Asymptotic expansion for random tensor models

ADRIAN TANASĂ

LaBRI and LIPN

in collaboration with:

V. Bonzom, T. Krajewski, V. Nador, T. Muller

Villetaneuse, 4 avril 2023

Plan

1. Introduction
   - Combinatorial Physics
   - Quantum Field Theory (QFT) and Combinatorial QFT
   - Random matrices
2. The multi-orientable (MO) tensor model
   - Ribbon jackets and the large N limit
   - The dominant order
   - The double scaling limit
3. The quartic $O(N)^3-$invariant model
4. The prismatic tensor model
5. Conclusions

Combinatorial Physics

Combinatorial Physics

- problems in Theoretical Physics successfully tackled using Combinatorial methods
- problems in Combinatorics successfully tackled using Theoretical Physics methods

(most of) *this talk*: example of the first case

combinatorial techniques:

- analysis of the general term in an asymptotic expansion
- analytic analysis of the singularities of the relevant generating series

physical problem: implementation of the celebrated double scaling mechanism for various random tensor models

Quantum Field Theory

Quantum Field Theory (QFT) - quantum description of particles and their interactions

description compatible with Einstein's theory of special relativity

QFT formalism applies to:

- Standard Model of elementary particle physics
- statistical physics (statistical QFT)
- condensed matter physics
- *etc.*

great experimental success!

QFT - built-in combinatorics

(real or complex) fields - $\Phi : \mathbb{R}^4 \to \mathbb{R}$ or $\mathbb{C}$ (4−dimensional QFT)

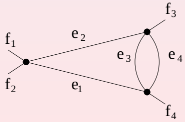
action of a QFT model ($S(\phi)$)
quadratic part (propagation) + non-quadratic part (cubic, quartic, *etc.*)

partition function: $Z = \int \mathcal{D}\Phi\, e^{-S(\Phi)}$

perturbative expansion (Taylor expansion) of the partition function Z in the coupling constant λ

Feynman graphs associated to the terms of the expansion

example of a Feynman graph of the Φ^4 *model*:



Feynman graphs → Feynman amplitudes

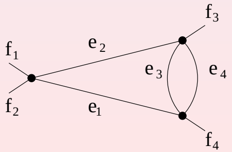
Combinatorial QFT

Combinatorial QFT - 0−*dimensional QFT*
the scalar field ϕ is not a function of space-time
(there is no space-time)!
real field $\phi \in \mathbb{R}$ (or complex field $\phi \in \mathbb{C}$)

partition function:

$$Z = \int_{\mathbb{R}} d\phi \, e^{-\frac{1}{2}\phi^2 + \frac{\lambda}{4!}\phi^4}.$$

perturbation theory - formal series in λ
$\rightarrow$ (abstract) Feynman graphs and Feynman amplitudes



One (still) needs to evaluate integrals of type

$$\frac{\lambda^n}{n} \int d\phi \, e^{-\phi^2/2} \left(\frac{\phi^4}{4!}\right)^n .$$

one can (still) use standard QFT techniques:

$$\int d\phi \, e^{-\phi^2/2} \phi^{2k} = \frac{\partial^{2k}}{\partial J^{2k}} \int d\phi \, e^{-\phi^2/2 + J\phi}|_{J=0} = \frac{\partial^{2k}}{\partial J^{2k}} e^{J^2/2}|_{J=0}.$$

J - the source

0−dimensional QFT - interesting "laboratories" for testing theoretical physics tools

V. Rivasseau and Z. Wang, *J. Math. Phys.* (2010), arXiv:1003.1037

I. Klebanov, F. Popov and G. Tarnopolsky, arXiv:1808.09434, TASI Lectures

From scalars to matrices

Definition

A random matrix is a matrix of given type and size whose entries consist of random numbers from some specified distribution.

Random matrices & combinatorics:

counting maps theorems (*via* matrix integral techniques)

$$\int f(\text{matrix of dim } N) \quad = \quad \sum_g N^{2-2g} A_g$$

A_g - some weighted sum encoding maps of genus g
(this depends on the choice of f - the physical model)

A. Zvonkine, *"Computers & Math. with Applications: Math. & Computer Modelling"*, (1997)

J. Bouttier, in *"The Oxford Handbook of Random Matrix Theory", 2011, arXiv:1104.3003*

Ph. Di Francesco *et. al., Phys. Rept.* (1995), arXiv:hep-th/9306153

Random matrices in mathematics & physics

- mathematics
  - non-commutative probabilities

    D. Voiculescu, *et. al. Free random variables* CRM Monograph (1992)
  - the Kontsevich matrix model - the Witten conjecture: rigorous approach to the moduli space of punctured Riemann surfaces

