Clifford representation of an algebra related to spanning forests

Università degli Studi di Milano

Andrea Sportiello
work in collaboration with S. Caracciolo and A.D. Sokal

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Potts and $O(n)$ non-linear σ-models

- Potts Model: variables $\sigma_i \in \{0, 1, \dots, q-1\}$;
  $\exp(-\beta\mathcal{H}(\sigma)) = \exp\left[\sum_{\langle ij\rangle} J_{ij}\delta(\sigma_i, \sigma_j)\right]$
  Symmetry: 'global' permutations in $\mathcal{S}_q$.
- $O(n)$ non-linear σ-model: variables $\vec{\sigma}_i \in \mathbb{R}^n$;
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  Symmetry: 'global' rotations in $O(n)$ (continuous!).
- If $\frac{1}{2}\left((\vec{\sigma}_i \cdot \vec{\sigma}_j)^2 - 1\right)$ instead of $(\vec{\sigma}_i \cdot \vec{\sigma}_j - 1)$:
  extra 'local' $\mathbb{Z}_2$ symmetry $\vec{\sigma}_i \to \epsilon_i\vec{\sigma}_i$, with $\epsilon = \pm 1$.
  In other words, the $\vec{\sigma}$'s are in the projective space: RP^{n-1}.
  $\left[\ \mathrm{RP}^{n-1} := \left\{\vec{x} \in \mathbb{R}^n \setminus \{0\}\right\} / \vec{x} \sim \lambda\vec{x}\ \right]$

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- Find relations between Potts and $O(n)$ non-lin. σ-models, and with combinatorial "generating functions" (i.e. countings of graphical structures);
- Understand analytic continuation in q for Potts Model, and in n for $O(n)$;
- Understand computational complexity for the generating function (and existence of FPRAS), as a fn. of q and of n;
- Understand asymptotic freedom in a geometric and non-perturbative way, in $D = 2$ Euclidean lattice, for our 'favourite' model: Potts $[q \to 0;\ J/q \text{ fixed}]$ $\equiv O(n)$ non-lin σ-model $[n \to -1] \equiv$ Spanning Forests.

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Analytic continuation is easy for Potts...

[Fortuin-Kasteleyn (1972), relating Potts p.fn. to the Tutte Poly.]

$$Z_G = \sum_{\sigma} e^{-\beta \mathcal{H}(\sigma)} = \sum_{\sigma} \prod_{(ij)} \left(1 + v_{ij}\, \delta(\sigma_i, \sigma_j)\right) \qquad \left[v_{ij} := e^{J_{ij}} - 1 \right]$$

$$= \sum_{H \subseteq G} \prod_{(ij) \in E(H)} v_{ij} \left(\sum_{\sigma} \prod_{(ij) \in E(H)} \delta(\sigma_i, \sigma_j) \right)$$

$$= \sum_{H \subseteq G} q^{K(H)} \prod_{(ij) \in E(H)} v_{ij} \,. \qquad \left[K(H) = \# \left\{ \begin{matrix} \text{comp.} \\ \text{in } H \end{matrix} \right\} \right]$$

Recognize the (slightly reparametrized and rescaled) multivariate Tutte Polynomial of G, and even better on next slide...

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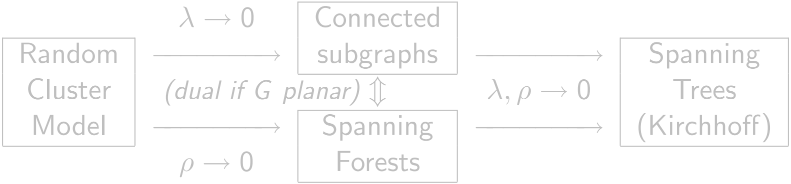
...and leads to the Random Cluster Model

Recall:

- $L(H)$, the *cyclomatic number*, is the number of linearly-independent cycles in H.
- Euler formula states that $V - K = E - L$.

$$Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho) = \sum_{H \subseteq G} \lambda^{K(H)-K(G)} \, \rho^{L(H)} \prod_{(ij) \in E(H)} w_{ij} \qquad \begin{bmatrix} \lambda\rho = q \\ w_{ij} = v_{ij}/\rho \end{bmatrix}$$

Tutte: $w = 1$; $x := Z[\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet] = 1 + \lambda$ and $y := Z[$  $] = 1 + \rho$.



