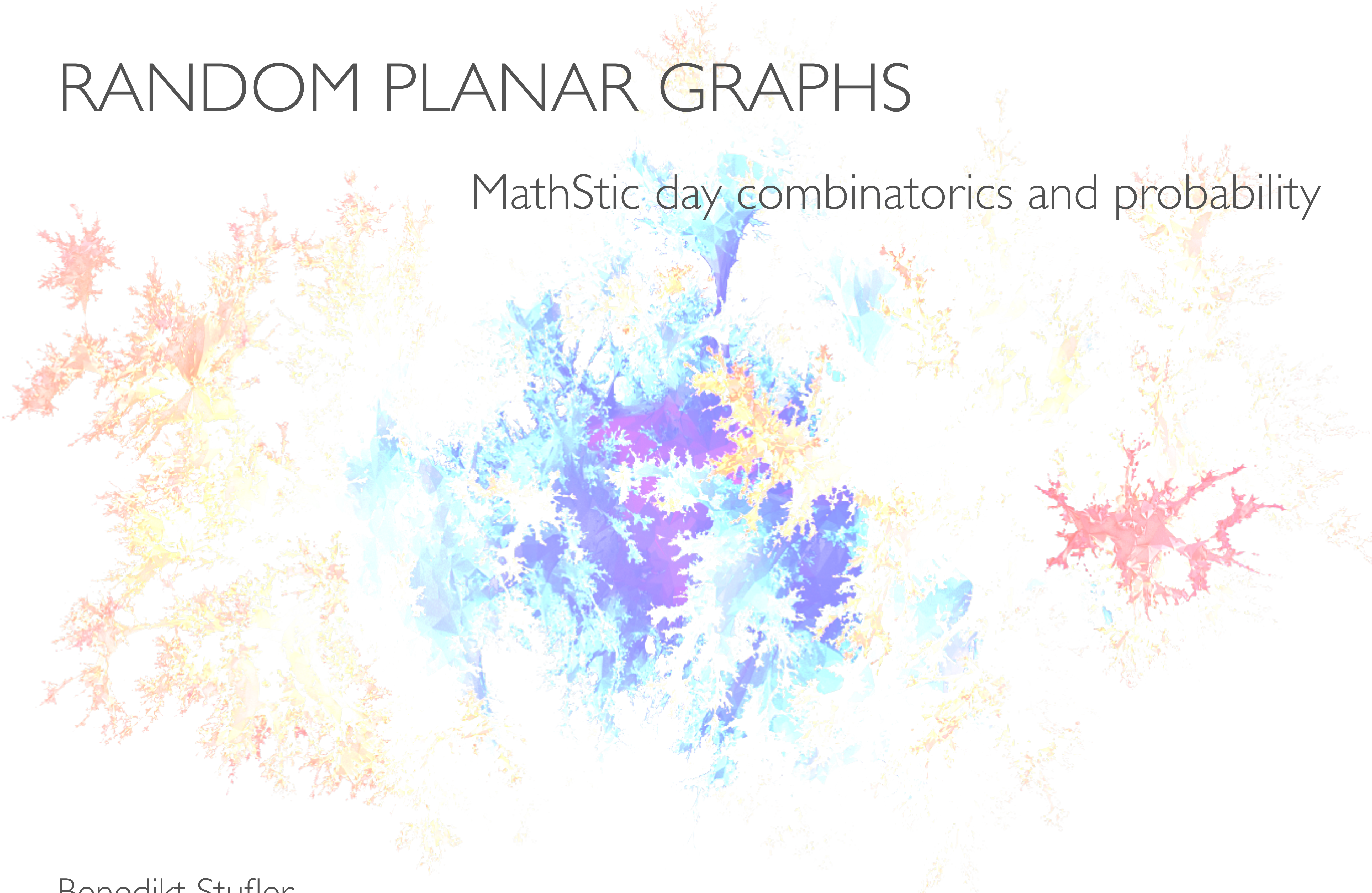
RANDOM PLANAR GRAPHS

MathStic day combinatorics and probability



Benedikt Stufler
Technische Universität Wien
dmg.tuwien.ac.at/stufler

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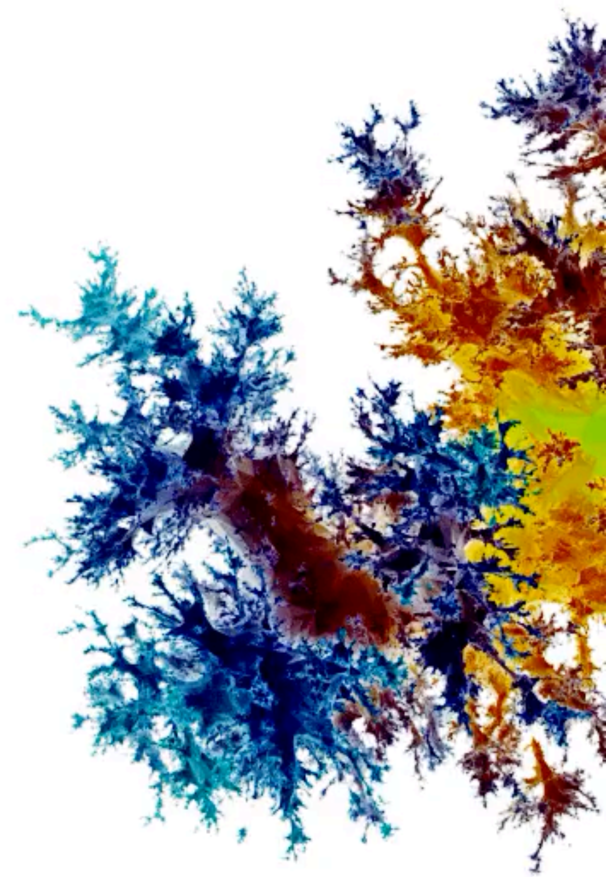

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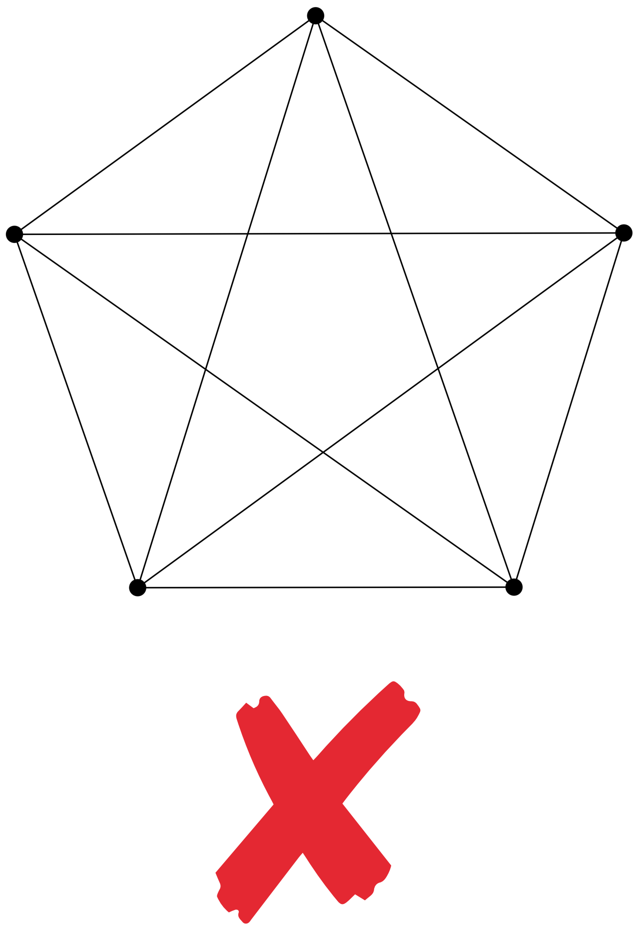
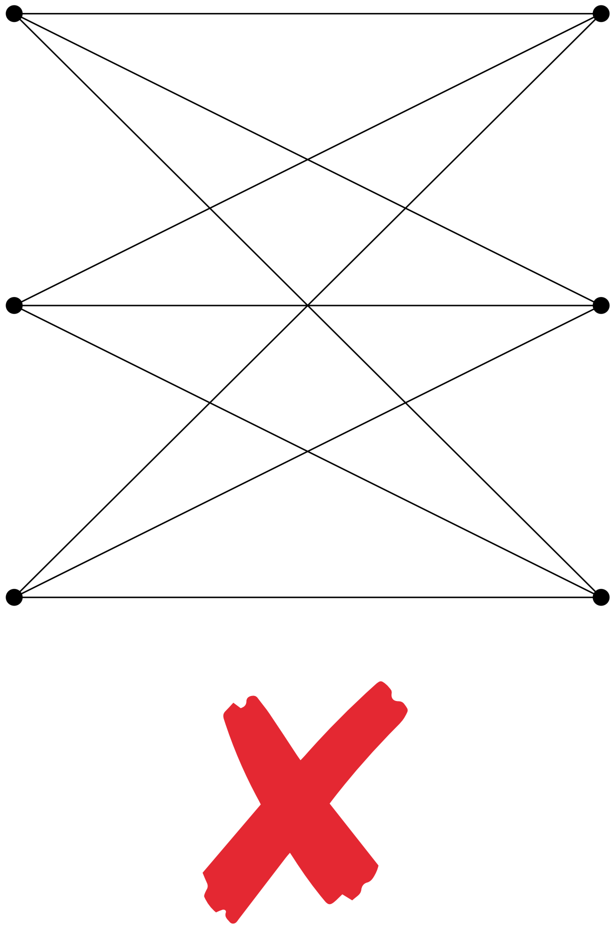
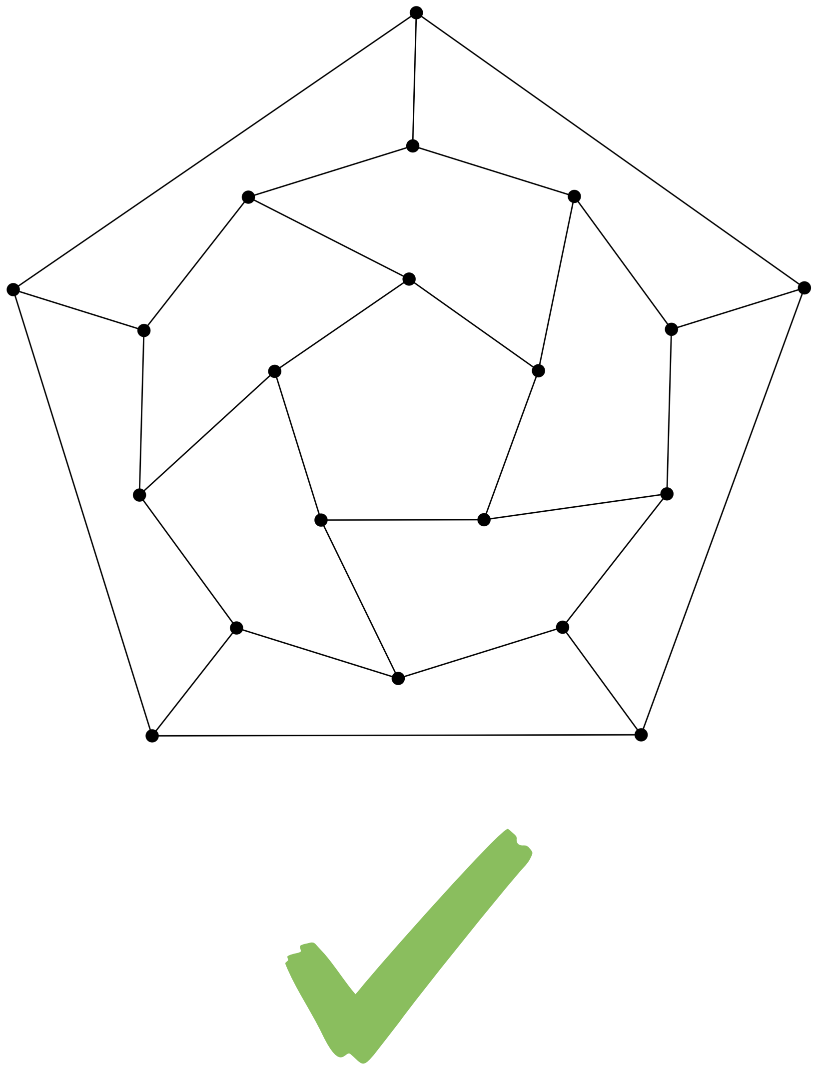
DEFINITION OF PLANAR GRAPHS







Question: How many planar graphs with n vertices are there?

- Giménez, Noy (2009): The number g_n of planar graphs with vertices labelled from 1 to n satisfies
$$g_n \sim g n^{-7/2} \rho_G^{-n} n!$$
for constants $g, \rho_G > 0$.

- The asymptotic behaviour of the number $\tilde{g}_n$ of unlabelled planar graphs is unknown.

