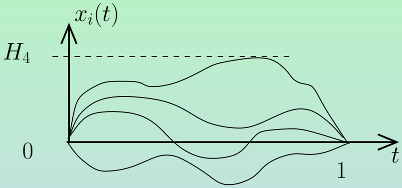
- The case $N = 2$: a reflection principle

$$\begin{aligned}\mathcal{P}_2(v_1, v_2, t_2 | u_1, u_2, t_1) &= \mathbf{P}_1(v_1, t_2 | u_1, t_1)\mathbf{P}_1(v_2, t_2 | u_2, t_1) \\ &\qquad - \mathbf{P}_1(v_2, t_2 | u_1, t_1)\mathbf{P}_1(v_1, t_2 | u_2, t_1) \\ &= \mathbf{P}_2(v_1, v_2, t_2 | u_1, u_2, t_1) - \mathbf{P}_2(v_2, v_1, t_2 | u_1, u_2, t_1)\end{aligned}$$

where $\mathbf{P}_2(.|.) \equiv$ transition probability for two free particles (allowed to cross each other)

Watermelons configurations : regularization procedure

- Brownian motion has an infinite density of zero-crossings



Such configurations are ill-defined for Brownian motion :

$$x_i(0) = x_{i+1}(0)$$
$$\text{AND } x_i(t = 0^+) < x_{i+1}(t = 0^+)$$

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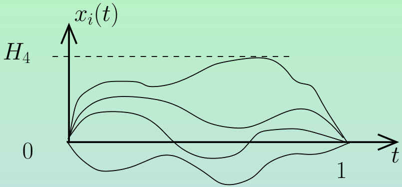
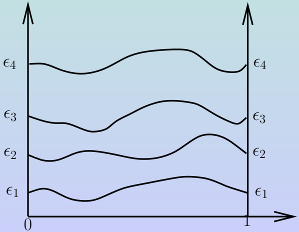


Such configurations are ill-defined for Brownian motion :

$$x_i(0) = x_{i+1}(0)$$
$$\text{AND } x_i(t = 0^+) < x_{i+1}(t = 0^+)$$

- A need for regularization : introduce cut-offs ϵ_i's

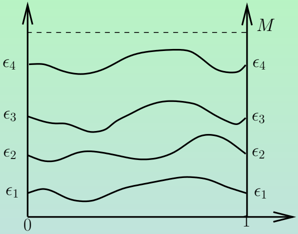


Only at the end take the limit $\epsilon_i \to 0$

Distribution of the maximal height : without wall

- Cumulative distribution of the maximal height

$$F_N(M) = \text{Proba.}[x_N(\tau) \leq M \ , \ \forall 0 \leq \tau \leq 1]$$



- Path integral (Feynman-Kac formula) for free fermions

$$F_N(M) = \frac{2^{-\binom{N}{2}}}{\prod_{j=0}^{N-1} j!} \det_{1\leq i,j \leq N} \left[(-1)^{i-1} H_{i+j-2}(0) - H_{i+j-2}(\sqrt{2}M) e^{-2M^2} \right]$$

where $H_n(M) \equiv$ Hermite Polynomials   see also T. Feierl '08

Distribution of the maximal height : without wall

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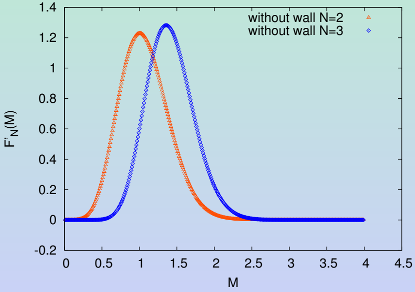
- Shape and asymptotic behavior

$$F_N(M) \propto M^{N^2+N} \ , \ M \to 0$$
$$1 - F_N(M) \sim e^{-2M^2} \ , \ M \to \infty$$