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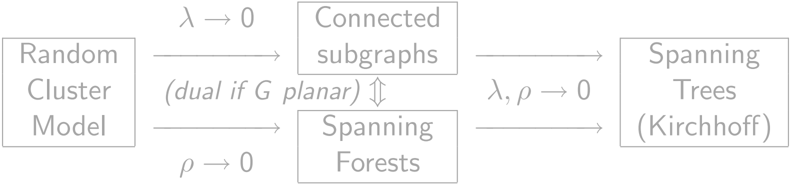

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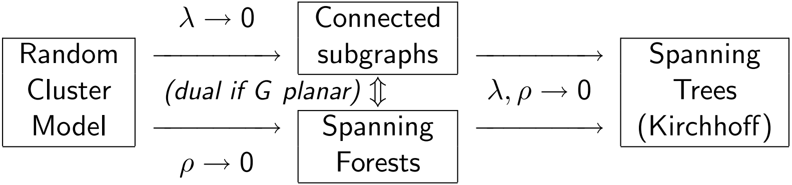
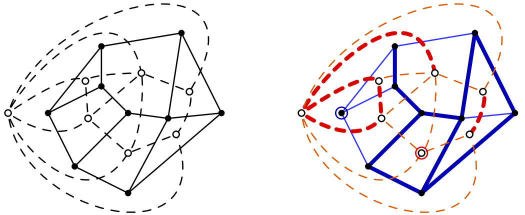


Planar duality

If graph G is connected and planar:

- Spanning Forests and Connected Subgraphs are dual;
- Trees are self-dual, and the intersection of the two sets.

More generally: $E(\widehat{H}) = \widehat{E(H)}^{c}$, and $L(\widehat{H}) = K(H) - 1$, so duality acts as $\lambda \leftrightarrow \rho$ and $w_{ij} \leftrightarrow 1/w_{ij}$.



Temperley-Lieb Algebra with parameter $\sqrt{\lambda\rho}$ plays a role.

Comput. complexity of Random-Cluster Partition Function

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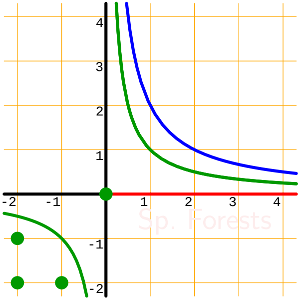

- Trivial if $\lambda\rho = q = 1$ (percolation);
- Computable in poly-time as a Pfaffian if $\lambda\rho = 2$ (Ising) and G is planar [Kasteleyn; Kač, Ward; 60's]
- Computable in poly-time at exceptional special points $(\lambda, \rho) = (-2, -2), (-2, -1), (-1, -2)$ and $(0, 0)$.

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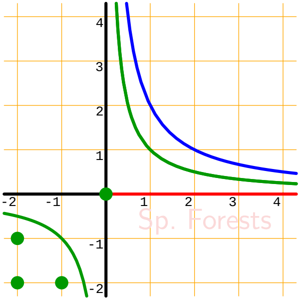

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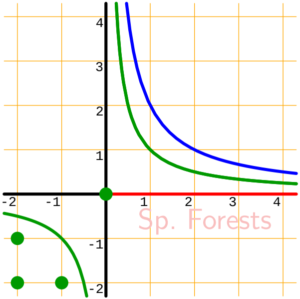

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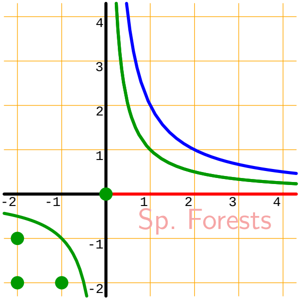

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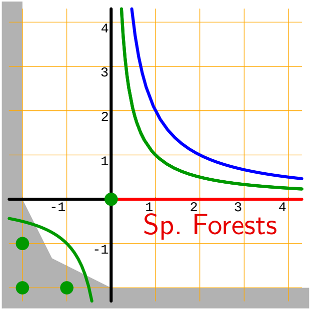