NUMBER OF LABELLED PLANAR GRAPHS

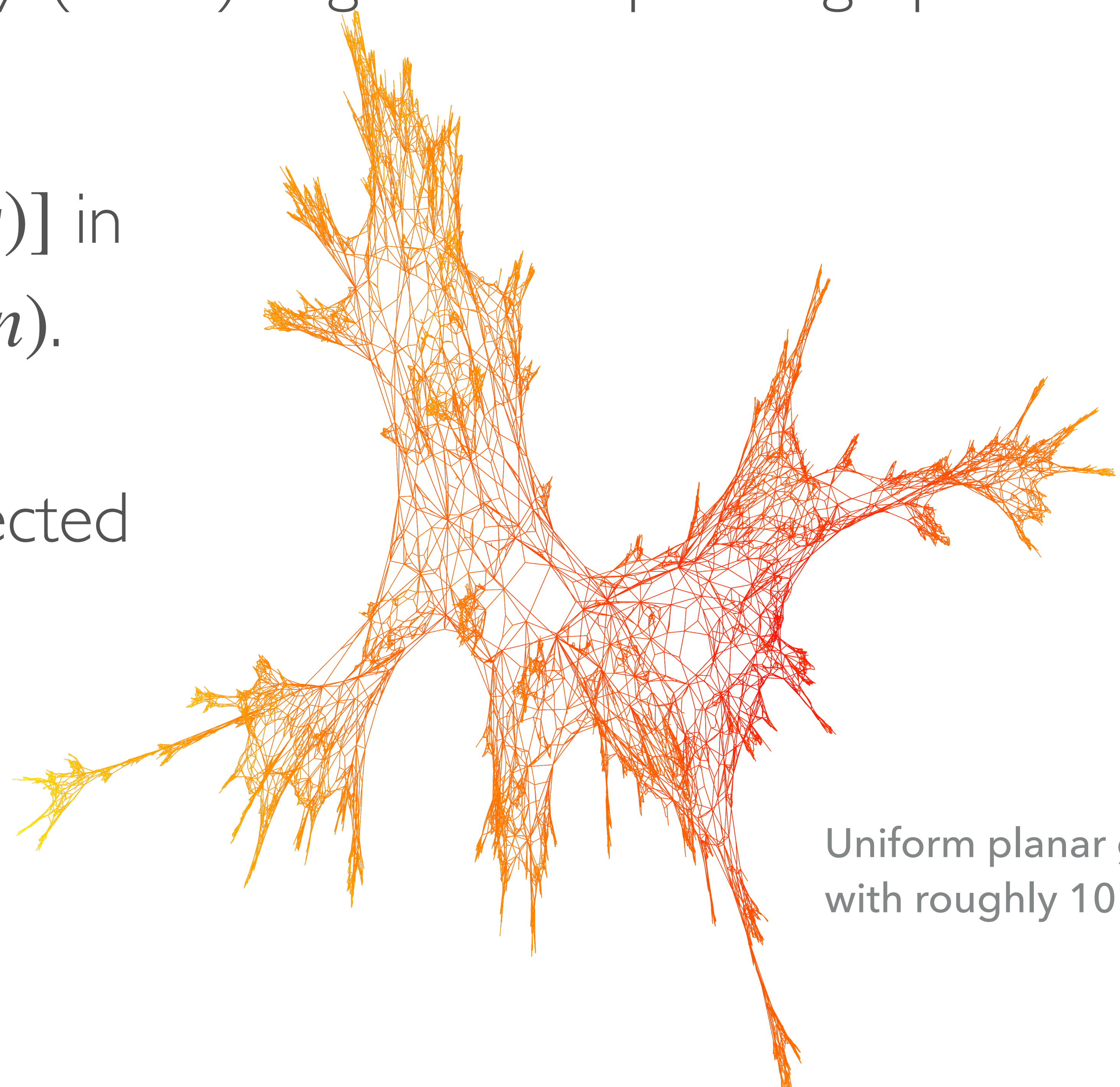
- Denise, Vasconcellos, Welsh (1996): $g_n \leq n!(75.8)^{n+o(n)}$, $(g_n/n!)^{1/n}$ converges
- Bender, Gao, Wormald (2002): $g_n \geq n!(26.1)^{n+o(n)}$, $b_n \sim bn^{-7/2}\rho_B^{-n}n!$
- Osthus, Prömel, Taraz (2003): $g_n \leq n!(37.3)^{n+o(n)}$
- (Further estimates of growth constants... sorry for omitting those)
- Giménez, Noy (2009): $g_n \sim gn^{-7/2}\rho_G^{-n}n!$ by analytic integration
- Chapuy, Fusy, Kang, Shoilekova (2008): "combinatorial integration", purely combinatorial approach to get analytic specification by Giménez and Noy.
- Stufler (2019+) : recover $g_n \sim gn^{-7/2}\rho_G^{-n}n!$ without integration, random walk approach, uses large deviation results by Denisov, Dieker, Shneer (2008)

Question: What are the properties of a uniform random planar graph $\mathscr{P}_n$ with n labelled vertices?

SIMULATION OF RANDOM PLANAR GRAPHS

Fastest known sampling algorithm was invented and implemented by Fusy (2008). It generates planar graphs...

- with size in $[n(1-\epsilon), n(1+\epsilon)]$ in expected time $O(n)$.
- with size n in expected time $O(n^2)$.



Uniform planar graph with roughly 10k vertices

GIANT CONNECTED COMPONENT

- McDiarmid (2008): Giant connected component, remainder admits a finite Boltzmann-Poisson Random Graph as limit
- McDiarmid (2009): Universality: in general, random graphs from proper addable minor-closed classes of graphs have a remainder with a Boltzmann-Poisson Random Graph as limit.
- Stufler (2018): Universality: Small block-stable classes of graphs. (If such a class fails to be small, the random graph is connected with high probability. See for example Stufler (2020).)

MAXIMUM DEGREE

- McDiarmid and Reed (2008): The maximum degree Δ_n satisfies whp $c_1 \log n < \Delta_n < c_2 \log n$ for suitable constants $0 < c_1 < c_2$.

- Drmota, Giménez, Noy, Panagiotou, Steger (2012): whp $|\Delta_n - c \log n| = O(\log \log n)$ for a constant $c > 0$.

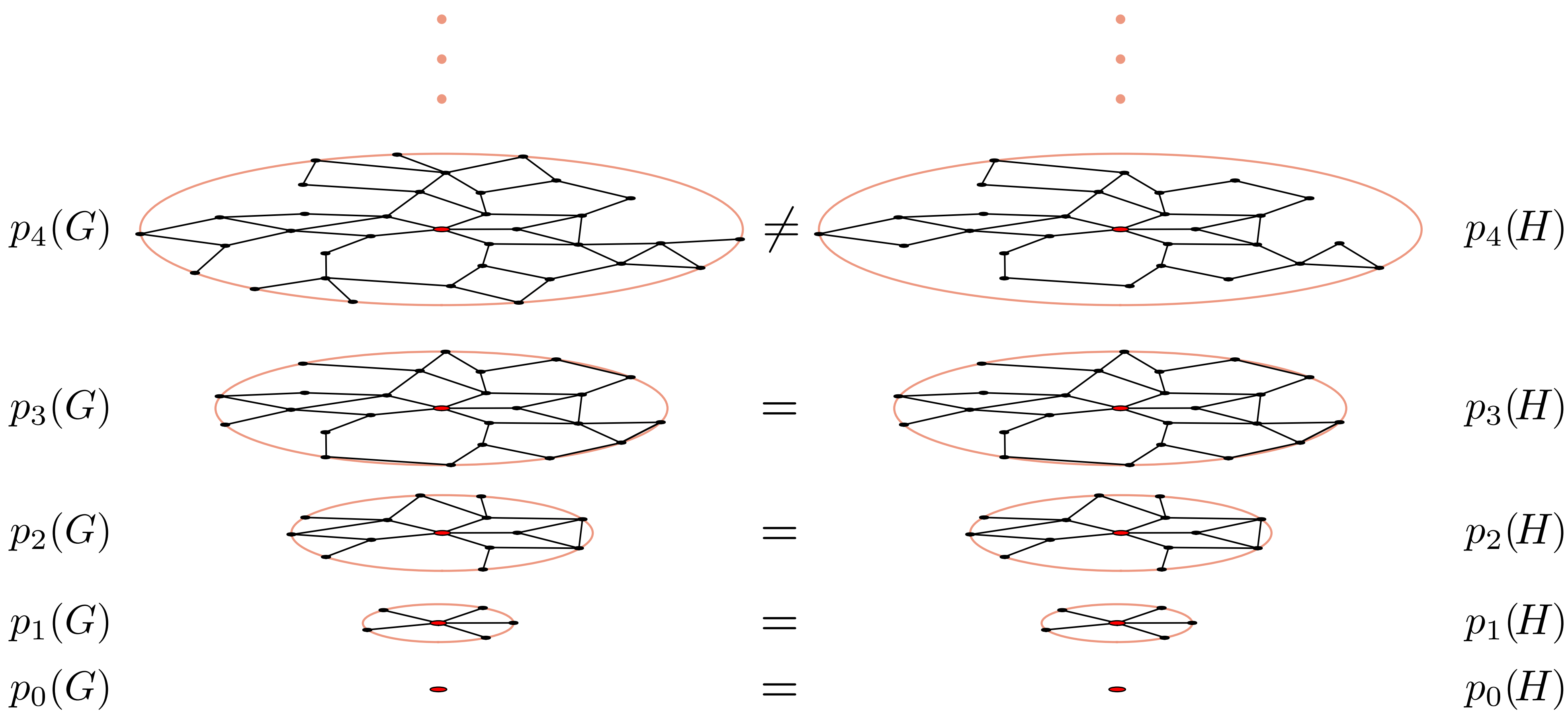
DEGREE DISTRIBUTION

- McDiarmid, Steger, Welsh (2004): Number $d_k(n)$ of vertices of a degree k is $\Theta(n)$
- Drmota, Giménez, Noy (2011): Degree of a random vertex has a limit distribution
- Panagiotou, Steger (2011): Recovered degree distribution via different methods
- Stufler (2019+): Degree of a random vertex converges to the degree of the root of a new Uniform Infinite Planar Graph (UIPG)

LOCAL DISTANCE

$\mathfrak{M}$ = collection of vertex-rooted locally finite unlabelled graphs

$p_k : \mathfrak{M} \to \mathfrak{M}$ projection to k-neighbourhood of the root vertex



$$d_{\mathrm{loc}}(G,H) = \frac{1}{1+\sup\{k \in \mathbb{N}_0 \mid p_k(G)=p_k(H)\}}$$

$(\mathfrak{M}, d_{\mathrm{loc}})$ is a Polish space

LOCAL CONVERGENCE: UIPG

Annealed Version (Stufler 2019+):

The uniform n-vertex planar graph $\mathscr{P}_n$ rooted at a uniformly selected vertex v_n admits a distributional limit $\hat{\mathscr{P}}$.

We call $\hat{\mathscr{P}}$ the Uniform Infinite Planar Graph (UIPG).

LOCAL CONVERGENCE: UIPG

Annealed Version (Stufler 2019+):

The uniform n-vertex planar graph $\mathscr{P}_n$ rooted at a uniformly selected vertex v_n admits a distributional limit $\hat{\mathscr{P}}$.

We call $\hat{\mathscr{P}}$ the Uniform Infinite Planar Graph (UIPG).

Quenched Version (Stufler 2019+):

The regular conditional law $\mathscr{L}((\mathscr{P}_n, v_n) \mid \mathscr{P}_n)$ satisfies

$$\mathscr{L}((\mathscr{P}_n, v_n) \mid \mathscr{P}_n) \xrightarrow{p} \mathscr{L}(\hat{\mathscr{P}}).$$

THE UIPG IS ALMOST SURELY RECURRENT

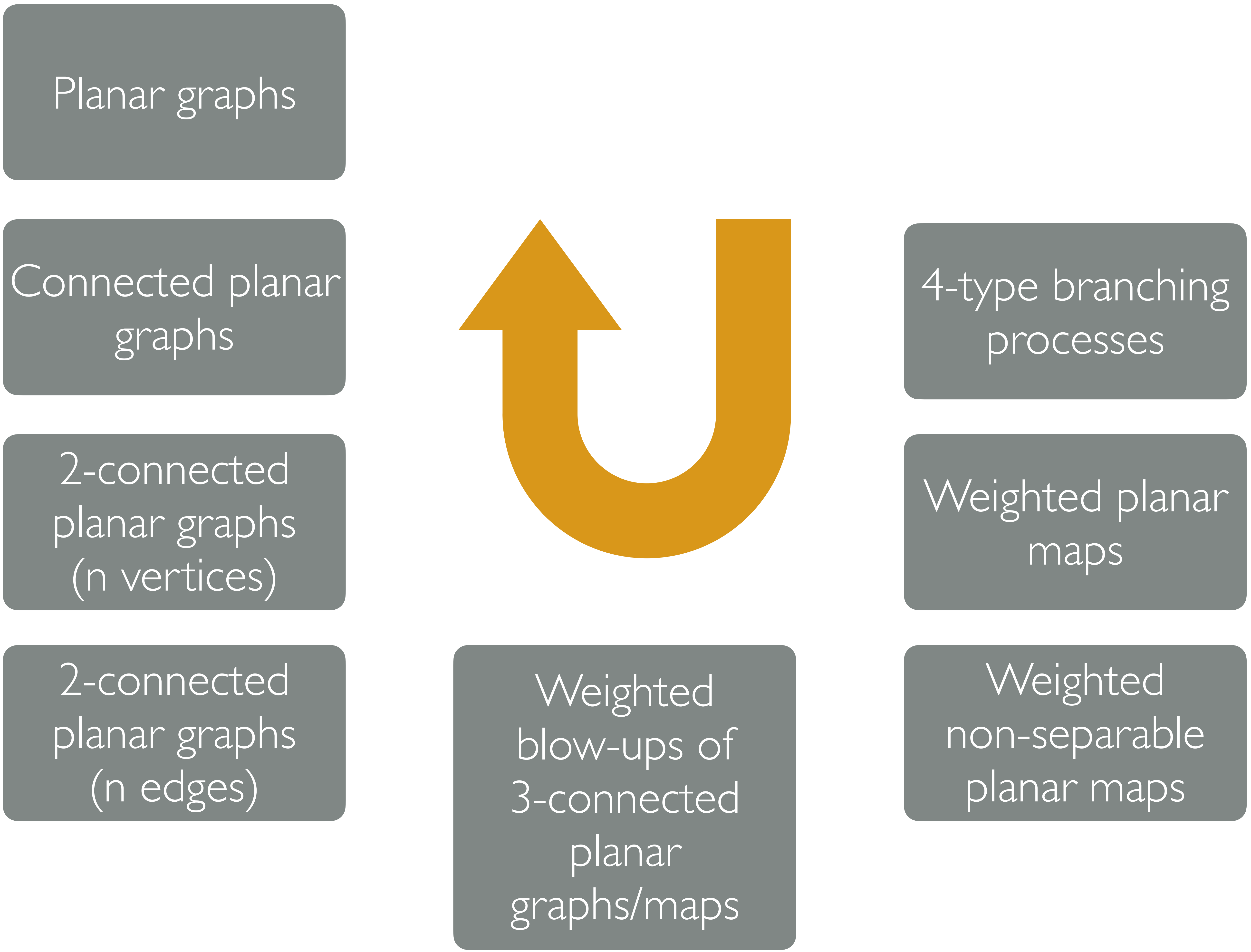
- (Benjamini and Schramm, 2001) Let $M < \infty$. **If** a random locally finite rooted graph G is a distributional limit of rooted random unbiased finite planar graphs (not necessarily uniform) with degrees bounded by M, **then** with probability one G is recurrent.