    E. Witten, *Nucl. Phys.* **B** (1990),

    M. Kontsevich, *Commun. Math. Phys.* (1992)
  - *etc.*
- physics: nuclear physics (spectra of heavy atoms), particle physics (quantum chromodynamics), 2-dimensional quantum gravity, string theory *etc.*

Wishart, *Biometrika* (1928)

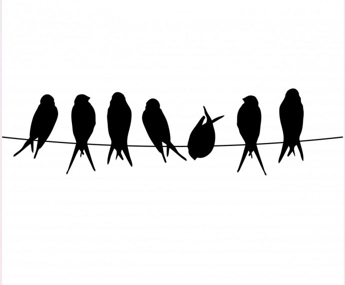
Wigner, *Annals Math.* (1955)

M. L. Mehta, *Random Matrices*, Elsevier ('04)

G. Anderson, A. Guionnet, O. Zeitouni, *An Introduction to Random Matrices*, Cambridge Univ. Press ('09)

Other applications of random matrices

- spacing between perched birds (parked cars)



P. Seba, *J. Phys.* **A** (2009)
A.Y. Abul-Magd *Physica* **A** (2006)

S. Rawal, G.J. Rodgers *Physica* **A** (2005)

G. Akermann, J. Baik and Ph. Di Francesco, *The Oxford Hadbook of Random Matrix Theory*, Oxford (2015)

More on matrix integral techniques

Ph. Di Francesco *et. al.*, *Phys. Rept.* (1995), hep-th/9306153,

B. Eynard, "*Counting Surfaces*" (Springer) *etc.*

M - $N \times N$ Hermitian matrix

The partition function:

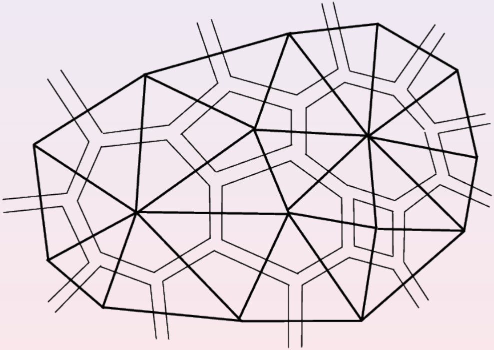
$$Z := \int dM e^{-\frac{1}{2}\mathrm{Tr}M^2 + \frac{\lambda}{\sqrt{N}}\mathrm{Tr}M^3}.$$

$dM := \prod_i dM_{ii} \prod_{i<j} d\,\mathrm{Re}M_{ij}\,\mathrm{Im}M_{ij}$ (the measure)

QFT perturbative expansion in λ - Feynman ribbon graphs (dual to 2-dimensional triangulations)

The partition function Z generates random triangulations - a generating function

Duality ribbon graphs $<->$ $2D$ random triangulations



the triangulation building block: the triangle (the $2D$ simplex)

dual of a triangle - a ribbon vertex of valence 3

Asymptotic expansion of matrix models - dominant graphs

Feynman graphs of matrix models are ribbon graphs or $(2D)$ maps

the matrix amplitude can be combinatorially computed - in terms of number of vertices (V), edges and faces (F) of the graph

$$\mathcal{A} = \lambda^V N^{-\frac{1}{2}V+F} = \lambda^V N^{2-2g}$$

(since $E = \frac{3}{2}V$)
The partition function supports a $1/N$ *expansion*:

$$Z = N^2 Z_0(\lambda) + Z_1(\lambda) + \ldots = \sum_{g=0}^{\infty} N^{2-2g} Z_g(\lambda)$$

Z_g gives the contribution from surfaces of genus g

In the $N \to \infty$ limit, only planar surfaces survive
- dominant graphs - *(triangulations of the sphere $\mathcal{S}^2$)*

E. Brézin *et al., Commun. Math. Phys.* ('78),

V. A. Kazakov, *Phys. Lett.* **B** ('85), F. David, *Nucl. Phys.* **B** ('85)

The double scaling limit for matrix models

The successive coefficient functions $Z_g(\lambda)$ as well diverge at the same critical value of the coupling $\lambda = \lambda_c$
the leading singular piece of Z_g:

$$Z_g(\lambda) \propto f_g(\lambda_c - \lambda)^{(2-\gamma_{\rm str})\chi/2} \text{ with } \gamma_{\rm str} = -\frac{1}{2} \text{ (pure gravity)}$$

contributions from higher genera ($\chi < 0$) are enhanced as $\lambda \to \lambda_c$
$\kappa^{-1} := N(\lambda - \lambda_c)^{(2-\gamma_{\rm str})/2}$
the partition function expansion:

$$Z = \sum_g \kappa^{2g-2} f_g$$

double scaling limit: $N \to \infty$, $\lambda \to \lambda_c$ while holding fixed κ
coherent contribution from all genus surfaces

M. Douglas and S. Shenker, *Nucl. Phys.* **B** ('90), E. Brézin and V. Kazakov, *Phys. Lett.* **B**, *Nucl. Phys.* **B** ('90),

D. Gross and M. Migdal, *Phys. Rev. Lett.*, *Nucl. Phys.* **B** ('90)

Question:

How much of these celebrated $2D$ results generalize to $3D$?