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The Matrix-Tree Theorem [Kirchhoff, 1848]

$$Z_{\mathrm{RC}}(G; \vec{w}; \lambda = \rho = 0) = \sum_{\substack{T \subseteq G \\ \text{trees}}} \prod_{(ij) \in E(T)} w_{ij} = \det L(i_0)$$

where i_0 is any vertex of G (the 'root'), $L(i_0)$ is the minor of L with row and col. i_0 removed, and L is the graph Laplacian matrix:

$$L_{ij} = \begin{cases} -w_{ij} & (ij) \in E(G) \\ 0 & (ij) \notin E(G) \\ \sum_{k \sim i} w_{ik} & i = j \end{cases} \qquad L \sim -\nabla^2$$

From Gaussian Integral formula in complex Grassmann Algebra:

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A digression on Grassmann Calculus

For $i = 1, .., n$, introduce the formal symbols θ_i, with $\theta_i \theta_j = -\theta_j \theta_i$, and symbols $\partial_i \equiv (\int \mathrm{d}\theta_i)$, with formal rules:

$$\{\partial_i, \theta_j\} = \delta_{ij} \qquad \text{(cfr. with} \quad [\frac{\mathrm{d}}{\mathrm{d}x_i}, x_j] = \delta_{ij})$$

$$\{\partial_i, \partial_j\} = \{\theta_i, \theta_j\} = 0 \qquad [\frac{\mathrm{d}}{\mathrm{d}x_i}, \frac{\mathrm{d}}{\mathrm{d}x_j}] = [x_i, x_j] = 0$$

$$\int \mathrm{d}\theta_i (\theta_i a + b) = a \quad \text{(so that} \quad \int \mathrm{d}\theta f(\theta + \chi) = \int \mathrm{d}\theta f(\theta)) \,.$$

As $\theta_i^2 = 0$, the most general monomial $\prod_i \theta_i^{n_i}$ has $n_i = 0, 1$ (this justifies the name 'fermion'). Remark

$$\int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \prod_{i=1,\dots,n} \theta_i^{n_i} = \begin{cases} 1 & n_i = 1 \quad \forall i \\ 0 & \text{otherwise} \end{cases}$$

Special application, for $n \times n$ antisymmetric matrix A,

$$\int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \exp\left(\tfrac{1}{2}\theta A \theta\right) = \mathrm{pf} A = (\det A)^{\frac{1}{2}} .$$

A "complex" structure is natural: consider the case of $2n$ symbols $\bar{\psi}_1, \ldots, \bar{\psi}_n$ and $\psi_1, \ldots, \psi_n$, and $\mathcal{D}(\psi, \bar{\psi}) := \mathrm{d}\psi_n \mathrm{d}\bar{\psi}_n \cdots \mathrm{d}\psi_1 \mathrm{d}\bar{\psi}_1$. Then, for any matrix A

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In the following we will prove an extension to arbitrary λ of Kirchhoff Formula ($\lambda \to 0$)

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Generalize $O(n)$ to $\mathrm{OSP}(n|2m)$ models:

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For $n \geq 1$ and $m \geq 0$, analytic continuation should depend on $n - 2m$ only. [Parisi-Sourlas, 1979; Cardy, 1983]

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Generalize $O(n)$ to $\mathrm{OSP}(n|2m)$ models:

$$\vec{\sigma} = (\phi^{(a)})_{a=1,\dots,n} \qquad |\vec{\sigma}|^2 = \sum_{a=1}^{n} (\phi^{(a)})^2$$

$$\Downarrow$$

$$\vec{\sigma} = (\underbrace{\phi^{(a)}}_{B}; \underbrace{\bar{\psi}^{(b)}, \psi^{(b)}}_{F})_{\substack{a=1,\dots,n \\ b=1,\dots,m}} \qquad |\vec{\sigma}|^2 = \sum_{a=1}^{n} (\phi^{(a)})^2 + 2\lambda \sum_{a=1}^{m} \bar{\psi}^{(b)} \psi^{(b)}$$

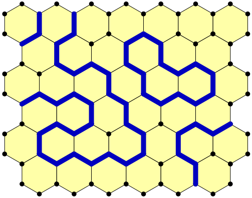
For $n \geq 1$ and $m \geq 0$, analytic continuation should depend on $n - 2m$ only. [Parisi-Sourlas, 1979; Cardy, 1983]

Simplest non-trivial choice: $\mathrm{OSP}(1|2)$, i.e. $\vec{\sigma} = (\phi; \bar{\psi}, \psi)$.