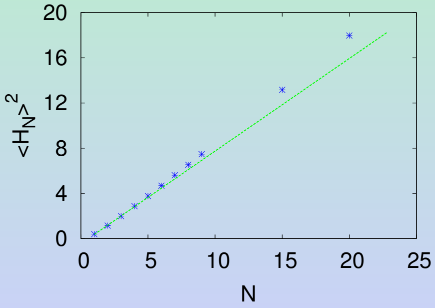
Exact value of the $\langle H_N \rangle$

$$
\begin{aligned}
\langle H_1 \rangle &= \frac{\sqrt{\pi}}{4}\sqrt{2} \\
\langle H_2 \rangle &= \frac{\sqrt{\pi}}{4}\left(1+\sqrt{2}\right) \\
\langle H_3 \rangle &= \frac{\sqrt{\pi}}{96}\left(45+36\sqrt{2}-8\sqrt{6}\right) \\
\langle H_4 \rangle &= \frac{\sqrt{\pi}}{20736}\left(17091+5184\sqrt{2}-1888\sqrt{6}\right) \\
&\dots
\end{aligned}
$$

Comparison with numerics by Bonichon & Mosbah

Numerical estimate by Bonichon & Mosbah

$$\langle H_N \rangle^2_{\text{num}} \sim 0.82N$$



Exact behavior at large N

$$\langle H_N \rangle^2 \sim N$$

Distribution of the maximal height : with a wall

- Cumulative distribution of the maximal height

G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling '08

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^N n_i^2}$$

$$A_N = \frac{\pi^{2N^2+N}}{2^{N^2-N/2}\prod_{j=0}^{N-1}\Gamma(2+j)\Gamma(\frac{3}{2}+j)} \text{ cf Selberg integral}$$

see also T. Feierl '08 and '12, N. Kobayashi *et al.* '08

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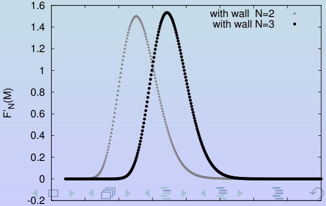
see also T. Feierl '08 and '12, N. Kobayashi *et al.* '08

- Shape and asymptotic behavior

$$F_N(M) \sim \frac{\alpha_N}{M^{2N^2+N}} e^{-\frac{\pi^2}{12M^2}N(N+1)(2N+1)}, \quad M \to 0$$

$$1 - F_N(M) \sim e^{-2M^2}, \quad M \to \infty$$



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see also T. Feierl '08 and '12, N. Kobayashi *et al.* '08

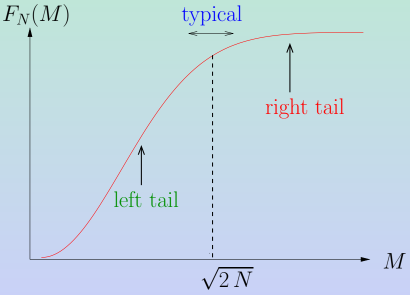
What about the asymptotic behavior of $F_N(M)$ for $N \to \infty$?

Outline

Large N limit for watermelons with a wall

- Large N analysis of

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\le j<k\le N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^N n_i^2}$$



$$\lim_{N\to\infty} -\frac{1}{N^2} \log F_N\left(x = \frac{M}{\sqrt{2N}}\right) = \begin{cases} \phi_-(x)\ , \ x < 1 \\ 0\ ,\ x > 1 \end{cases}$$

$$\phi_-(x) \sim \frac{16}{3}(1-x)^3$$ third order phase transition

Typical fluctuations via orthogonal polynomials

- Large N analysis of

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\le j<k\le N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^N n_i^2}$$

- Discrete orthogonal polynomials

$$\sum_{n=-\infty}^{\infty} p_k(n) p_{k'}(n) e^{-\frac{\pi^2}{2M^2} n^2} = \delta_{k,k'} h_k \; ,$$

$$p_k(n) = n^k + \cdots$$

- Useful expression for asymptotic analysis

$$F_N(M) = \frac{B_N}{M^{2N^2+N}} \prod_{k=1}^{N} h_{2k-1}$$

Large N limit for $F_N(M)$

- For large N, in the "double-scaling limit" P. J. Forrester, S. N. Majumdar, G. S. '11

$$\frac{d^2}{dt^2}\log F_N\left(\sqrt{2N}(1+t/(2^{7/3}N^{2/3}))\right) = -\frac{1}{2}\left(q^2(t)-q'(t)\right)$$
$$q''(t) = 2q^3(t) + tq(t)\ ,\ q(t) \sim \mathrm{Ai}(t)\ ,\ t \to \infty$$

i.e.