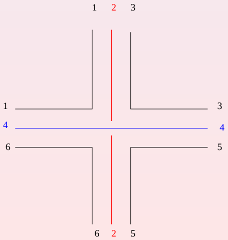
From matrices to tensors

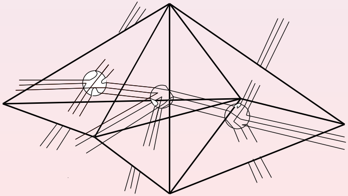
Tensor models were introduced already in the 90's - replicate in dimensions higher than 2 the success of **random matrix models**:
J. Ambjorn *et. al.*, *Mod. Phys. Lett.* ('91),

N. Sasakura, *Mod. Phys. Lett.* ('91), M. Gross *Nucl. Phys. Proc. Suppl.* ('92)

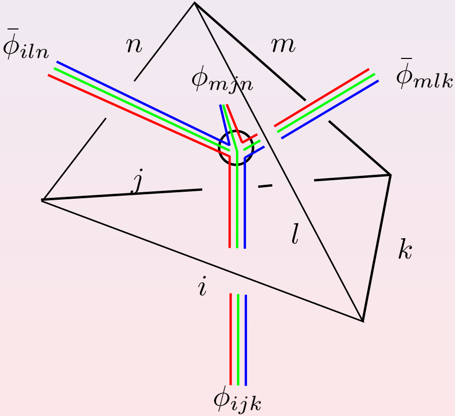
natural generalization of matrix models

matrix → rank three tensor





From a tetrahedron to a 4−valent tensor vertex



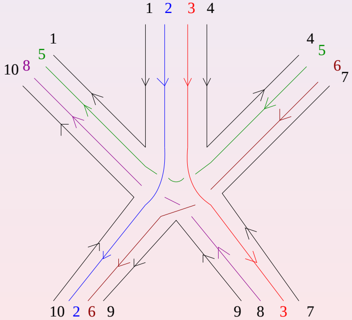
tensor graphs - $3D$ maps

the triangulation building block: the tetrahedron (the $3D$ simplex)

dual of a tetrahedron - a tensor vertex of valence 4

4−dimensional models

4D vertex (dual image of a 4−simplex (5−cell)):



QFT-inspired simplification - the colored tensor model

highly non-trivial combinatorics and topology
→ a QFT simplification of these models - colored tensor models

(R. Gurău, Commun. Math. Phys. (2011), arXiv:0907.2582)

a quadruplet of complex fields $\left(\phi^0, \phi^1, \phi^2, \phi^3\right)$;

$$\begin{aligned} S[\{\phi^i\}] &= S_0[\{\phi^i\}] + S_{int}[\{\phi^i\}] \\ S_0[\{\phi^i\}] &= \frac{1}{2}\sum_{p=0}^{3}\sum_{i,j,k,=1}^{N} \overline{\phi^p_{ijk}}\phi^p_{ijk} \\ S_{int}[\{\phi^i\}] &= \frac{\lambda}{4}\sum_{i,j,k,i',j',k'=1}^{N} \phi^0_{ijk}\phi^1_{i'j'k}\phi^2_{i'jk'}\phi^3_{k'j'i} + \text{ c. c.}, \end{aligned} \tag{1}$$

the indices $0, \ldots, 3$ - color indices.

R. Gurau, "*Random Tensors*", Oxford Univ. Press (2016)

Various results

- double-scale limit mechanism
  1. combinatorial methods - analysis of the general term of the large N asymptotic expansion and analytic analysis of the singularities of the relevant generating series

     G. Schaeffer and R. Gurău, arXiv:1307.5279, Annales IHP **D** Comb. Phys. & Interactions (2016)
  2. QFT methods S. Dartois *et. al.*, JHEP (2013), V. Bonzom *et. al.*, JHEP (2014)
- Connes-Kreimer Hopf algebraic reformulation of tensor renormalizability

  M. Raasakka and A. Tanasă, *Sém. Loth. Comb.* (2014)
- loop vertex expansion of the perturbative series

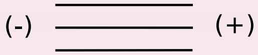
  T. Krajewski & R. Gurau, *Annales IHP* **D** *- Combinatorics, Phys. & their Interactions* (2015)
- *etc.*

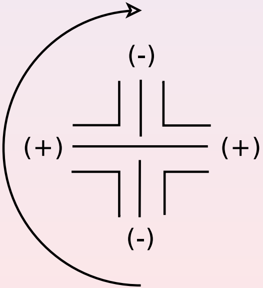
Another (QFT-inspired) simplification of tensor models