A different approach: Nienhuis Loop-Gas Model

Nienhuis [1982] considers an $O(n)$-invariant model with a logarithmic action:

$$\exp(-\beta\mathcal{H}(\sigma)) = \exp\left[\sum_{\langle ij\rangle}\log\left(1+\frac{w_{ij}}{n}\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\right]$$



Then, on a cubic graph, and for any one-body measure (also the Gaussian one...), reduces to a combinatorial model of loop gas, with a topological weight n per loop.

- ✔ Easy analytic continuation in n, through a geometric model;
- ✘ Log-action: many terms; Blind to one-body measure... issues of universality?

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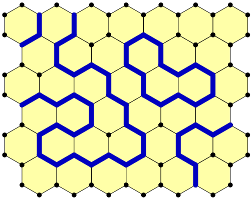

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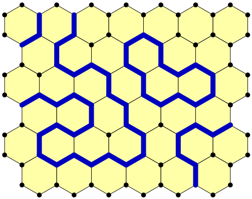

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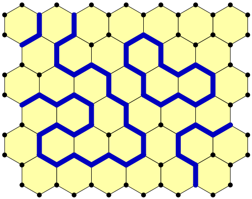

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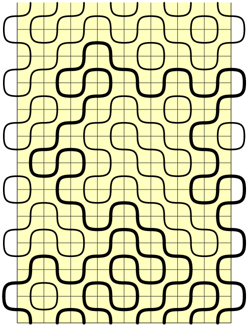
Dense $O(n)$ Loops, Potts, and Temperley-Lieb algebra



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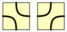
❶ fill the square lattice with 

❷ give weight n to each cycle.

This model of dense loops has special algebraic properties ➽ TL Algebra

$$e_i^2 = n\, e_i \qquad e_i e_{i\pm1} e_i = e_i$$
$$[e_i, e_j] = 0 \quad \text{if } |i-j| > 1.$$

➽ Potts Model on the square lattice (rot. 45°), for $n = \sqrt{q}$

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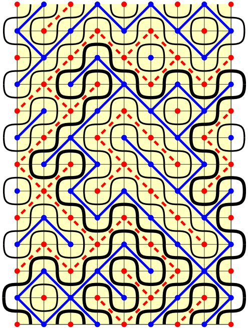

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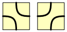
❶ fill the square lattice with 

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$\mathrm{OSP}(1|2)$ – Spanning-Forest correspondence

Theorem: the $\mathrm{OSP}(1|2)$ non-linear σ-model partition function is related to the Random Cluster partition function at $\rho = 0$

$$Z_{\mathrm{OSP}(1|2)}(G; -\vec{w}/\lambda) = Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho = 0)$$

at a perturbative level. For the $\mathrm{RP}^{0|2}$ model, the relation is non-perturbative.

...Let's prove it...

From the δ's, for each i we have $\phi_i^2 + 2\lambda\bar{\psi}_i\psi_i = 1$.

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Forget about ϵ's (say, all $+1$). [this why 'perturbative'...]

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The action, in both cases

$$\mathrm{OSP}(1|2): \qquad \mathcal{S} = -\sum_{(ij)} \frac{w_{ij}}{\lambda} \left(1 - \vec{\sigma}_i \cdot \vec{\sigma}_j\right)$$

$$\mathrm{RP}^{0|2}: \qquad \mathcal{S} = -\sum_{(ij)} \frac{w_{ij}}{2\lambda} \left(1 - (\vec{\sigma}_i \cdot \vec{\sigma}_j)^2\right)$$

gives the peculiar expression

$$\mathcal{S} = \sum_{(ij)} w_{ij} f_{ij}^{(\lambda)} \qquad f_{ij}^{(\lambda)} := (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) - \lambda \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j$$

and we are left to prove our "generalized Matrix-Tree theorem":

$$\int \mathcal{D}(\psi, \bar{\psi}) \exp\left[\lambda \bar{\psi}\psi + \sum_{(ij)} w_{ij} f_{ij}^{(\lambda)}\right] = Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho = 0)$$