- (Gurel-Gurevich and Nachmias, 2013) Instead of a uniform bound on the degrees, it suffices to assume that degree of the root of G has an exponential tail.

- Consequence: the UIPG is almost surely recurrent

NON-EXHAUSTIVE LIST OF MODELS WITH LOCAL LIMITS

- Kesten's tree: Simply generated trees (Kennedy, 1975)
- UIPT: Planar Triangulations (Angel, Schramm 2003)
- UIPQ: Planar Quadrangulations (Krikun 2005)
- UIPM: Planar Maps (Ménard, Nolin 2013)
- UI3PM: 3-connected Planar Maps (Addario-Berry 2014)
- IBPM: Boltzmann Maps (Björnberg, Stefánsson 2014, Stephenson 2018)
- PSHT: Triangulations with a high genus (Budzinsky, Louf 2020)
- UIPG, UI2PG, UI2PM: Planar Graphs (S. 2019+)


Planar graphs
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Weighted non-separable planar maps
Weighted planar maps
4-type branching processes

LOCAL CONVERGENCE: UI2PM

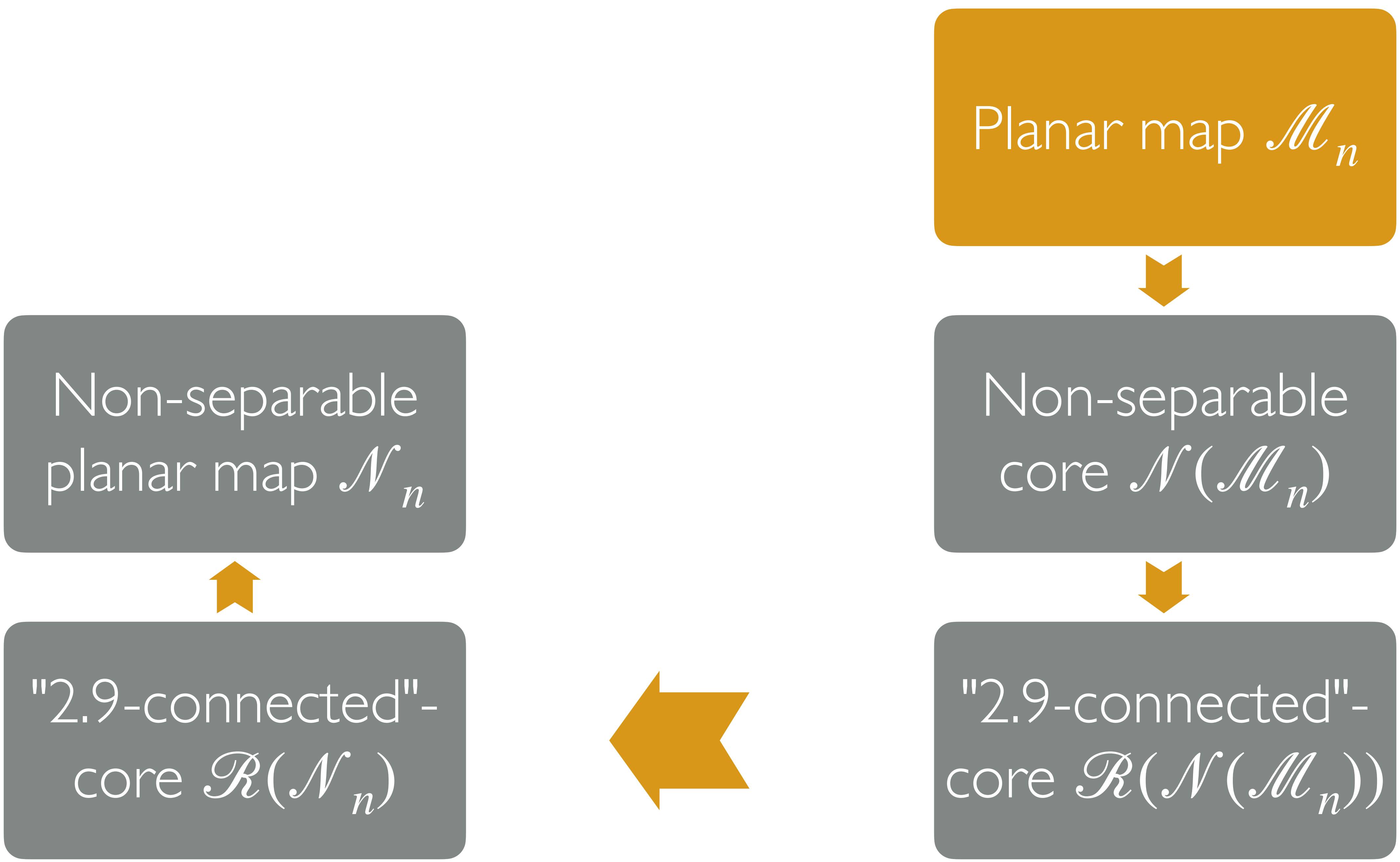
Non-separable Maps (Stufler 2019+):

The uniform n-edge 2-connected (= non-separable) planar map $\mathcal{N}_n$ rooted at a uniformly selected corner c_n admits a novel Uniform Infinite 2-connected Planar Map (UI2PM) $\hat{\mathcal{N}}$ as quenched local limit

$$\mathcal{L}((\mathcal{N}_n, c_n) \mid \mathcal{N}_n) \xrightarrow{p} \mathcal{L}(\hat{\mathcal{N}}) \,.$$

NEW NON-BIJECTIVE PROOF METHOD

"Two steps down, one step up"



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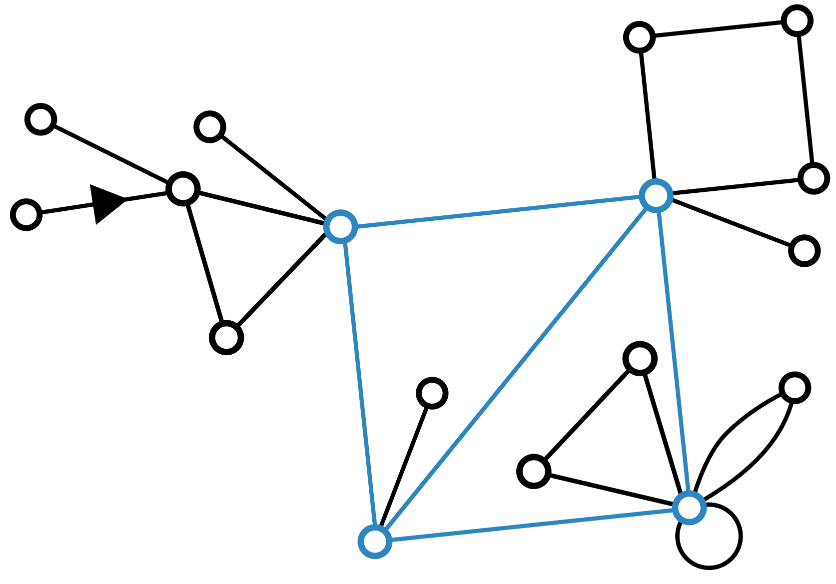
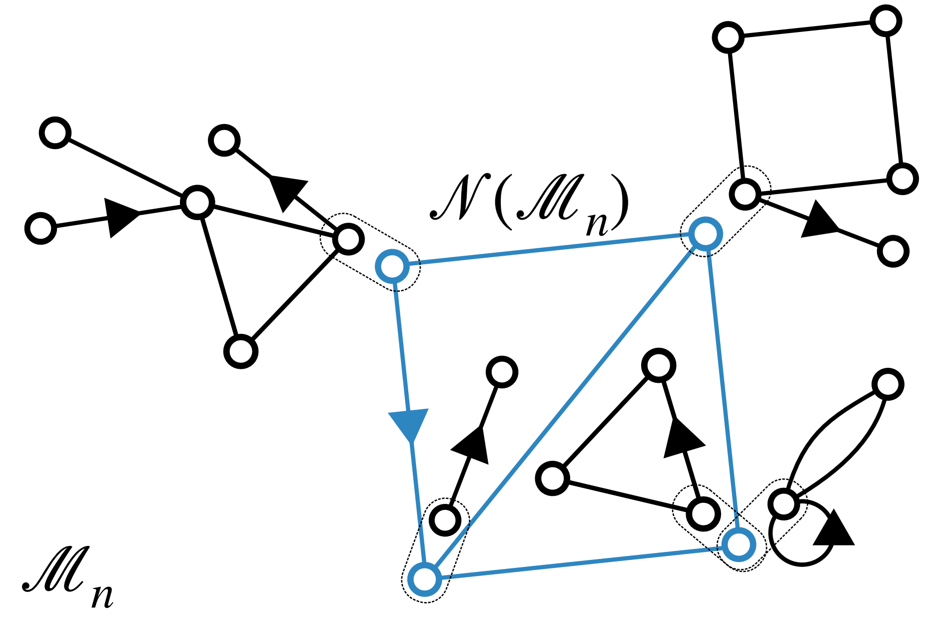
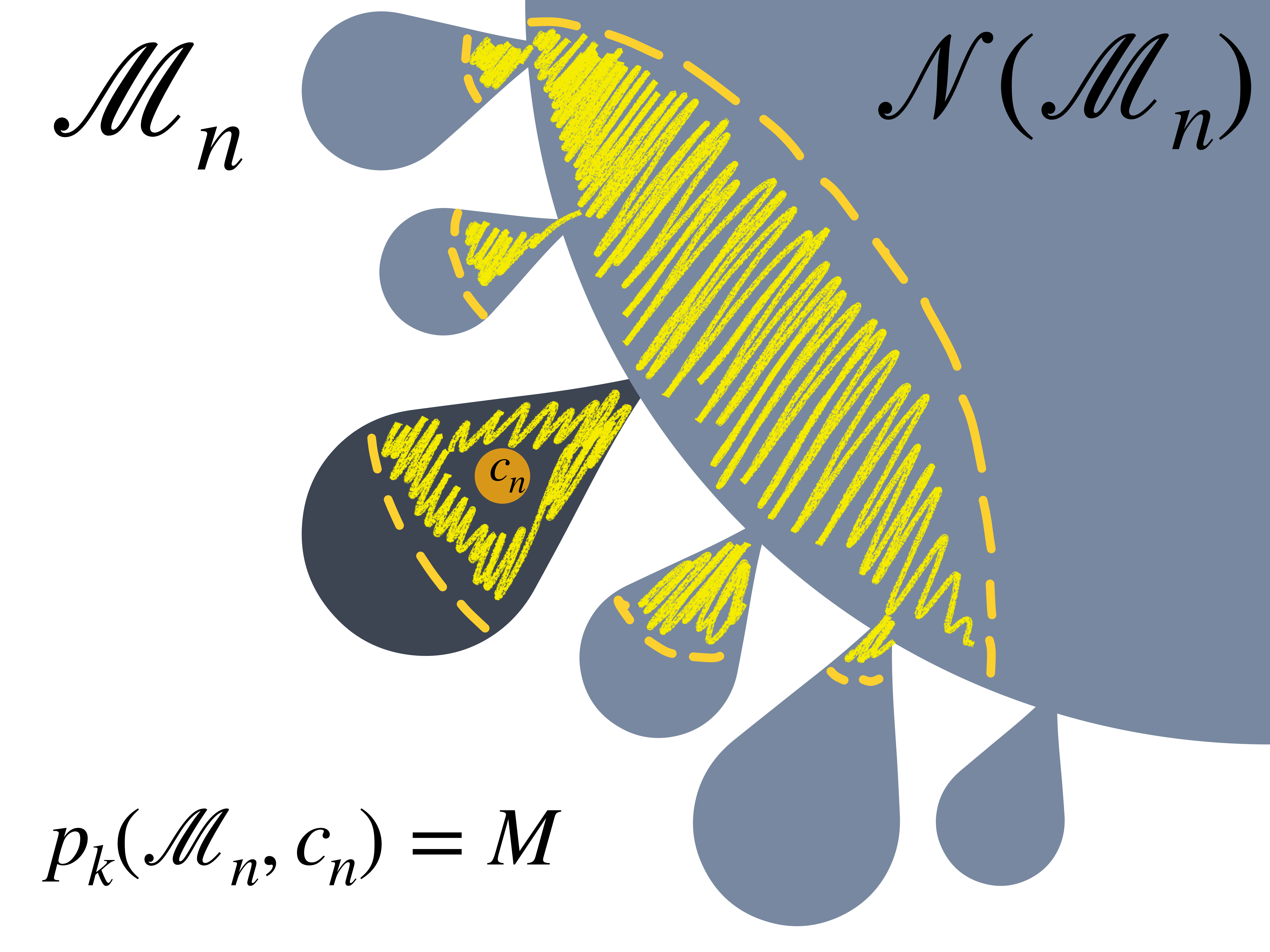
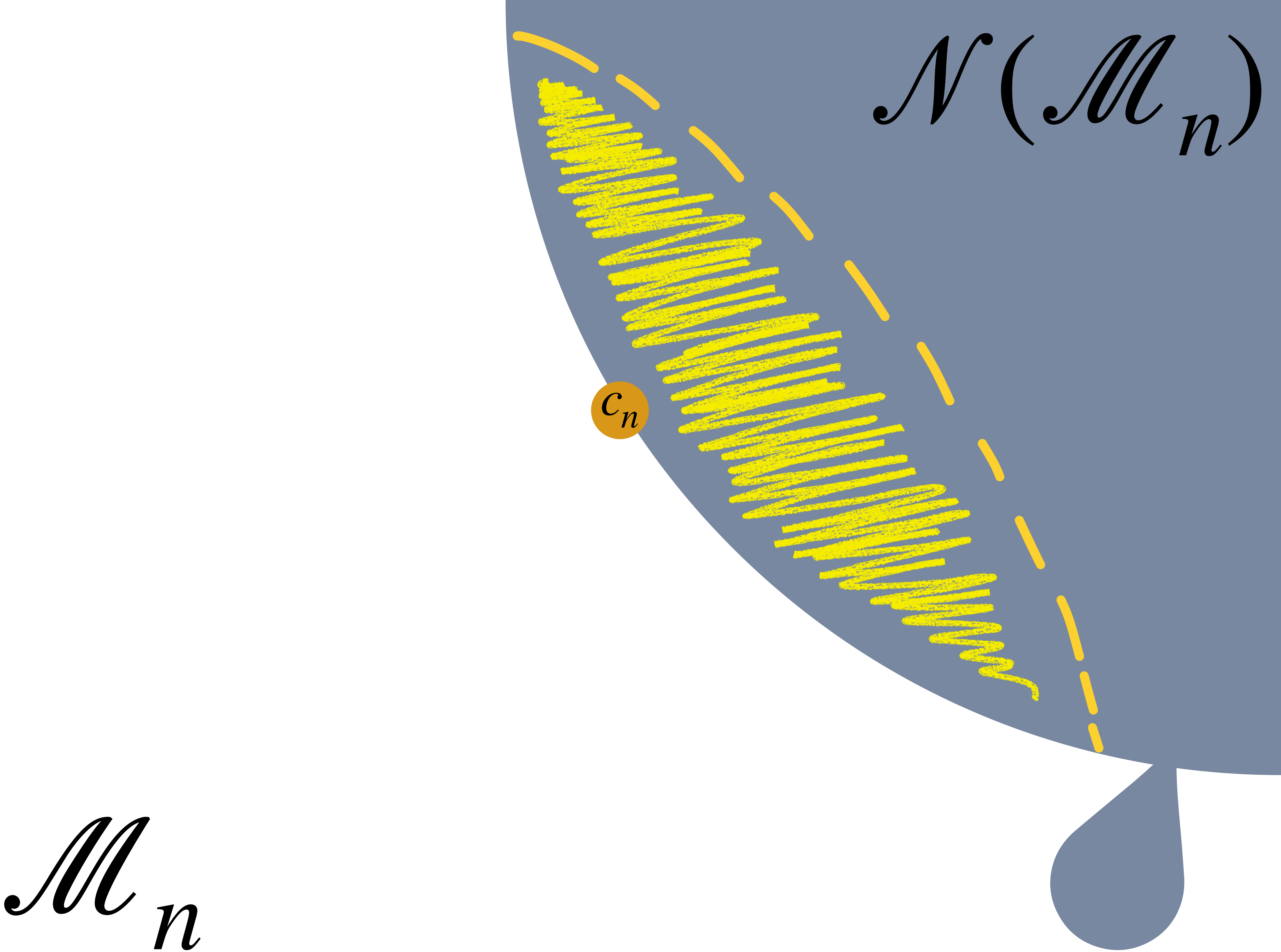