Multi-Orientable (MO) models

A. Tanasă, J. Phys. **A** (2012) arXiv:1109.0694[math.CO]

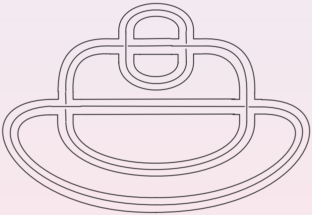
edge and (valence 4) vertex of the model:





(Feynman) MO tensor graphs

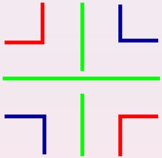
Example of an MO tensor graph:



Combinatorial and topological tools - jacket ribbon subgraphs

S. Dartois *et. al.*, *Annales Henri Poincaré* (2014)

three pairs of opposite corner strands

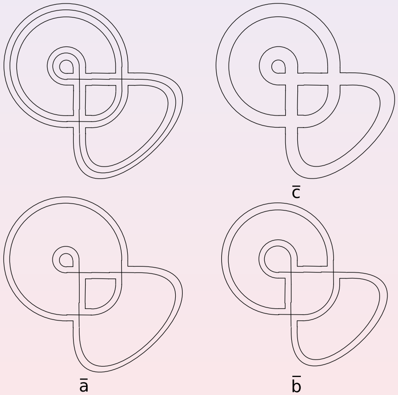


A **jacket of an MO graph** is the graph made by excluding one type of strands throughout the graph. The *outer* jacket $\bar{c}$ is made of all outer strands, or equivalently excludes the inner strands (the green ones); jacket $\bar{a}$ excludes all strands of type a (the red ones) and jacket $\bar{b}$ excludes all strands of type b (the blue ones).

$\hookrightarrow$ such a splitting is always possible

Example of jacket subgraphs

A MO graph with its three jackets $\bar{a}$, $\bar{b}$, $\bar{c}$



one can prove that each jacket of an MO tensor graph is a ribbon graph (or $2D$ map)

Euler characteristic & degree of MO tensor graphs

ribbon graphs - orientable or non-orientable surfaces.

Euler characteristic formula:

$$\chi(\mathcal{J}) = V_{\mathcal{J}} - E_{\mathcal{J}} + F_{\mathcal{J}} = 2 - k_{\mathcal{J}},$$

$k_{\mathcal{J}}$ is the non-orientable genus,
$V_{\mathcal{J}}$ is the number of vertices,
$E_{\mathcal{J}}$ the number of edges and
$F_{\mathcal{J}}$ the number of faces.

If the surface is orientable, k is even and equal to twice the orientable genus g

the degree of an MO tensor graph $\mathcal{G}$:

$$\omega(\mathcal{G}) := \sum_{\mathcal{J}} \frac{k_{\mathcal{J}}}{2} = 3 + \frac{3}{2} V_{\mathcal{G}} - F_{\mathcal{G}},$$

the sum over $\mathcal{J}$ running over the three jackets of $\mathcal{G}$.

Large N expansion of the MO tensor model

generalization of the random matrix asymptotic expansion in N

One needs to count the number of faces of the tensor graph

This can be achieved using the graph's jackets (ribbon subgraphs)

The tensor partition function writes as a formal series in $1/N$:

$$\sum_{\omega \in \mathbb{N}/2} C^{[\omega]}(\lambda) N^{3-\omega},$$

$$C^{[\omega]}(\lambda) = \sum_{\mathcal{G}, \omega(\mathcal{G})=\omega} \frac{1}{s(\mathcal{G})} \lambda^{v_{\mathcal{G}}}.$$

the role of the genus is played by the degree

Dominant graphs of the large N expansion

dominant graphs:

$$\omega = 0.$$

Theorem

The MO model admits a $1/N$ expansion whose dominant graphs are the "melonic" ones.

More on melonic tensor graphs

- they maximize the number of faces for a given number of vertices.
- they correspond to a particular class of triangulations of the sphere $\mathcal{S}^3$.

Combinatorial analysis of the general term of the expansion

- for the colored tensor model
  R. Gurău and G. Schaeffer, arXiv:1307.5279[math.CO],

  Annales IHP **D** *Comb., Phys. & their Interactions* (2016)

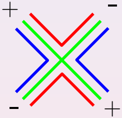
- for the MO tensor model

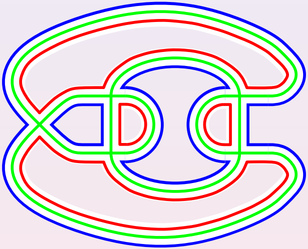
  E. Fusy and A. Tanasă, arXiv:1408.5725[math.CO], *Elec. J. Comb.* (2015)

  adaptation of the Gurău-Schaeffer combinatorial approach for the MO case

  combinatorial analysis leading to the implementation of the double scaling mechanism

(Types of) strands





An external strand is called **left (L)** if it is on the left side of a positive half-edge or on the right side of a negative half-edge.
An external strand is called **right (R)** if it is on the right side of a positive half-edge or on the left side of a negative half-edge.