- Define $\tau_A := \prod_{i\in A} \bar\psi_i\psi_i$. Generalize f_{ij} to f_A, with $A \subseteq V(G)$:

$$f_A = \lambda(1-|A|)\tau_A + \sum_{i\in A} \tau_{A\smallsetminus i} - \sum_{(i\neq j)\in A} \bar\psi_i\psi_j\tau_{A\smallsetminus\{i,j\}}$$

- Check the algebraic lemma, for $A \cap B \neq \varnothing$:

$$f_A f_B = \begin{cases} f_{A\cup B} & |A\cap B| = 1 \\ 0 & |A\cap B| \geq 2 \end{cases} \qquad (\text{corollary: } f_{ij}^2 = 0)$$

- Expand the action: $\exp\left(\sum_{(ij)} w_{ij} f_{ij}\right) = \sum_{E'\subseteq E(G)} \prod_{(ij)\in E'} w_{ij} f_{ij}$

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So, our fermionic integral has already been reduced to a sum over spanning forests, and factors w_{ij} are appropriate. We still have to prove that the remaining fermionic integral of each summand gives exactly $\lambda^{K(F)}$.

Of course, the integral factorizes on various $V(T_\alpha)$, and we can concentrate on a single component, with $V(T_\alpha) = U$:

$$\int \mathcal{D}(\psi,\bar\psi) \prod_i (\overset{\spadesuit}{1} + \underbrace{\lambda\bar\psi_i\psi_i}_{\clubsuit}) \left[\overbrace{\lambda(1-|U|)\tau_U}^{\spadesuit} + \sum_i \underbrace{\tau_{U\setminus i}}_{\clubsuit} - \sum_{(i\neq j)} \bar\psi_i\psi_j\tau_{U\setminus\{i,j\}} \right]$$

Term $\spadesuit$ contributes $\lambda(1-|U|)$. Terms $\clubsuit_i$ contribute λ each. So we get a factor $\lambda(1-|U|+\sum_{i\in U} 1) = \lambda$, as claimed. □

Conclusions in the "continuum limit"

$$Z_{\mathrm{OSP}(1|2)} = \int \mathcal{D}(\psi, \bar{\psi})\, e^{\lambda \bar{\psi}\psi + \bar{\psi}\nabla^2\psi + \frac{\lambda}{2}\bar{\psi}\psi\nabla^2\bar{\psi}\psi} = Z_{\mathrm{RC}}(\lambda, \rho = 0)$$

which generalizes Kirchhoff Theorem

$$Z' \begin{bmatrix} \text{massless} \\ \text{fermion} \end{bmatrix} = \int \mathcal{D}(\psi, \bar{\psi})\, \bar{\psi}_0 \psi_0\, e^{\bar{\psi}\nabla^2\psi} = Z_{\mathrm{RC}}(\lambda = 0, \rho = 0)$$

Of course, the theory at $\lambda = 0$ (Spanning Trees) is critical. In $D = 2$, it is a $c = -2$ logarithmic CFT.

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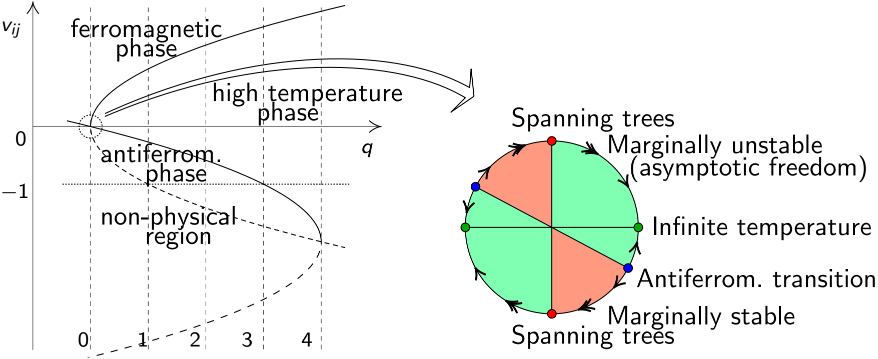
E.g., our present understanding for Potts on the square lattice (combined with Baxter solution):