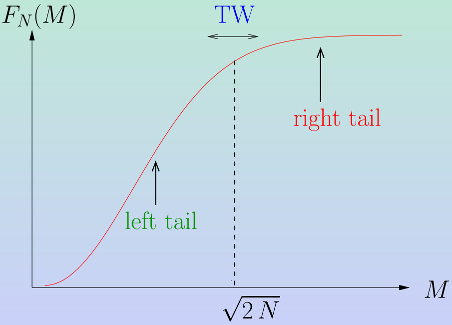
$$\begin{aligned} F_N(M) &\to \mathcal{F}_1\left(2^{11/6}p^{1/6}\left|M-\sqrt{2N}\right|\right) \\ \mathcal{F}_1(t) &= \exp\left(-\frac{1}{2}\int_t^\infty \left((s-t)\,q^2(s)-q(s)\right)\,ds\right) \\ &\equiv \text{Tracy-Widom distribution for GOE} \end{aligned}$$

Large N limit for $F_N(M)$

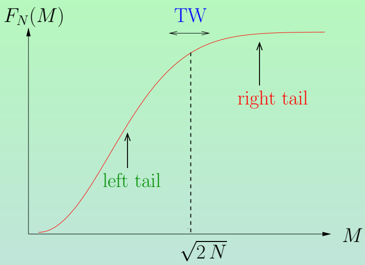
$$F_N(M) \rightarrow \mathcal{F}_1\left(2^{11/6} p^{1/6} \left| M - \sqrt{2N} \right|\right)$$

$$\mathcal{F}_1(t) = \exp\left(-\frac{1}{2}\int_t^\infty \left((s-t)\, q^2(s) - q(s)\right)\, ds\right)$$

$$\equiv \text{Tracy-Widom distribution for GOE}$$



Large N limit for $F_N(M)$ G. S., S. N. Majumdar, A. Comtet, P. J. Forrester '12



$$\begin{cases} F_N(M) & \sim \exp\left[-N^2\,\phi_-\left(M/\sqrt{2N}\right)\right] \;,\; M < \sqrt{2N} \;\&\; |M-\sqrt{2N}| \sim \mathcal{O}(\sqrt{N}) \\ F_N(M) & \sim \mathcal{F}_1\left[2^{\frac{11}{6}} N^{\frac{1}{6}}(M-\sqrt{2N})\right] \;,\; M \sim \sqrt{2N} \;\&\; |M-\sqrt{2N}| \sim \mathcal{O}(N^{-\frac{1}{6}}) \\ 1-F_N(M) & \sim \exp\left[-N\,\phi_+\left(M/\sqrt{2N}\right)\right] \;,\; M > \sqrt{2N} \;\&\; |M-\sqrt{2N}| \sim \mathcal{O}(\sqrt{N}) \end{cases}$$