(L - blue, straight (S) - green, R - red)

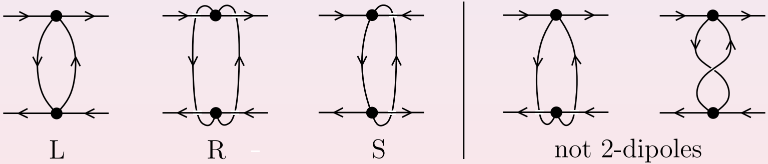
Main issue of a combinatorial analysis

There exists an infinite number of melon-free graphs of a given degree.

Nevertheless, some configurations can be repeated without increasing the degree.

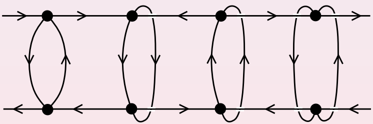
Dipoles

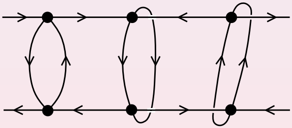
A **(two-)dipole** is a subgraph formed by a couple of vertices connected by two parallel edges which has a face of length two, which, if the graph is rooted, does not pass through the root.



Chains - ladder diagrams

In a Feynman graph, a **chain** is as a sequence of dipoles $d_1 \ldots, d_p$ such that for each $1 \leq i < p$, d_i and d_{i+1} are connected by two edges involving two half-edges on the same side of d_i and two half-edges on the same side of d_{i+1}.

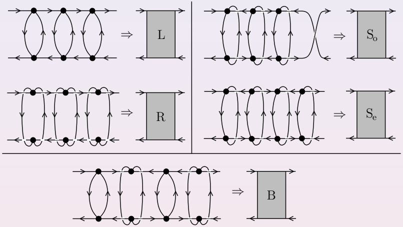




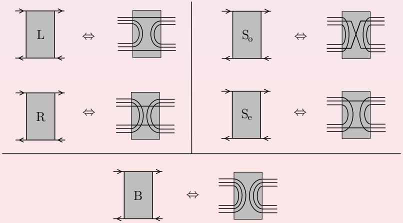
Some more definitions - (un)broken chains

- A chain is called **unbroken** if all the p dipoles are of the same type.
- A **proper chain** is a chain of at least two dipoles.
- A proper chain is called **maximal** if it cannot be extended into a larger proper chain.

Chains, chain-vertices and their strand configurations

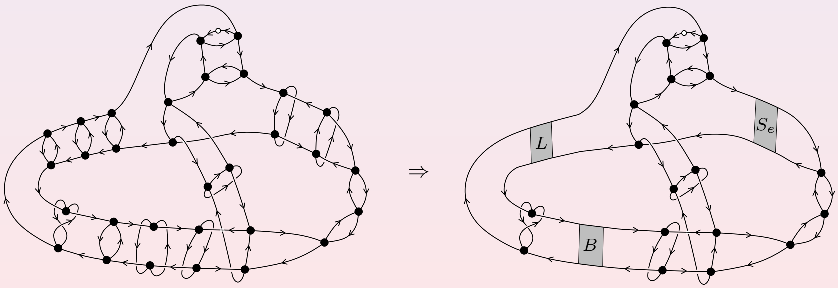


strand configurations:



Schemes

Let G be a rooted melon-free MO-graph. The **scheme** of G is the graph obtained by simultaneously replacing any maximal proper chain of G by a *chain-vertex*.



A **reduced scheme** is a rooted melon-free MO-graph with chain-vertices and with no proper chain.

By construction, the scheme of a rooted melon-free MO-graph (with no chain-vertices) is a reduced scheme.

Every rooted melon-free MO-graph is uniquely obtained as a reduced scheme where each chain-vertex is consistently substituted by a chain of at least two dipoles

Degree conservation

Proposition

Let G be an MO-graph with chain-vertices and let G' be an MO-graph with chain-vertices obtained from G by consistently substituting a chain-vertex by a chain of dipoles. Then the degrees of G and G' are the same.

Proof. Carefully counting the number of faces, vertices and connected components and using the formula:

$$2\omega = 6c + 3V - 2F.$$

Analysis of the general term of the large N expansion

Finiteness of the set of reduced schemes of a given degree

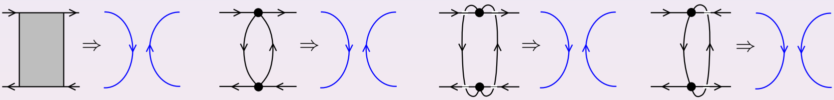
Theorem

For each $\omega \in \frac{1}{2}\mathbb{Z}_+$, the set of reduced schemes of degree ω is finite.