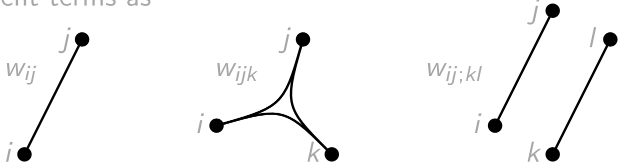


Robustness of $\mathrm{OSP}(1|2)$ symmetry for interacting forests

The set of $\{f_{ij}^{(\lambda)}\}_{1\leq i<j\leq n}$ generates all functions of scalar products $\{\vec{\sigma}_i \cdot \vec{\sigma}_j\}$ for n unit vectors in $\mathrm{RP}^{0|2}$, as an algebra of polynomials. So the most general function $\mathcal{S}(\bar{\psi},\psi)$ invariant under $\mathrm{OSP}(1|2)$ global rotation is of the form

$$\mathcal{S}(\bar{\psi},\psi) = \sum_{(ij)} w_{ij} f_{ij} + \sum_{(ijk)} w_{ijk} f_{ijk} + \cdots + \sum_{(ij;kl)} w_{ij;kl}\, f_{ij} f_{kl} + \cdots$$

Represent terms as



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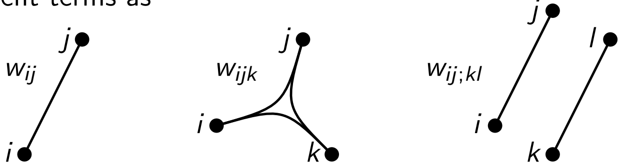

then

$$\int \mathcal{D}(\psi, \bar{\psi})\, e^{\lambda\bar{\psi}\psi + \mathcal{S}(\bar{\psi},\psi)} = \sum_{\substack{F \subseteq G \\ \text{hyperforests}}} \lambda^{K(F)}\, P(w; F)$$

with G a hypergraph with edges $(i_1 \cdots i_k)$ corresponding to k-uples such that some coefficient w is non-zero, and $P(w; F)$ is a polynomial in the w's whose k-uples appear as hyper-edges in F.

Even for the most general $\mathrm{OSP}(1|2)$-invariant action, restriction to cycle-free sub-(hyper)graphs, i.e. forests, appears as an algebraic consequence of symmetry, and even at the level of the Grassmann sub-algebra of f_{ij}'s, before integration.

Linear-space dimension of the polynomial algebra of f_{ij}'s

As $f_i = 1$ and $f_\varnothing = \lambda$, the most general monomial in the polynomial algebra generated by f_{ij}'s is labeled by a partition $\mathcal{C} = (C_1, \ldots, C_k)$ of $[n]$:

$$\mathcal{C} \in \Pi(n) : \qquad f_{\mathcal{C}} := f_{C_1} \cdots f_{C_k}$$

They must be a redundant basis for the linear space of global $\mathrm{OSP}(1|2)$-invariant functions, as $|\Pi(n)| = B_n \sim n!$, while the whole Grassmann Algebra has dimension only 4^n...

- Which dimension has the linear space?
- There is any natural non-redundant basis of $f_{\mathcal{C}}$'s?
- Which relations do generate the kernel?

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a few answers...