- $\mathcal{M}_n$ consists of 2-con. core $\mathcal{N}(\mathcal{M}_n)$ and components $(\mathbf{M}_i(\mathcal{M}_n))_{1 \le i \le |\mathcal{N}(\mathcal{M}_n)|}$
- *For the purpose of proving local convergence*, we may pretend that the components are i.i.d. copies of a Boltzmann map
- Waiting time paradox: the component containing a uniformly selected corner c_n follows a *size-biased distribution*



$\mathscr{M}_n$
$\mathscr{N}(\mathscr{M}_n)$
c_n
$p_k(\mathscr{M}_n, c_n) = M$



$\mathcal{N}(\mathcal{M}_n)$
c_n
$\mathcal{M}_n$

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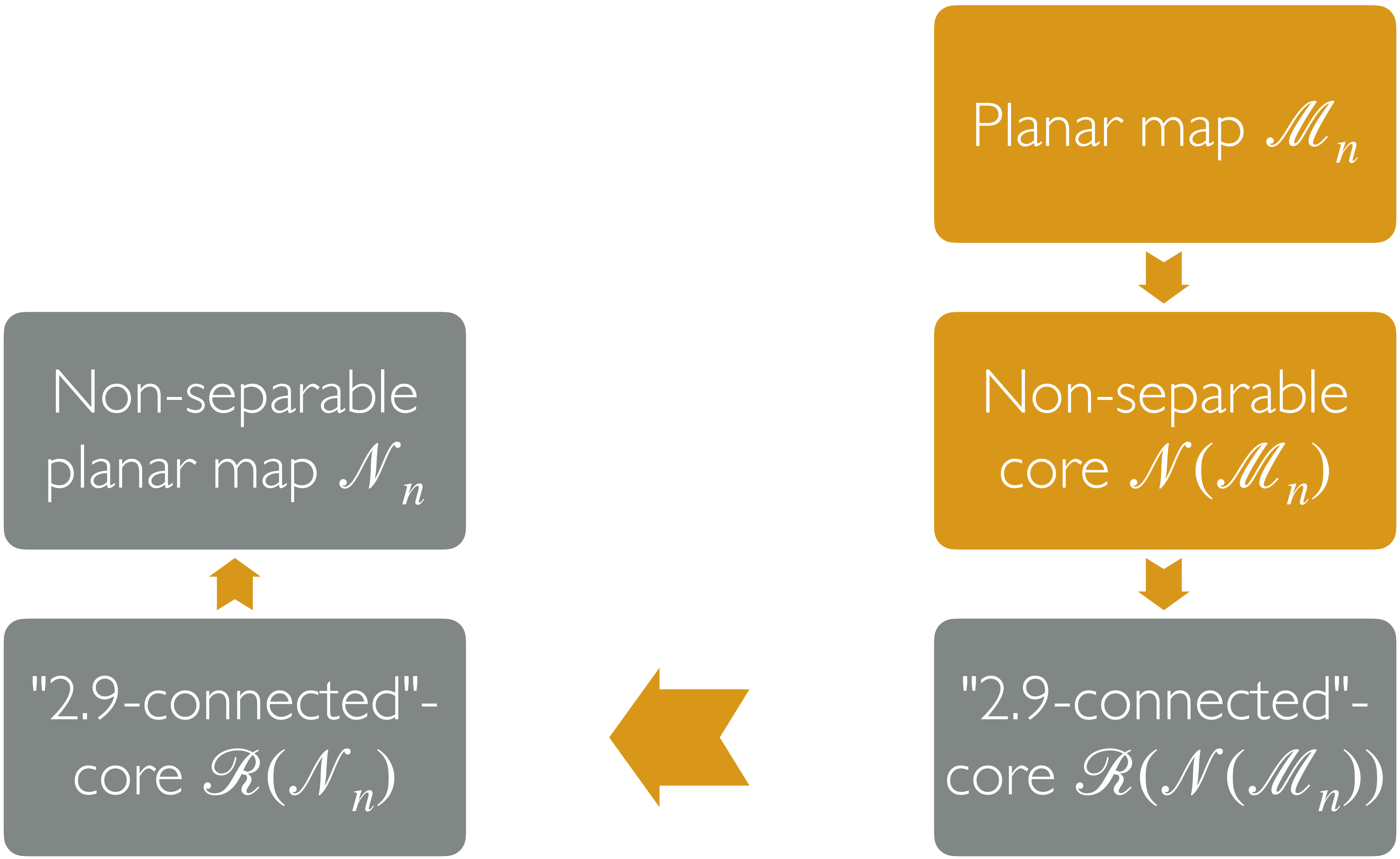

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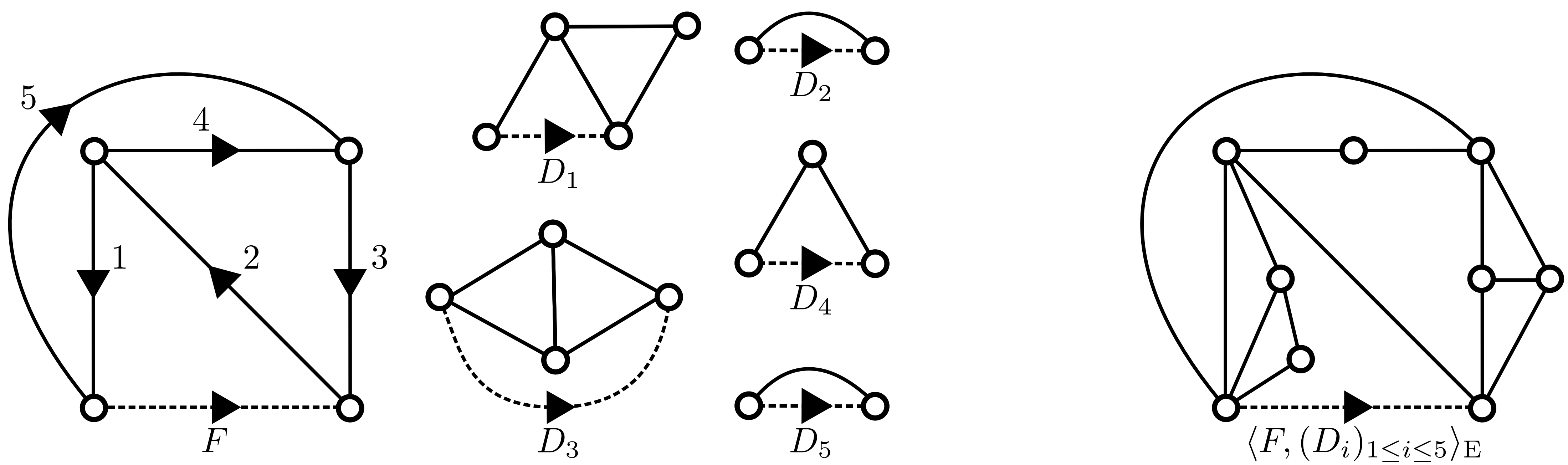
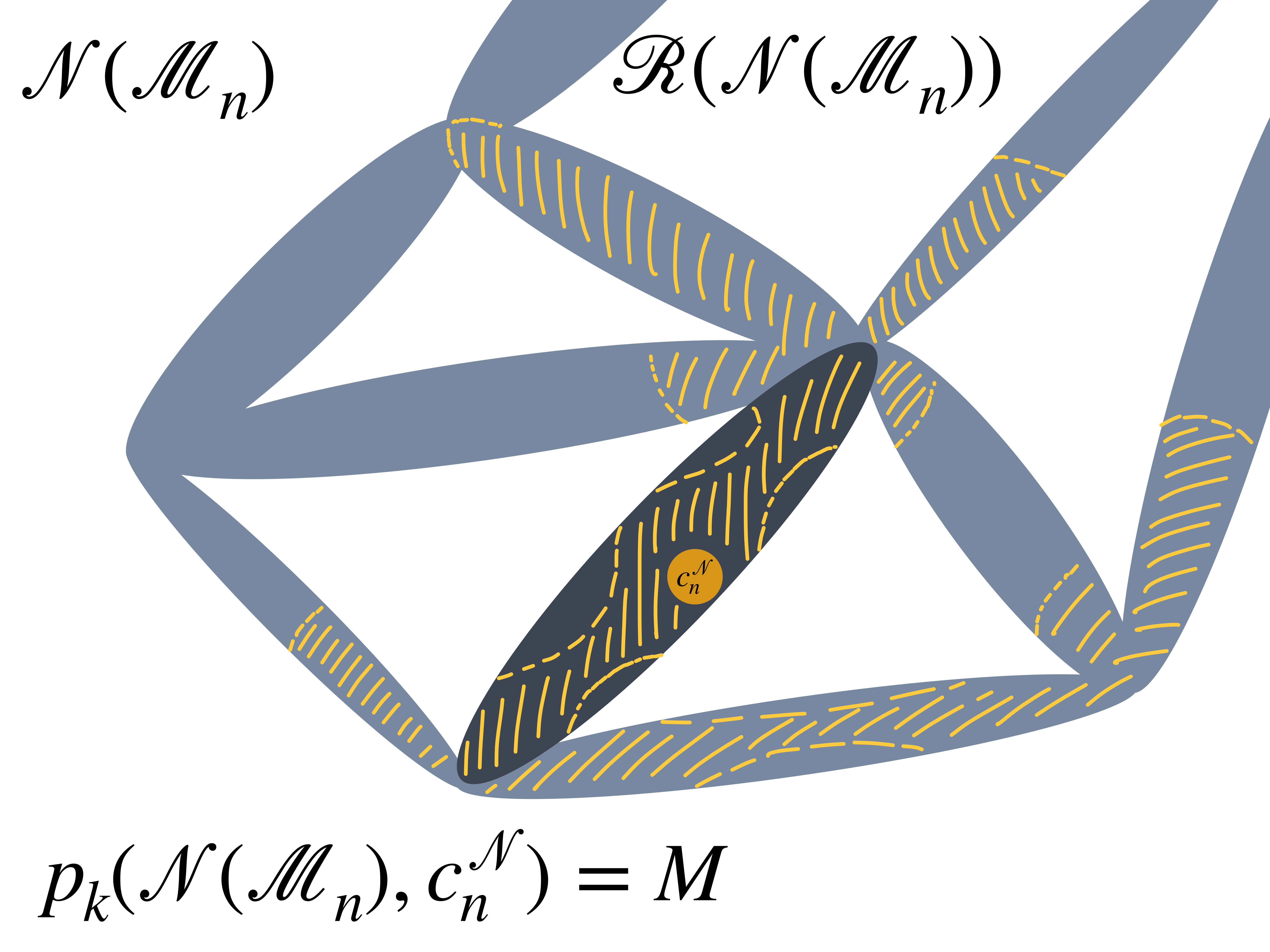
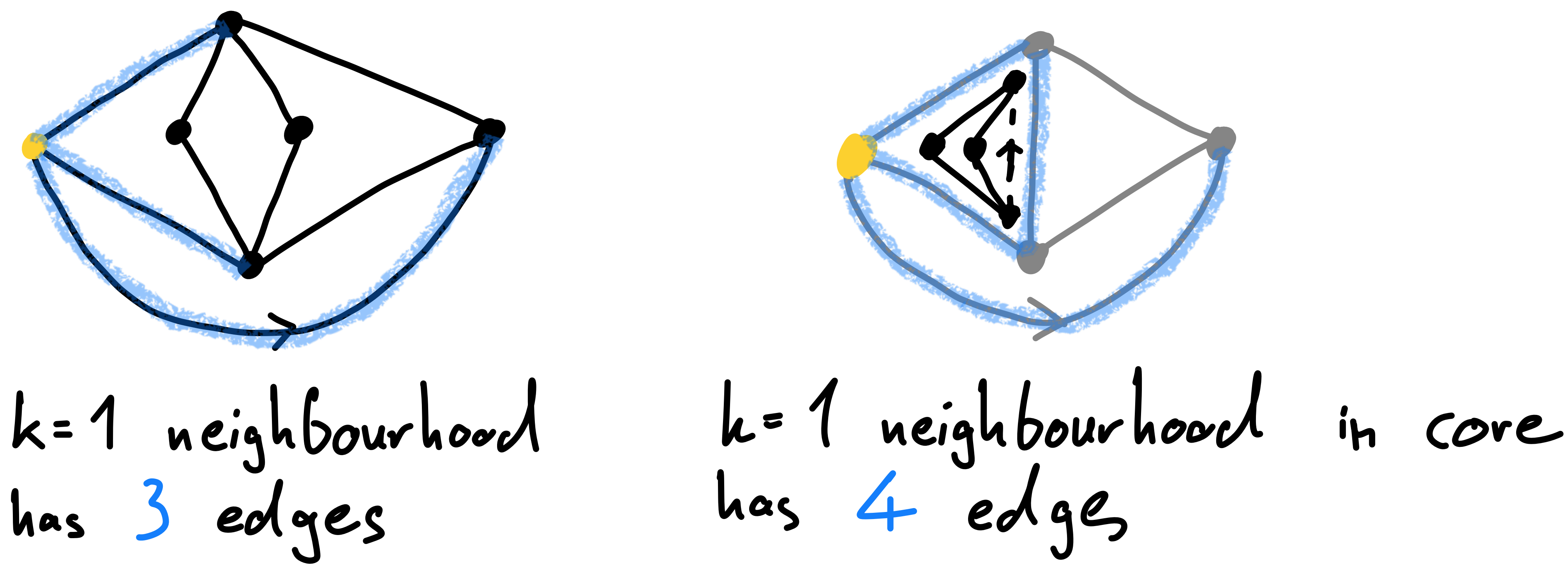