Proof.

1. For each reduced scheme of degree ω, the sum $N(G)$ of the numbers of dipoles and chain-vertices satisfies $N(G) \leq 7\omega - 1$.
2. For $k \geq 1$ and $\omega \in \frac{1}{2}\mathbb{Z}_+$, there is a constant $n_{k,\omega}$ s. t. any connected unrooted MO-graph of degree ω with at most k dipoles has at most $n_{k,\omega}$ vertices.

Proof - dipole and chain-vertex reductions



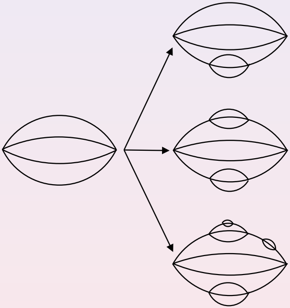
- removal of a chain-vertex (of any type)
- removal of a dipole of type L, R and S.

2 types of chain-vertices (and dipoles):

1. separating
2. non-separating

(if the number of connected components is conserved or not after removal)

Some analytic combinatorics - melonic generating function



the *generating function of melonic graphs*:

$$T(z) = 1 + z\,(T(z))^4\,.$$

Generating functions of our objects

u marks half the number of vertices
(i.e., for $p \in \frac{1}{2}\mathbb{Z}_+$, u^p weight given to a MO Feynman graph with $2p$ vertices)

generating function for:

- unbroken chains of type L (or R)

$$u^2 \frac{1}{1-u} = u^2 + u^3 + \dots$$

- even straight chains

$$u^2 \frac{1}{1-u^2} = \frac{u^2}{1-u^2} = u^2 + u^4 + u^6 + \dots$$

- odd straight chains

$$u^3 \frac{1}{1-u^2} = \frac{u^3}{1-u^2} = u^3 + u^5 + u^6 + \dots$$

etc.

More generating functions

putting together the generating functions of all contributions $\Longrightarrow G_S^{(\omega)}(u)$ - the generating function of rooted melon-free MO-graphs of reduced scheme S of degree ω,

$$G_S^{(\omega)}(u) = u^p \frac{u^{2a}}{(1-u)^a} \frac{u^{2s_e}}{(1-u^2)^{s_e}} \frac{u^{3s_o}}{(1-u^2)^{s_o}} \frac{6^b u^{2b}}{(1-3u)^b (1-u)^b}.$$

b - the number of broken chain-vertices
a - the number of unbroken chain-vertices of type L or R
s_e - the number of even straight chain-vertices,
s_o - the number of odd straight chain-vertices.

MO generating functions

$F_S^{(\omega)}(z)$ - the generating function of graphs of reduced scheme S

$$F_S^{(\omega)}(z) = T(z)\frac{6^b U(z)^{p+2c+s_o}}{(1-U(z))^{c-s}(1-U(z)^2)^s(1-3U(z))^b},$$

$U(z) := zT(z)^4 = T(z) - 1$

$F^{(\omega)}(z)$ - the generating function of rooted MO-graphs of degree ω

$$F^{(\omega)}(z) = \sum_{S \in \mathcal{S}_\omega} F_S^{(\omega)}(z).$$

$\mathcal{S}_\omega$ - the (finite) set of reduced schemes of degree ω.

Singularity order - dominant schemes

$T(z)$ has its main singularity at

$$z_0 := 3^3/2^8,$$

$T(z_0) = 4/3$, and $1 - 3U(z) \sim_{z \to z_0} 2^{3/2} 3^{-1/2} (1 - z/z_0)^{1/2}$.

R. Gurău and G. Schaeffer, arXiv:1307.5279[math.CO]

$$\implies (1 - 3U(z))^{-b} \sim_{z \to z_0} (1 - z/z_0)^{-b/2}$$

$\implies$ the dominant terms are those for which b is maximized.

the larger b, the larger the singularity order

A reduced scheme S of degree $\omega \in \frac{1}{2}\mathbb{Z}_+$ is called **dominant** if it maximizes (over reduced schemes of degree ω) the number b of broken chain-vertices.