❶➽ The dimension of the linear space is $C_n = \frac{1}{n+1}\binom{2n}{n} \sim 4^n n^{-3/2}$, the n-th Catalan number;

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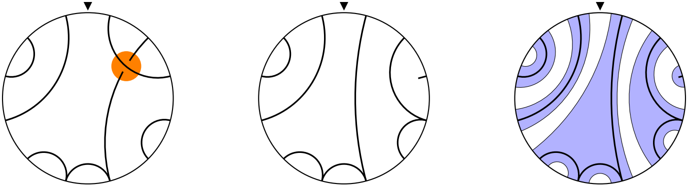

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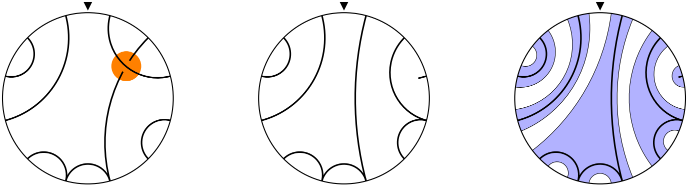

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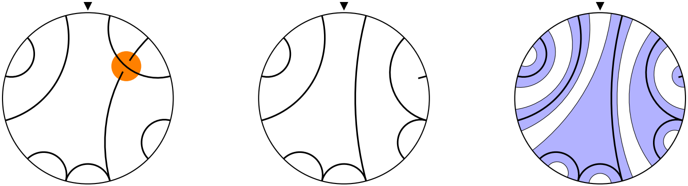

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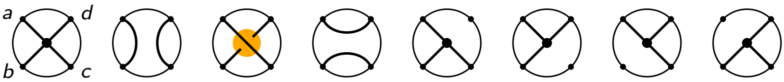

Can be used to recursively write a $f_{\mathcal{C}}$ with $\mathcal{C}$ crossing as a linear combination of $f_{\mathcal{C}'}$'s, with all $\mathcal{C}'$ non-crossing.

Consider Clifford Algebra. Other OSP(1|2)-invariant objects are:

$$p_i := \partial_i \bar{\partial}_i (1 + \lambda \bar{\psi}_i \psi_i) = \int \mathrm{d}\psi_i \mathrm{d}\bar{\psi}_i e^{\lambda \bar{\psi}_i \psi_i}$$

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$$p_i^2 = \lambda p_i \, ; \quad [p_i, p_j] = \underbrace{[p_i, f_{jk}] = 0}_{i \neq j,k} ; \quad (p_i f_A) = f_{A \setminus i} \quad \text{if } i \in A.$$

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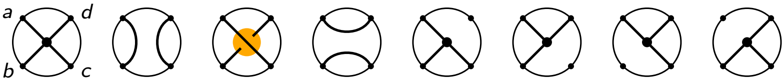

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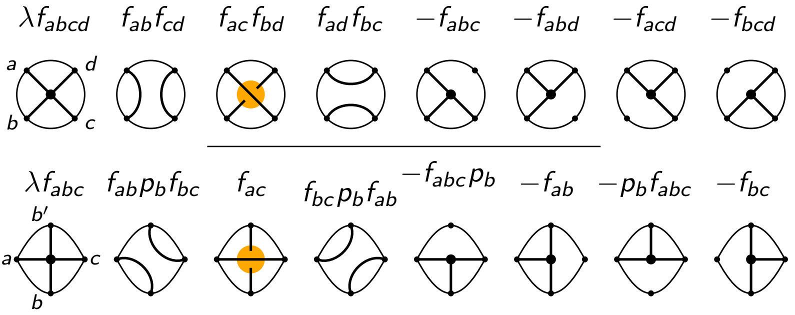
Clifford Algebra and $R_{ac}^{b} = 0$

With p_i's we get a three-point relation in Clifford Algebra: $R_{ac}^{b} = 0$.
It is an easy check that $R_{ac}^{b} f_{bd} = R_{abcd}$.

Compare the terms appearing in R_{abcd} and in R_{ac}^{b}:



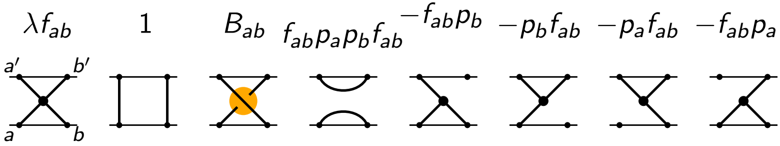
Exchange operator and $R^{ab} = 0$

Another interesting $\mathrm{OSP}(1|2)$-invariant in Clifford Algebra is the "exchange" operator

$$B_{ab} := \left(1 - (\bar\psi_a - \bar\psi_b)(\bar\partial_a - \bar\partial_b)\right)\left(1 - (\psi_a - \psi_b)(\partial_a - \partial_b)\right)$$

$$B_{ab} P(\bar\psi_a, \psi_a, \bar\partial_a, \partial_a, \bar\psi_b, \cdots) = P(\bar\psi_b, \psi_b, \bar\partial_b, \partial_b, \bar\psi_a, \cdots) B_{ab}$$

With B_{ab} we can build a two-point relation $R^{ab} = 0$:



and $R^{bc} f_{ab} f_{cd} = R_{abcd}$.