- $\mathscr{N}(\mathscr{M}_n)$ consists of 2.9-con. core $\mathscr{R}(\mathscr{N}(\mathscr{M}_n))$ and components that substitude its edges
- *For the purpose of proving local convergence*, we may pretend that the components are i.i.d. copies of a Boltzmann map
- Waiting time paradox: the component containing a uniformly selected corner c_n follows a *size-biased distribution*

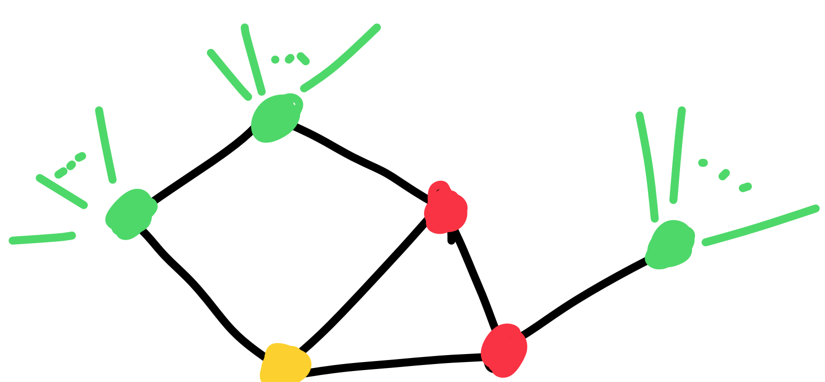

$\mathcal{N}(\mathcal{M}_n)$
$\mathcal{R}(\mathcal{N}(\mathcal{M}_n))$
$c_n^{\mathcal{N}}$
$p_k(\mathcal{N}(\mathcal{M}_n), c_n^{\mathcal{N}}) = M$

Problem: k-neighbourhood of core structure $\mathscr{R}(\mathscr{N}(\mathscr{M}_n))$ could have more edges than k-neighbourhood of $\mathscr{N}(\mathscr{M}_n)$



Induction does *not* work for *neighbourhoods*

Solution: Use *communities* instead of neighbourhoods



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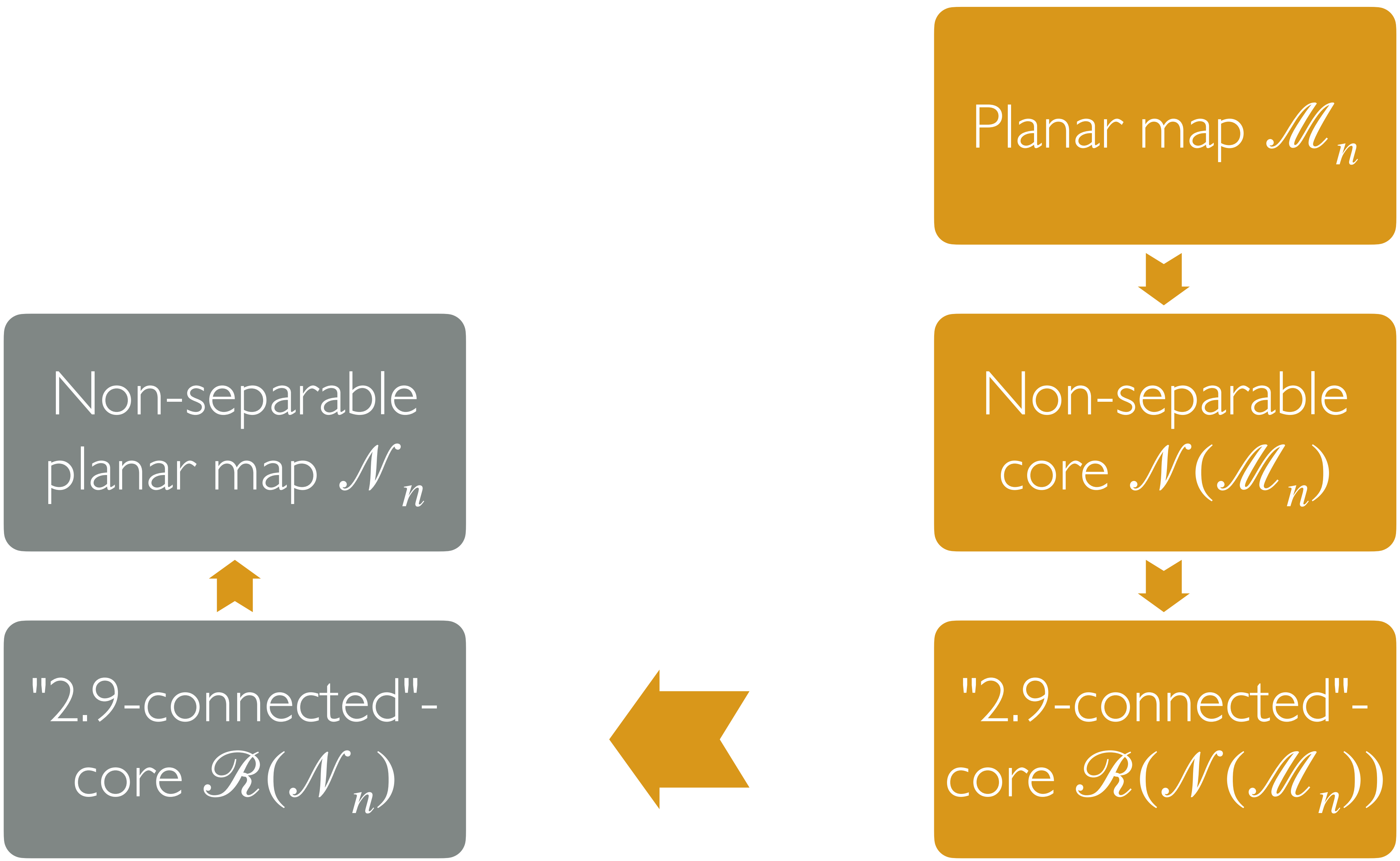

NEW NON-BIJECTIVE PROOF METHOD

- $\mathscr{R}(\mathscr{N}(\mathscr{M}_{3n}))$ and $\mathscr{R}(\mathscr{N}_n)$ are distributed like mixtures $\mathscr{R}_{X_n}$ and $\mathscr{R}_{Y_n}$.

- There are $\mu, a, b > 0$, h density of a $3/2$-stable law, such that uniformly for $\ell \in \mathbb{N}$

$$\mathbb{P}(X_n = \ell) = \frac{1}{an^{2/3}}\left(h\left(\frac{\mu n - \ell}{an^{2/3}}\right) + o(1)\right)$$

$$\mathbb{P}(Y_n = \ell) = \frac{1}{bn^{2/3}}\left(h\left(\frac{\mu n - \ell}{bn^{2/3}}\right) + o(1)\right)$$

- For any $\epsilon > 0$ there exists $M, c, C > 0$ such that $I_n := [\mu n - Mn^{2/3}, \mu n + Mn^{2/3}]$ satisfies for all large enough n

$$\mathbb{P}(X_n \notin I_n), \mathbb{P}(Y_n \notin I_n) < \epsilon$$

and uniformly for $\ell \in I_n$

$$c < \frac{\mathbb{P}(X_n = \ell)}{\mathbb{P}(Y_n = \ell)} < C$$

NEW NON-BIJECTIVE PROOF METHOD

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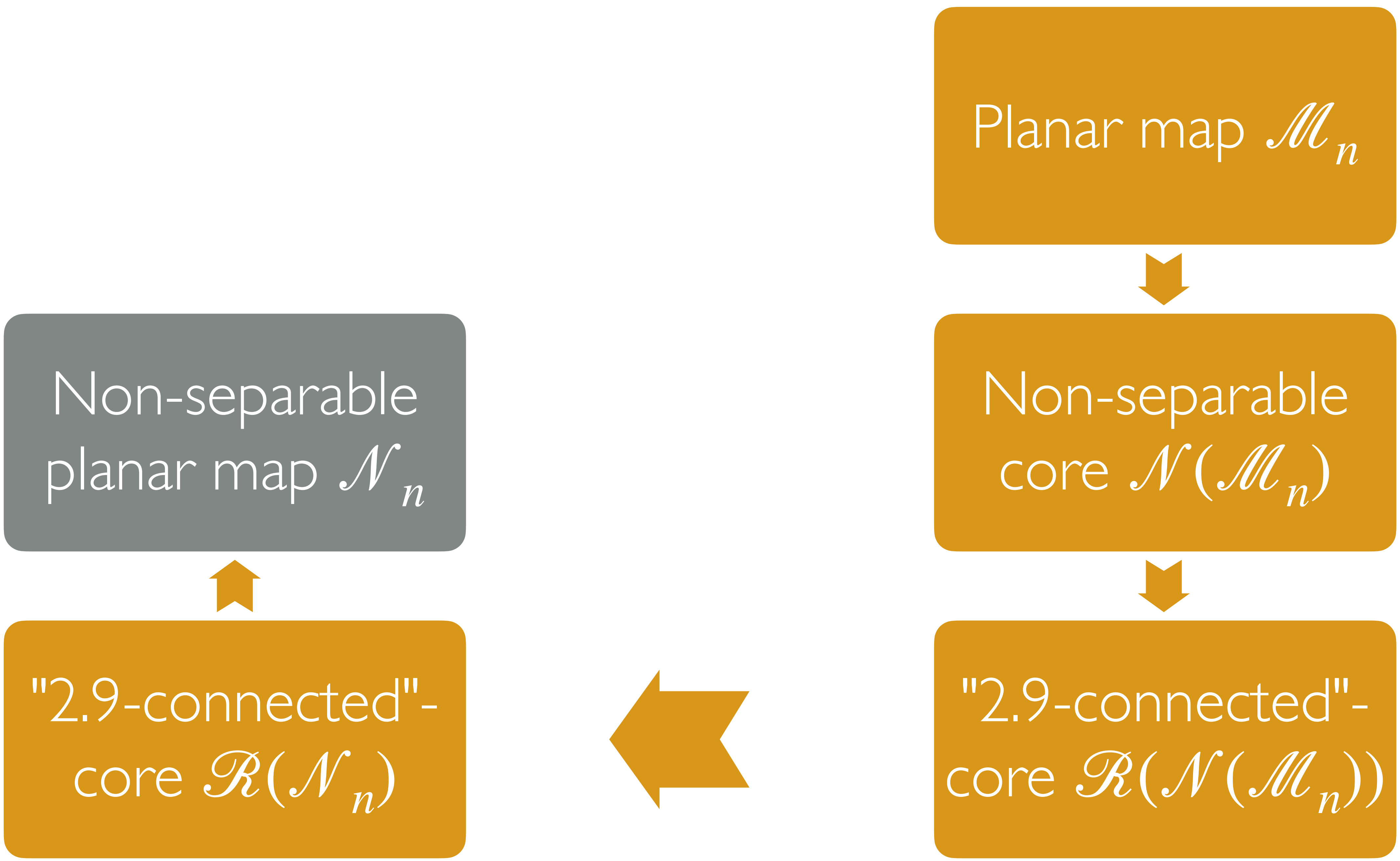