The double scaling limit of the MO tensor model

R. Gurău, A. Tanasă, D. Youmans, Europhys. Lett. (2015)

The dominant configurations in the double scaling limit are the dominant schemes

The successive coefficient functions $Z_g(\lambda)$ as well diverge at the same critical value of the coupling $\lambda = \lambda_c$
contributions from higher degree are enhanced as $\lambda \to \lambda_c$

$\kappa^{-1} := N^{\frac{1}{2}}(1 - \lambda/\lambda_c)$
the partition function expansion:

$$Z = \sum_{\omega} N^{3-\omega} f_{\omega}$$

double scaling limit: $N \to \infty$, $\lambda \to \lambda_c$ while holding fixed κ

contribution from all degree tensor graphs

similar behaviour to the matrix model double scaling limit

The quartic $O(N)^3$-invariant tensor model

The quartic $O(N)^3$-tensor model

1. model introduced in

   S. Carrozza, A. T., 2015 arXiv:1512.06718 *Lett. Math. Phys.* (2016)

   - The tensor ϕ_{abc} is invariant under the action of $O(N)^3$:

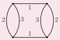
   $$\phi_{abc} \to \phi'_{a'b'c'} = \sum_{a,b,c=1}^{N} O^1_{a'a} O^2_{b'b} O^3_{c'c} \phi_{abc} \qquad O^i \in O(N)$$

   - quartic invariants:

   $$I_t(\phi) = \sum_{a,a',b,b',c,c'} \phi_{abc}\phi_{ab'c'}\phi_{a'bc'}\phi_{a'b'c} =$$

   $$I_{p,1}(\phi) = \sum_{a,a',b,b',c,c'} \phi_{abc}\phi_{a'bc}\ \phi_{ab'c'}\phi_{a'b'c'} =$$

2. the model above was extended to the 1−dimensional case:

   I. Klebanov, G. Tarnopolsky, arXiv:1611.08915 [hep-th], *Phys. Rev.* **D** (2017)

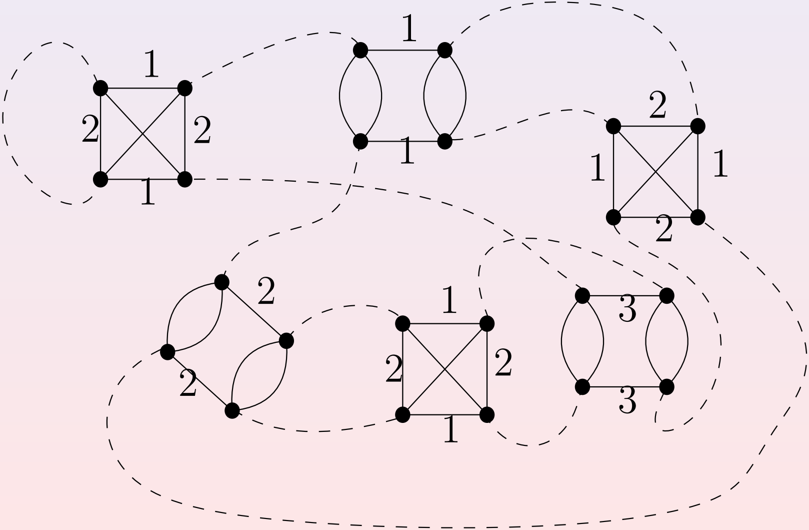
I. Klebanov, F. Popov, G. Tarnopolsky, TASI Lectures (2017)

The action of the model

The action of the quartic $O(N)^3$-invariant tenso model:

$$S_{CTKT}(\phi) = -\frac{N^2}{2}\phi^2 + N^{5/2}\frac{\lambda_1}{4}I_t(\phi) + N^2\frac{\lambda_2}{4}\Big(I_{p,1}(\phi) + I_{p,2}(\phi) + I_{p,3}(\phi)\Big)$$

An example of Feynman graph of the model



The large N limit expansion

The free energy admits a large N expansion

$$F_N(\lambda_1, \lambda_2) = \ln Z_N(\lambda_1, \lambda_2) = \sum_{\mathcal{G} \in \bar{\mathbb{G}}} N^{3-\omega(\mathcal{G})} \mathcal{A}(\mathcal{G}). \qquad (2)$$

where the degree is:

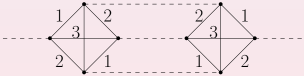
$$\omega(\mathcal{G}) = 3 + \frac{3}{2} n_t(\mathcal{G}) + 2 n_p(\mathcal{G}) - F(\mathcal{G}) \qquad (3)$$

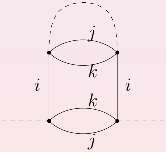
Two types of LO graphs

$$\omega(\mathcal{G}) = 3 + \frac{3}{2} n_t(\mathcal{G}) + 2 n_p(\mathcal{G}) - F(\mathcal{G})$$

Dominant graphs: $\omega(\mathcal{G}) = 0$

two types of interaction $\rightarrow$ two types of melonic graphs:

Type I: 

Type II: 

"melon-tadpoles" graphs

Back to double scaling limit - again on schemes

V. Bonzom, V. Nador and A.T., *J. Phys.* **A** (2022)

- Recall that a scheme (of degree ω) is a "blueprint" that tells us how to obtain graphs of the same degree ω.