Comments on R_{abcd}, R^b_{ac} and R^{ab}

The three relations $R_{abcd} = 0$, $R^b_{ac} = 0$ and $R^{ab} = 0$ are different forms of a single "fundamental" $\mathrm{OSP}(1|2)$ relation, which, at a level of diagrams, relates the only 4-point crossing partition to the other seven 2-block non-crossing ones.

They all involve eight fermions, and have eight terms, four positive and four negative.

A version of $R_{abcd} = 0$ for $\lambda = 0$ (thus with seven terms) was also in [Kenyon-Wilson, 2006].

An important completeness proof for the set of related observables is in [Ko-Smolinsky, 1991] and [Di Francesco-Golinelli-Guittier, 1996]. It is at $\lambda = 0$, but extends immediately from block-triangularity of the T-L Gram matrix.

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Recognizing even/odd Temperley-Lieb

We have seen some algebraic rules for f_{ij}'s and p_i's:

$$f_{i\,i+1}^2 = 0\,; \qquad [f_{i\,i+1}, f_{j\,j+1}] = 0\,; \qquad f_{i\,i\pm1}\, p_i\, f_{i\,i\pm1} = f_{i\,i\pm1}\,;$$
$$p_i^2 = \lambda p_i\,; \qquad [p_i, p_j] = 0\,; \qquad p_i\, f_{i\,i\pm1}\, p_i = p_i\,;$$
$$[p_i, f_{j\,j+1}] = 0 \qquad \text{if } j \neq i, i-1.$$

. . . look similar to Temperley-Lieb Algebra [1971],

$$e_i^2 = \lambda e_i\,; \qquad e_i e_{i\pm1} e_i = e_i\,; \qquad [e_i, e_j] = 0 \qquad \text{if } |i-j| \geq 2.$$

by identifying $e_{2i} = p_i$ and $e_{2i+1} = f_{i\,i+1}$, but $e_i^2 = \lambda_{\mathrm{parity}(i)}$ with $\lambda_{\mathrm{even}} = \lambda$ and $\lambda_{\mathrm{odd}} = 0$.

...comments on Temperley-Lieb

Indeed, T-L describes the transfer matrix of the Random Cluster Model, on planar graphs, at $\lambda = \rho = \sqrt{q}$, and allows to "integrate" the model, say on the square lattice, on Baxter critical parabola.

Instead, this algebra describes the line $\lambda > 0$, $\rho = 0$ corresponding to spanning forests.

As a result of $\rho = 0$, we do not need to deal with $L(H)$, and through $R_{abcd} = 0$ we can build a transfer matrix on $\mathrm{NC}(n)$ also for non-planar graphs.

This is related to a modification of Martin-Saleur Partition Algebra [1993], in which cycles are forbidden.

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Conclusions

- We put in correspondence the $\mathrm{OSP}(1|2)$ non-linear σ-model with Spanning Forests, i.e. Potts Model for $q \to 0$ and $v_{ij}/q = w_{ij}$ fixed.
- Even the most general $\mathrm{OSP}(1|2)$-invariant action admits a combinatorial expansion in terms of sub-hyperforests only (no cycles in subgraphs). The symmetry is a precious guideline when building proofs.
- Study of linear independence in the symmetric subalgebra led to a 'fundamental' relation $R_{abcd} = 0$, generalizing the one for spanning trees, i.e. free-fermion theory.
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  Probabilistic understanding of Ward identities.
- Raise to a $\mathrm{OSP}(1|2m)$–Spanning-Forest relation.
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- You can add a "vector field", and count unicyclics
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- Relation between Spanning Forests and Abelian Sandpile
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