Outline

Conclusion

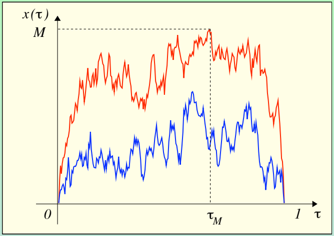
- Exact results for $\langle H_N \rangle$ for large N
- Exact result for distribution of the maximal height H_N using path integral techniques
- Connections with stochastic growth models for large N
- Large N limit

$$F_N(M) \to \mathcal{F}_1 \left(2^{11/6} N^{1/6} \left| M - \sqrt{2N} \right| \right) \ , \ N \to \infty$$

see also Liechty'11 for a recent rigorous proof

- Joint distribution of the maximum and its position

Joint pdf of the maximum and its position



$P_N(M, \tau_M) \equiv$ joint distribution of the maximum M and its position τ_M

J. Rambeau, G. S. '11

$$P_N(M,\tau_M) = \frac{A_{N,E}}{M^{N(2N+1)+3}} \sum_{\mathbf{n}, n'_N} \Bigg\{ (-1)^{n_N + n'_N} \, n_N^2 \, {n'_N}^2 \left(\prod_{i=1}^{N-1} n_i^2 \right) e^{-\frac{\pi^2}{2M^2} \sum_{i=1}^{N-1} n_i^2}$$

$$\Delta_N(n_1^2, \ldots, n_{N-1}^2, n_N^2) \, \Delta_N(n_1^2, \ldots, n_{N-1}^2, {n'_N}^2) e^{-\frac{\pi^2}{2M^2}\left[(1-\tau_M){n'_N}^2 + \tau_M n_N^2\right]} \Bigg\}$$

What about the large N limit of $P_N(M, \tau_M)$?

More recent results

- Large N asymptotics for $P_N(M, \tau_M)$ G. S. '12 ; Baik, Liechty, G. S. '12

$$\lim_{N\to\infty} 2^{-\frac{9}{2}} N^{-\frac{1}{2}} P_N(\sqrt{2N} + 2^{-\frac{11}{6}} s N^{-\frac{1}{6}}, \frac{1}{2} + 2^{-\frac{8}{3}} w N^{-\frac{1}{3}}) = P(s, w)$$

$$P(s, w) = \frac{\pi^2}{2^{\frac{20}{3}}} \mathcal{F}_1(s) \int_s^\infty f(x, w) f(x, -w)\, dx$$

$$f(x, w) = -\frac{2^{\frac{13}{2}}}{\pi^2} \int_0^\infty \zeta \Phi_2(\zeta, x) e^{-w\zeta^2}\, d\zeta$$

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$$P(s, w) = \frac{\pi^2}{2^{\frac{20}{3}}} \mathcal{F}_1(s) \int_s^\infty f(x, w) f(x, -w)\, dx$$

$$f(x, w) = -\frac{2^{\frac{13}{2}}}{\pi^2} \int_0^\infty \zeta \Phi_2(\zeta, x) e^{-w\zeta^2}\, d\zeta$$

$$\underbrace{\frac{\partial}{\partial \zeta}\Psi = A\Psi \;,\; \frac{\partial}{\partial x}\Psi = B\Psi}_{\text{Lax Pair}} \;,\; \Psi = \begin{pmatrix} \Phi_1(\zeta, x) \\ \Phi_2(\zeta, x) \end{pmatrix}$$

Lax Pair

More recent results

- Large N asymptotics for $P_N(M, \tau_M)$ G. S. '12 ; Baik, Liechty, G. S. '12

$$\lim_{N\to\infty} 2^{-\frac{9}{2}} N^{-\frac{1}{2}} P_N(\sqrt{2N} + 2^{-\frac{11}{6}}\, s\, N^{-\frac{1}{6}}, \frac{1}{2} + 2^{-\frac{8}{3}}\, w\, N^{-\frac{1}{3}}) = P(s,w)$$

$$P(s,w) = \frac{\pi^2}{2^{\frac{20}{3}}} \mathcal{F}_1(s) \int_s^\infty f(x,w) f(x,-w)\, dx$$

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$$A(\zeta,x) = \begin{pmatrix} 4\zeta q & 4\zeta^2 + x + 2q^2 + 2q' \\ -4\zeta^2 - x - 2q^2 + 2q' & -4\zeta q \end{pmatrix} \, , \, B(\zeta,x) = \begin{pmatrix} q & \zeta \\ -\zeta & -q \end{pmatrix}$$

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Lax Pair

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