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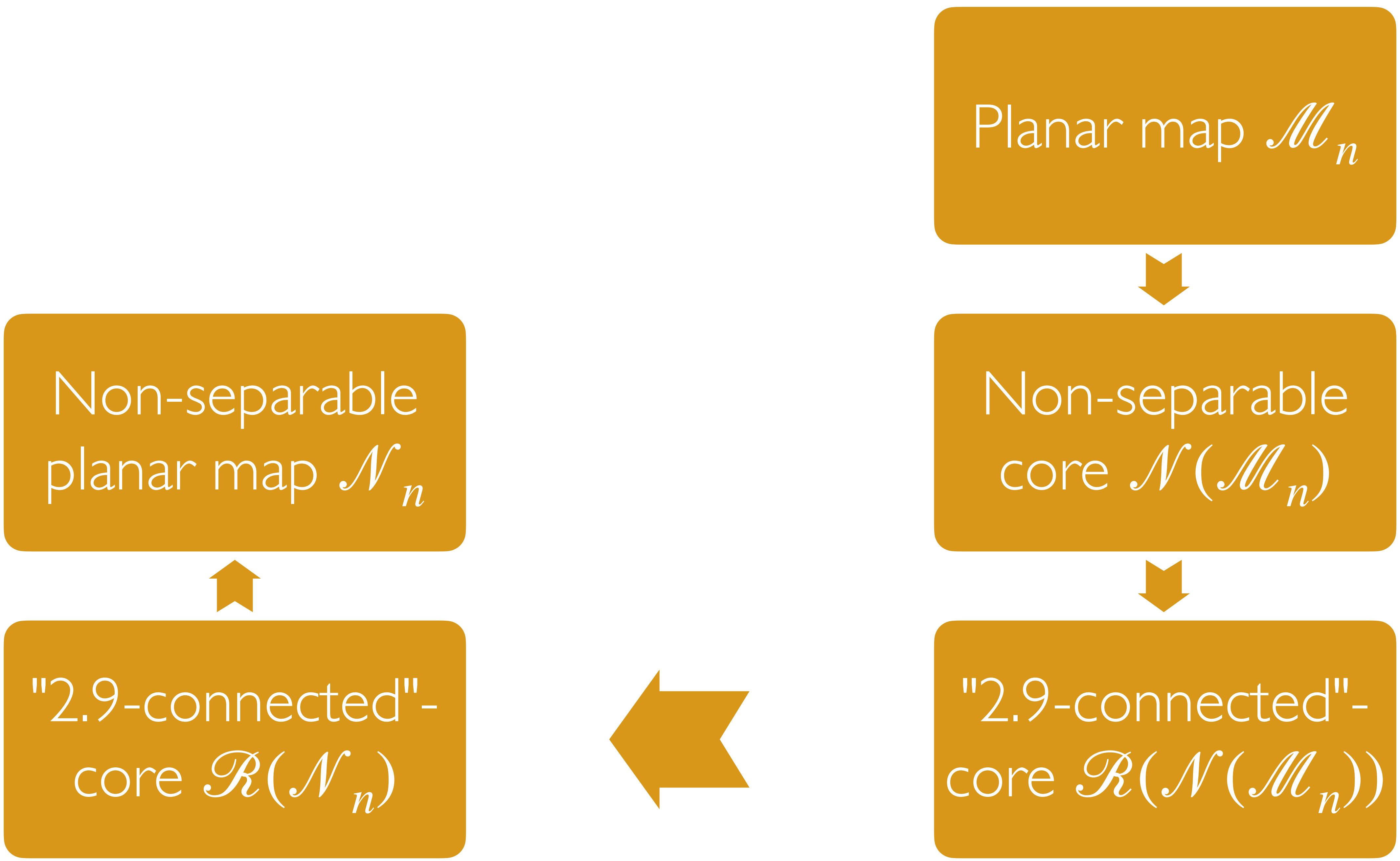

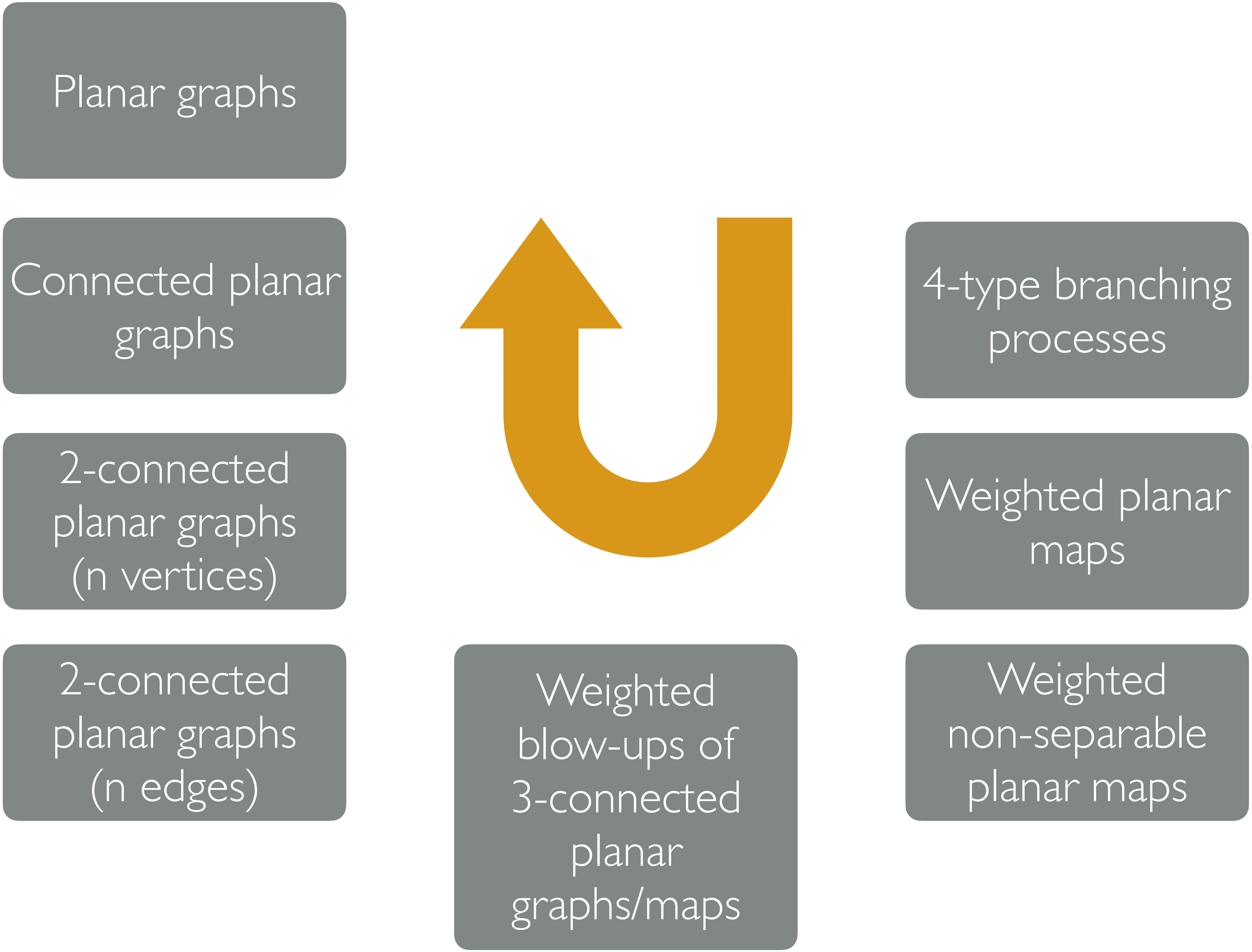

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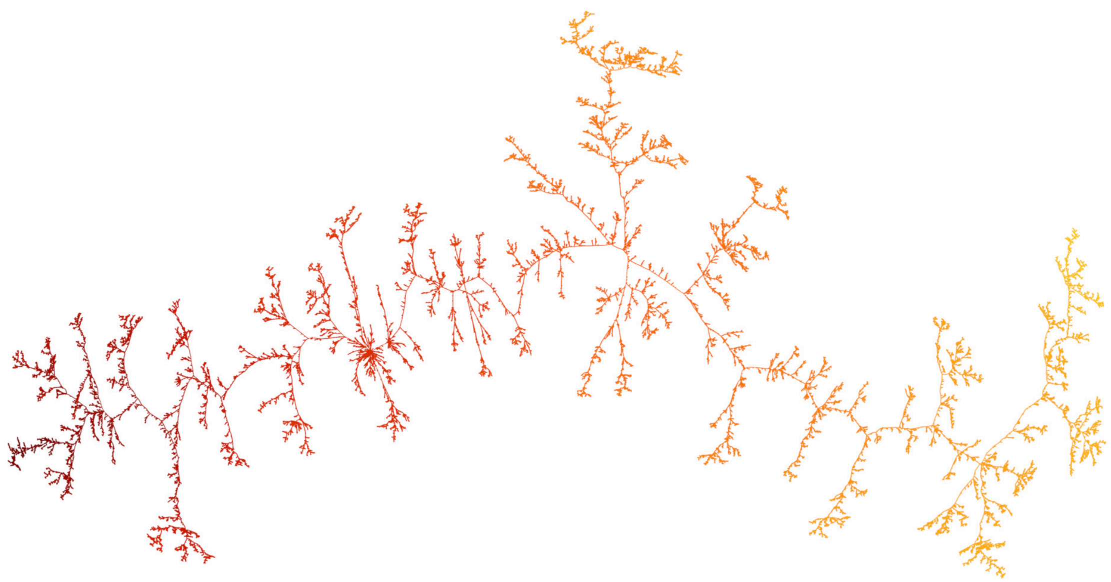
DIAMETER AND SCALING LIMITS

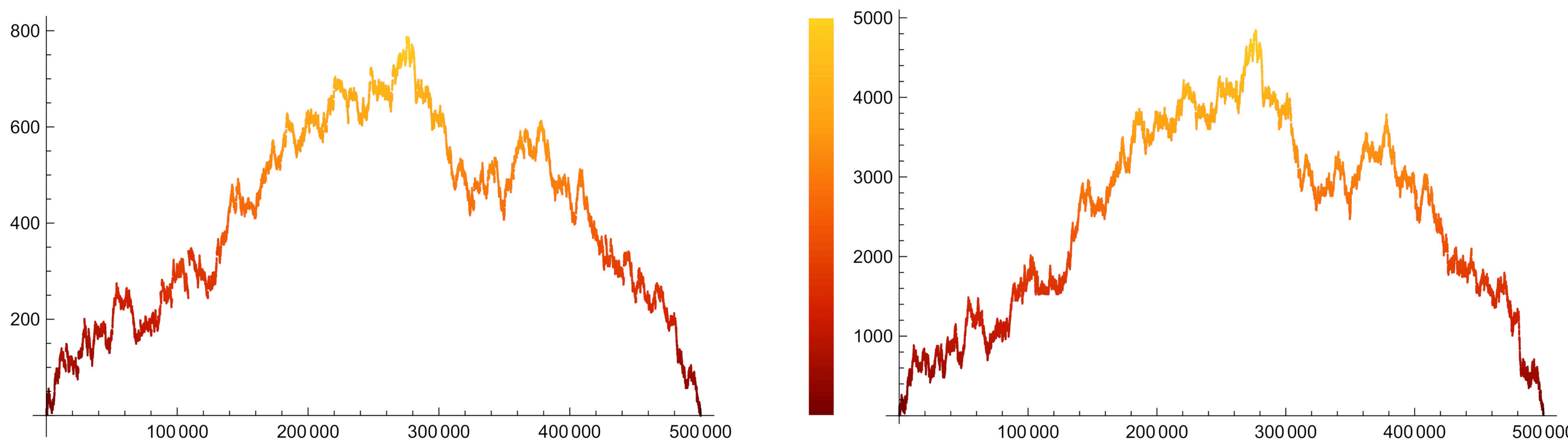
- (Chapuy, Fusy, Giménez, Noy) There exists a $c > 0$ such that the diameter $\mathrm{D}(\mathscr{P}_n)$ satisfies for each small enough $\epsilon > 0$ and all $n > n_0(\epsilon)$
$$\mathbb{P}(\mathrm{D}(\mathscr{P}_n) \notin [n^{1/4-\epsilon}, n^{1/4+\epsilon}]) < \exp(-n^{c\epsilon}).$$

- **Open problem:** What happens when we rescale distances by $n^{-1/4}$?

SCALING LIMITS

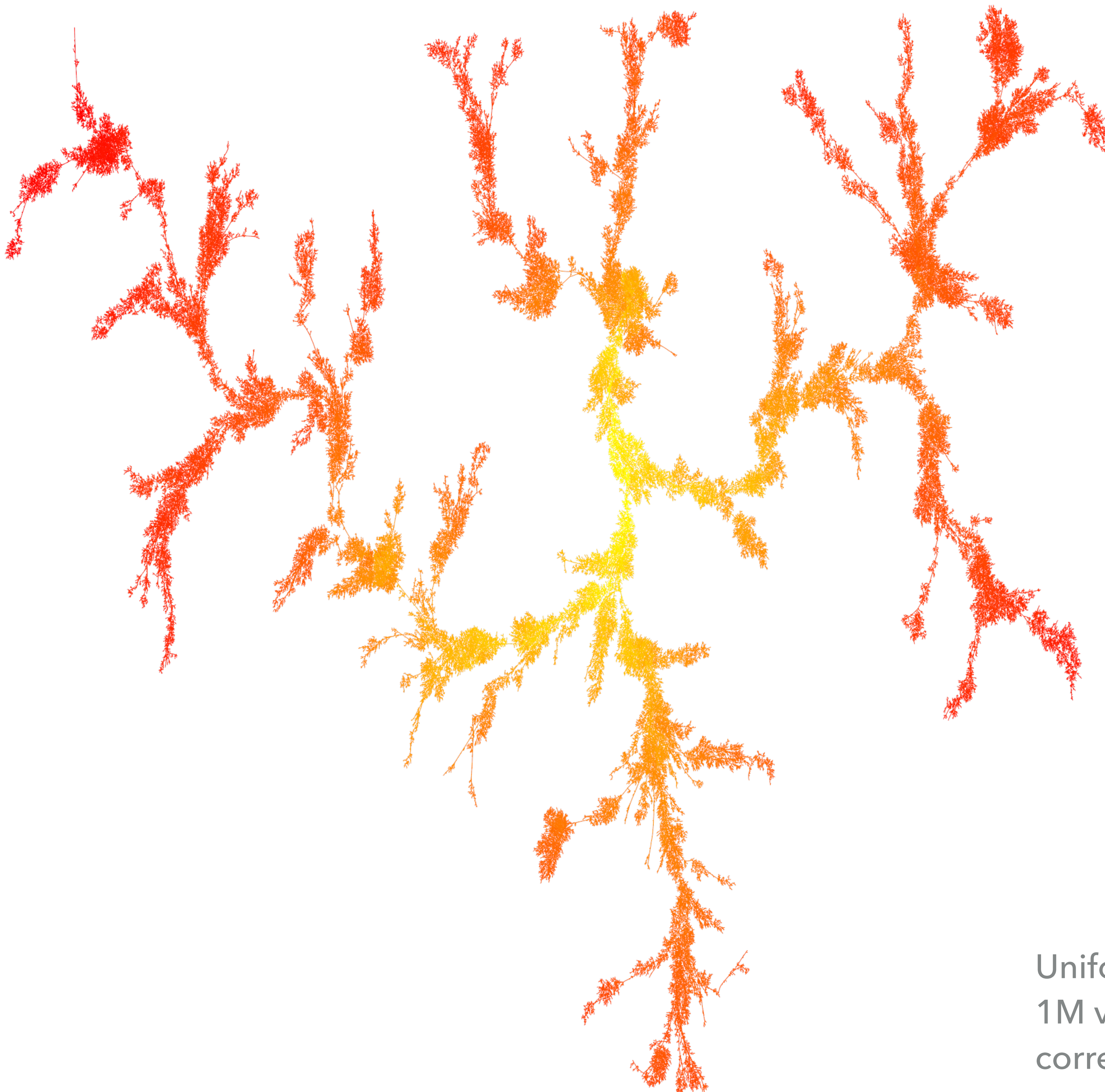


500k vertex simply generated tree in the universality class of the Brownian continuum random tree. Colours correspond to the height of the vertex.



Simulation: GRANT (Generate RANdom Trees), available here: http://github.com/BenediktStufler/grant

SCALING LIMITS



Uniform labelled tree with 1M vertices. Colours correspond to closeness centrality of the vertex.

SCALING LIMITS

Thm. (Aldous, 1991) The uniform labelled tree T_n with its graph distance d_{T_n} and the uniform measure μ_n on its vertices satisfies

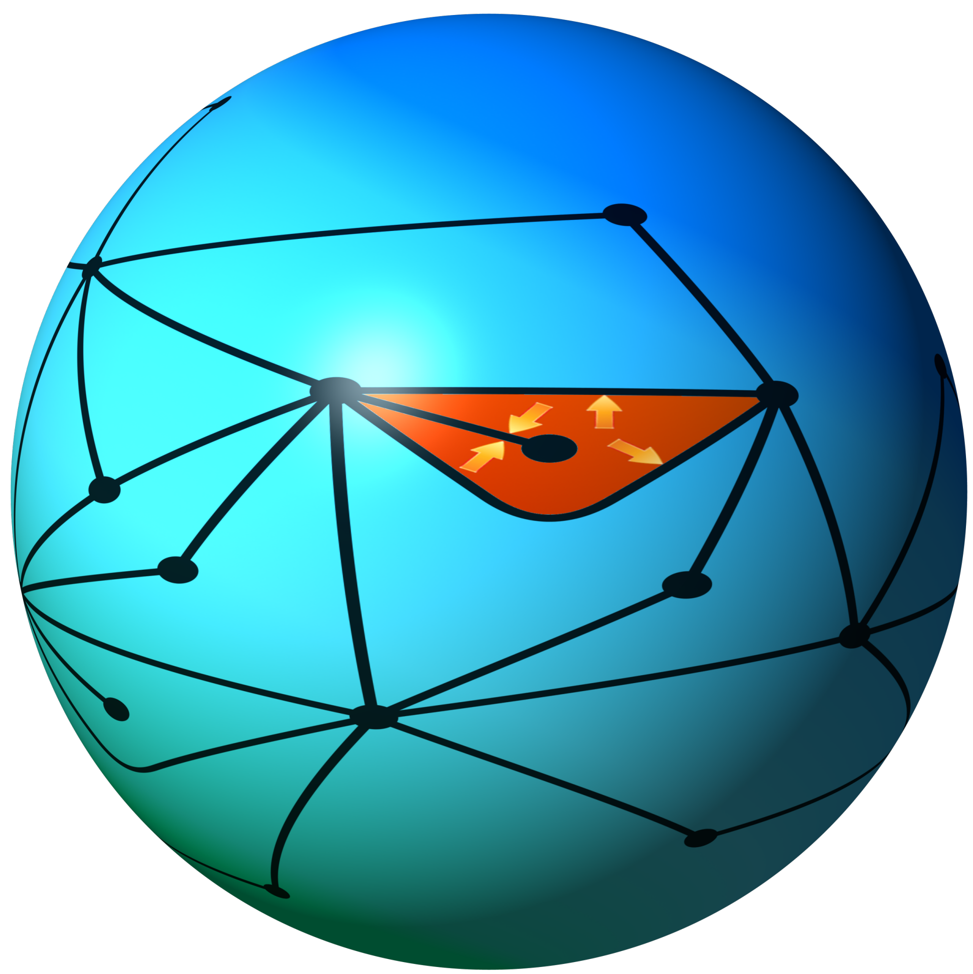
$$(T_n, \frac{1}{2\sqrt{n}} d_{T_n}, \mu_{T_n}) \to (T, d_T, \mu_T)$$

for a limiting random measured metric space (T, d_T, μ_T).

SCALING LIMITS

Thm. (Chassaing and Schaeffer, 2004) The height $H(Q_n)$ of a uniform random quadrangulation with n faces admits the width r of Aldous' one-dimensional ISE as scaling limit:

$$(8n/9)^{-1/4} H(Q_n) \to r$$



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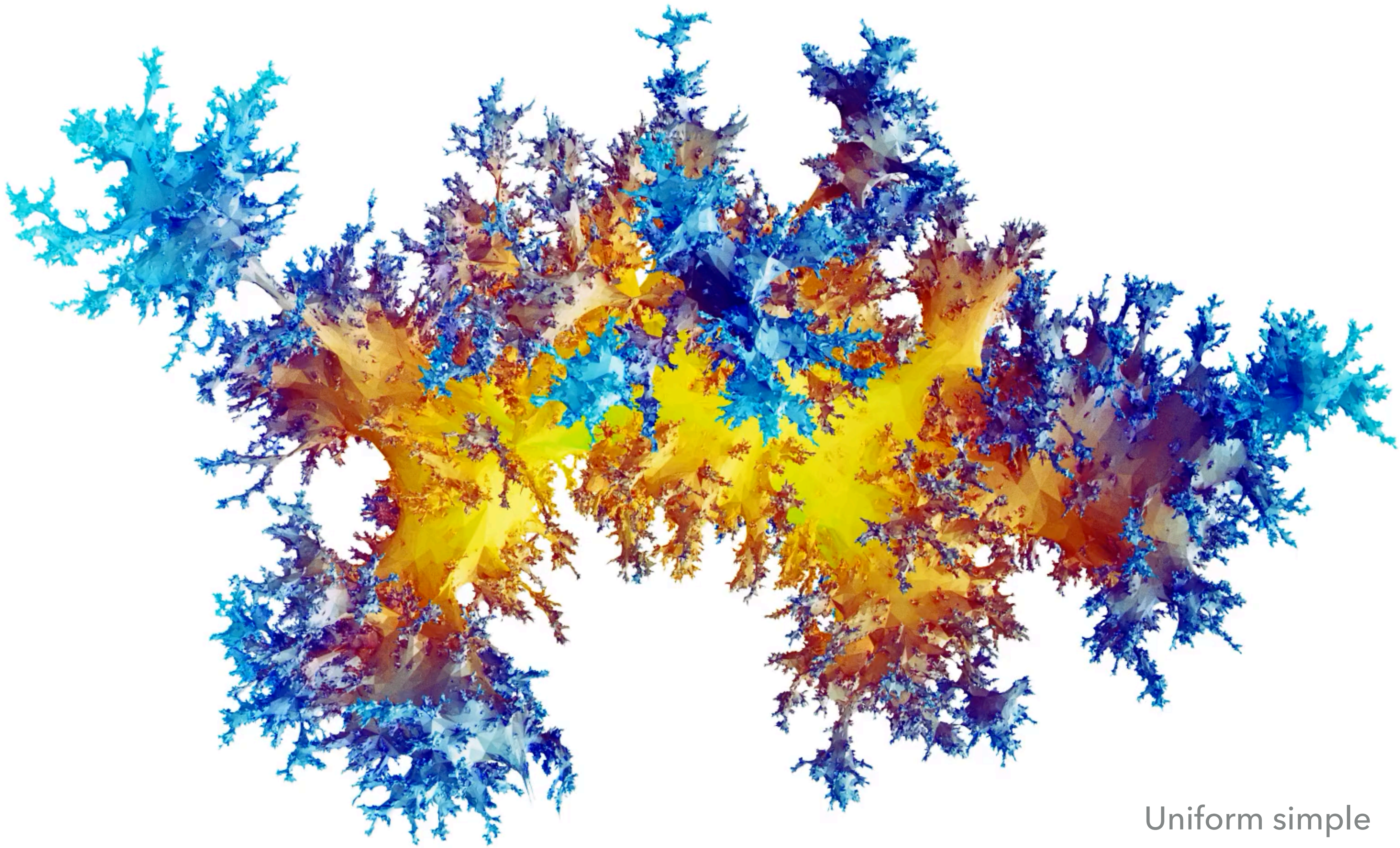
Miermont (2013), Le Gall (2013): GHP scaling limit called the Brownian map (M, d_M, μ_M):

$$(Q_n, (8n/9)^{-1/4}d_{Q_n}, \mu_{Q_n}) \to (M, d_M, \mu_M)$$



Uniform simple triangulation of the sphere with 1M faces

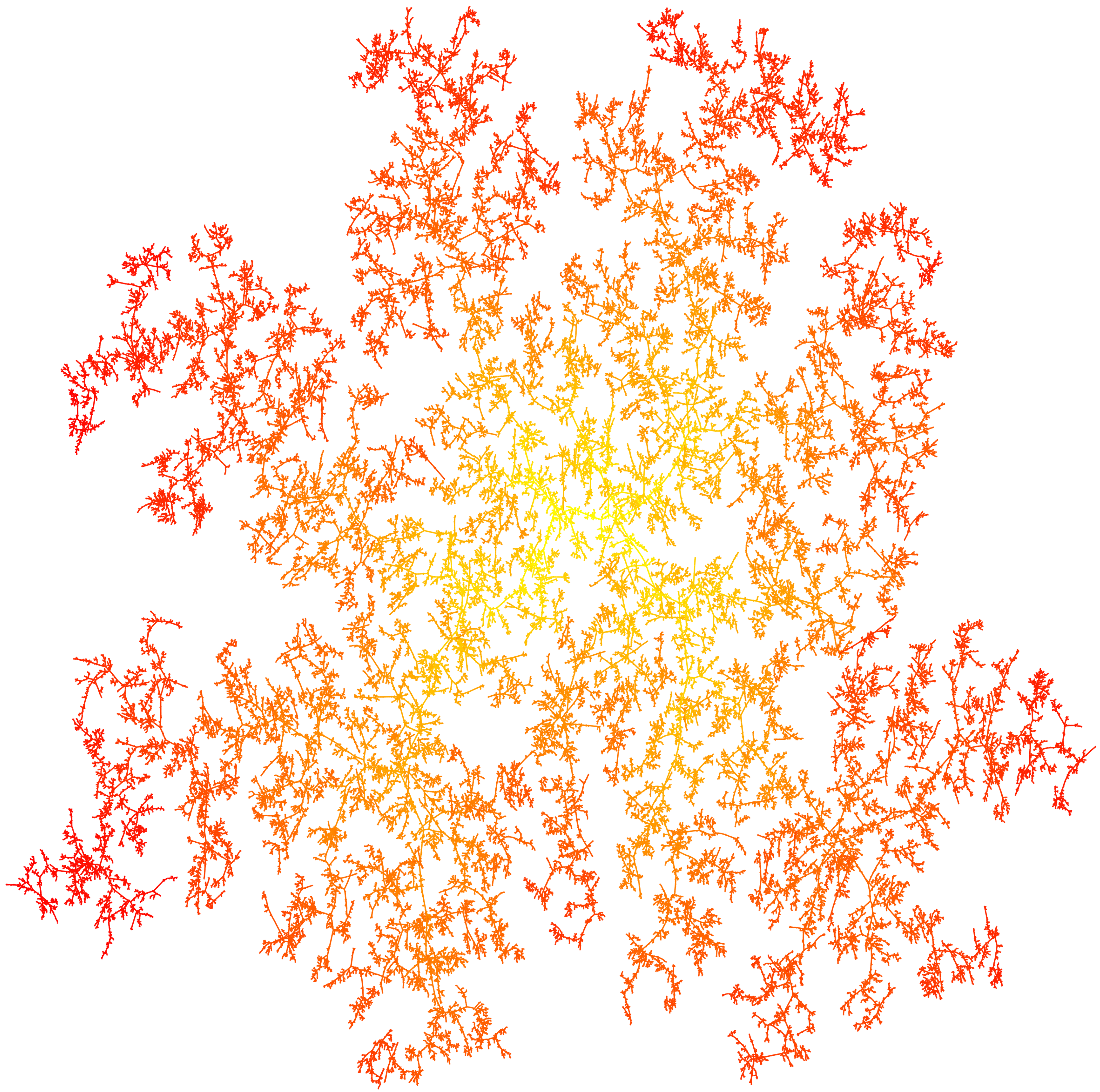
Simulation: SIMTRIA (Generate SIMple TRIAngulations): http://github.com/BenediktStufler/simtria,
SCENT (Calculate closeness centrality): http://github.com/BenediktStufler/scent
Mathematica, Blender

UNIFORM SPANNING TREE

UNIFORM SPANNING TREE

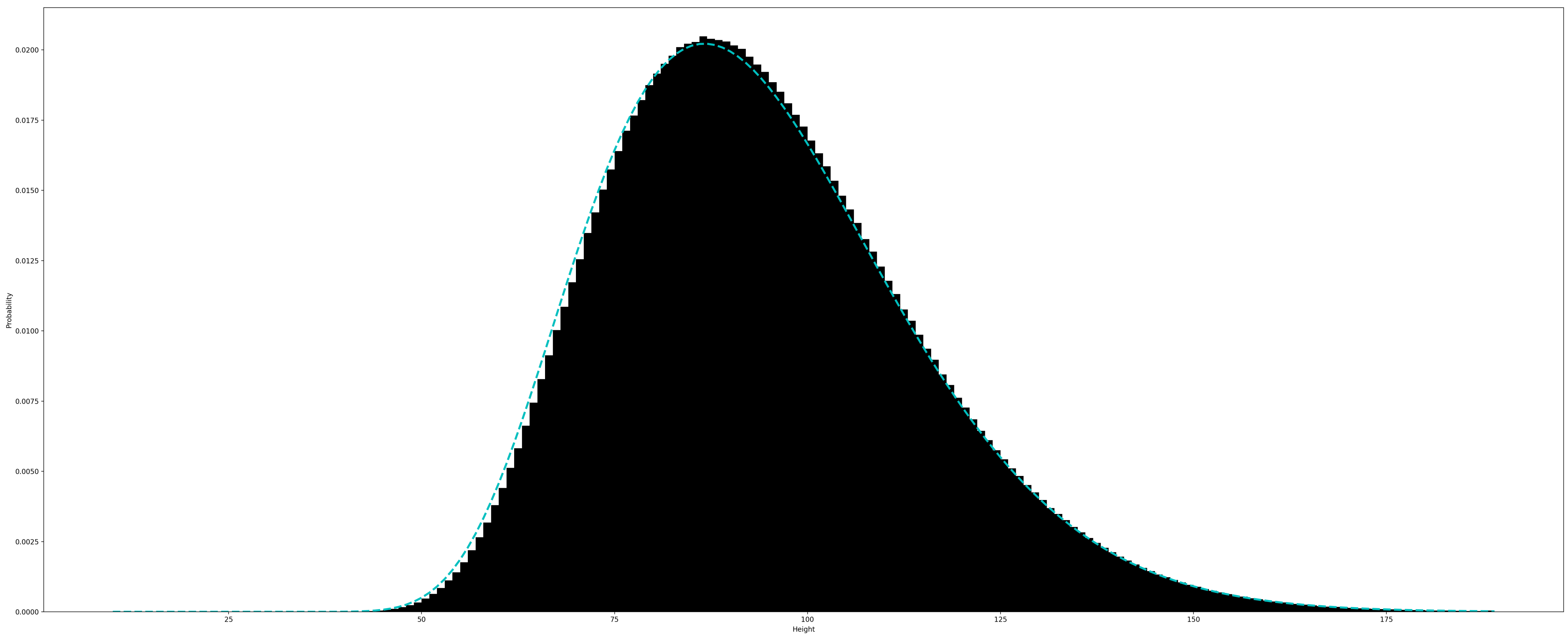
Question: What are the properties of a uniform random spanning tree of a uniform random planar graph $\mathscr{P}_n$ with n labelled vertices?

UNIFORM SPANNING TREE



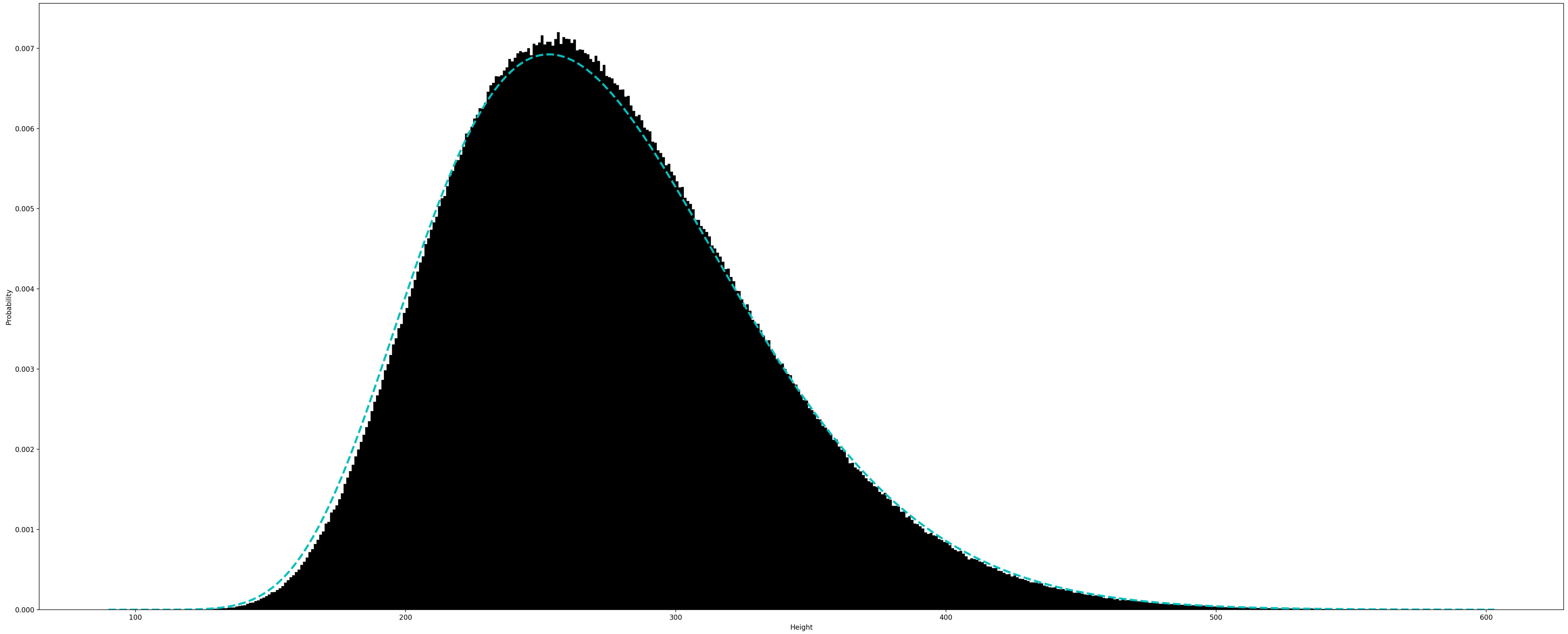
Uniform spanning tree of a uniform planar map with 1M edges

UNIFORM SPANNING TREE



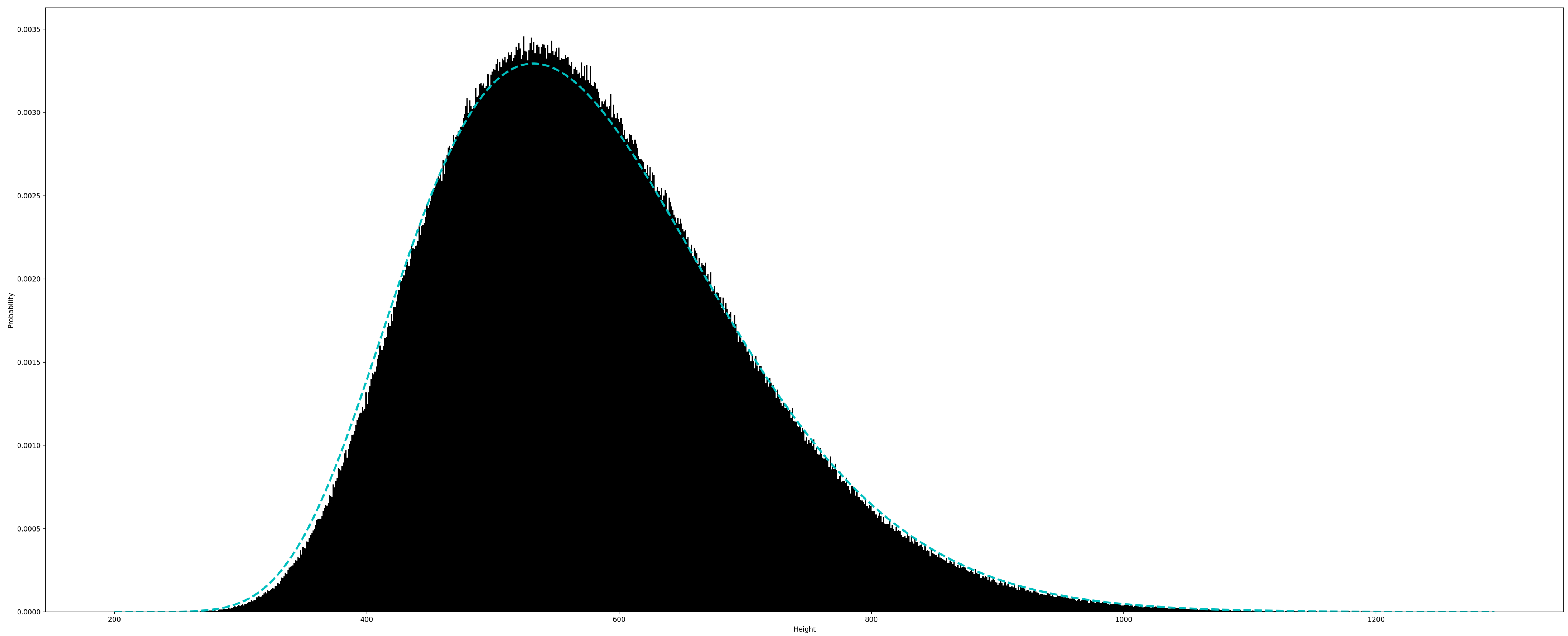
Histogram for the height of the UST of a uniform random planar **map** with $n = 10000$ edges

UNIFORM SPANNING TREE



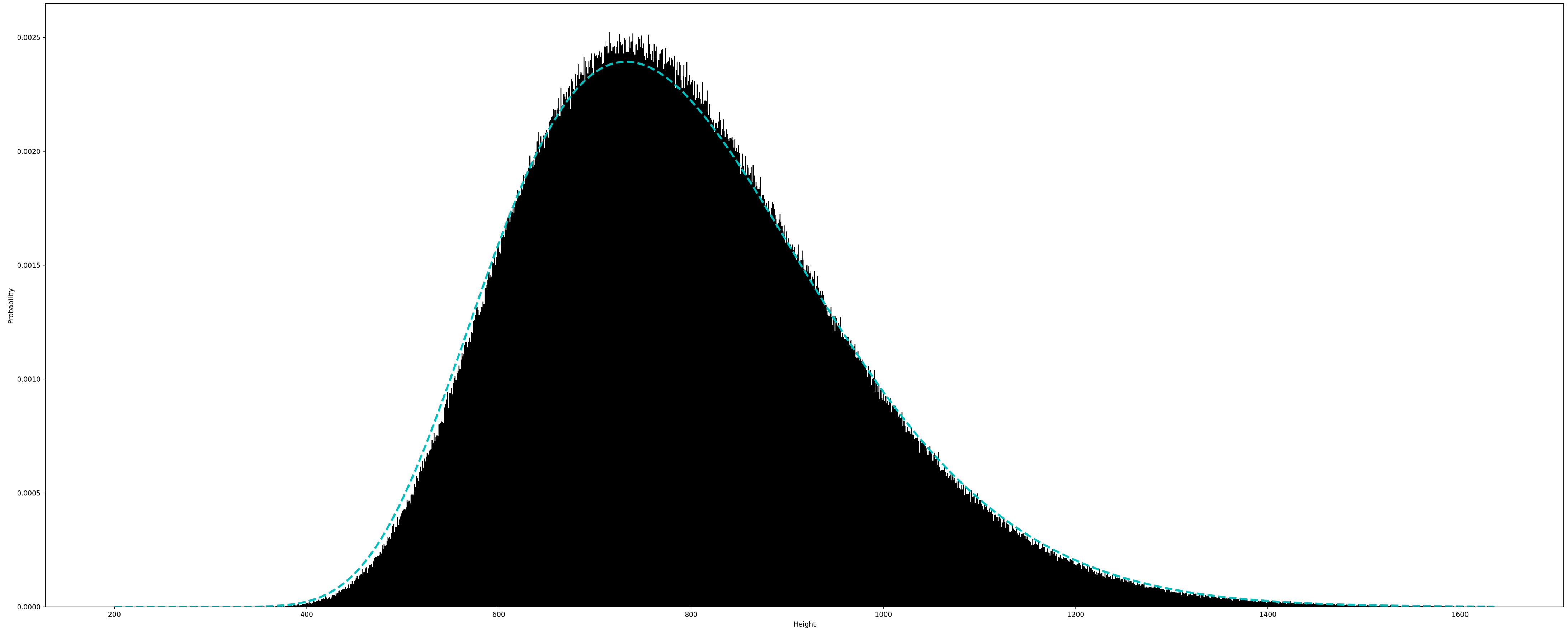
Histogram for the height of the UST of a uniform random planar **map** with $n = 100000$ edges

UNIFORM SPANNING TREE



Histogram for the height of the UST of a uniform random planar **map** with $n = 500000$ edges

UNIFORM SPANNING TREE



Histogram for the height of the UST of a uniform random planar **map** with $n = 1000000$ edges

UNIFORM SPANNING TREE

$h(n)$: average height of **simulations** of UST of uniform planar **map** with n edges.

$$\alpha(n) = \log(\frac{h(10n)}{h(n)})/\log n$$

n	10^3	10^4	10^5	10^6	10^7	10^8
h(n)	31.2812	93.8020	273.9275	792.7325	2285.815	6585.556
alpha(n)	0.476927	0.465423	0.461491	0.459914	0.459551	

UNIFORM SPANNING TREE

Non-rigorous Knizhnik-Polyakov-Zamolodchikov (KPZ) formula predicts:

$$\alpha = \frac{5-\sqrt{10}}{4} = 0.4594305...$$

Many thanks to Nathanaël Berestycki for explaining this to me.

n	10^3	10^4	10^5	10^6	10^7	10^8
h(n)	31.2812	93.8020	273.9275	792.7325	2285.815	6585.556
alpha(n)	**0.4**76927	**0.4**65423	**0.4**61491	**0.459**914	**0.459**551	

Thanks for your attention.