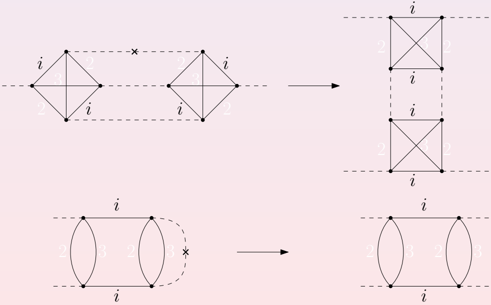
Recall the general idea: Identify operations that leave the degree invariant and use them to repackage all the graphs that differ only by the applications of these operations

Melonic moves are such graphic operations.

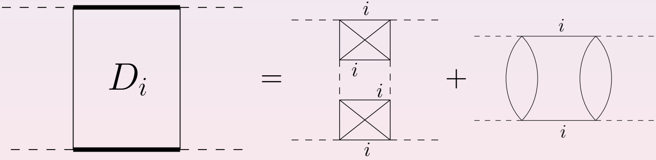
Dipoles

Definition

A dipole is a 4-point graph obtained by cutting an edge in an elementary melon.



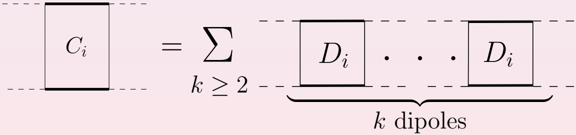
Dipoles



Chains - ladder diagrams

Definition

Chains are the 4-point functions obtained by connecting an arbitrary number of dipoles.



Schemes

Definition

The scheme $\mathcal{S}$ of a 2-point graph $\mathcal{G}$ is obtained by

1. Removing all melonic 2-point subgraphs in $\mathcal{G}$
2. Replacing all maximal chains with chain-vertices and all dipoles with dipole-vertex of the same color.

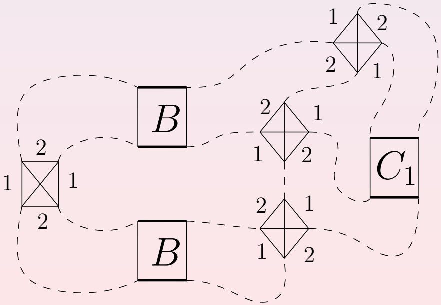


Figure: An example of scheme

Finiteness of the number of schemes

Theorem

(Bonzom-Nador-Tanasa (2022))
The set of schemes of a given degree is finite in the quartic $O(N)^3$-invariant tensor model.

Generating function of dominant scheme

The generating function associated to a dominant schemes is

$$G_{\mathcal{T}}^{\omega}(t,\mu) = (3t^{\frac{1}{2}})^{2\omega}(1+6t)^{2\omega-1}B(t,\mu)^{4\omega-1}$$
$$= (3t^{\frac{1}{2}})^{2\omega}(1+6t)^{2\omega-1}\frac{6^{4\omega-1}U^{8\omega-2}}{((1-U)(1-3U))^{4\omega-1}}$$

where B is the generation functions of broken chains and U is th generation function of dipoles.

Summing over the different trees
(in bijection with the dominant schemes):

$$G_{\text{dom}}^{\omega}(t,\mu) = \text{Cat}_{2\omega-1}\, M(t,\mu) G_{\mathcal{T}}^{\omega}(t,\mu)$$

where M is the generation functions of melonic graphs.

Double scaling parameter

Near critical point

$$G^{\omega}_{dom}(t,\mu) \underset{t\to t_c(\mu)}{\sim} N^{3-\omega} M_c(\mu)\,\mathrm{Cat}_{2\omega-1}\,9^{\omega}\,t_c^{\omega}\,(1+6t_c)^{2\omega-1}$$
$$\times\left(\frac{1}{\left(1-\frac{4}{3}t_c(\mu)\mu M_c(\mu)\right)K(\mu)\sqrt{1-\frac{t}{t_c(\mu)}}}\right)^{4\omega-1}$$

- The double scaling parameter $\kappa(\mu)$ is the quantity to hold fixed when sending $N\to+\infty, t\to t_c(\mu)$.
- dominant schemes of all degree ω contribute in the double scaling limit

One has

$$\kappa(\mu)^{-1}=\frac{1}{3}\frac{1}{t_c(\mu)^{\frac{1}{2}}\,(1+6t_c(\mu))}\left(\left(1-\frac{4}{3}t_c(\mu)\mu M_c(\mu)\right)K(\mu)\right)^2\left(1-\frac{t}{t_c(\mu)}\right)N^{\frac{1}{2}} \qquad (4)$$

2−point function in the double scaling limit

$$G_2^{DS}(\mu) = N^{-3} \sum_{\omega \in \mathbb{N}/2} G_{dom}^{\omega}(\mu)$$

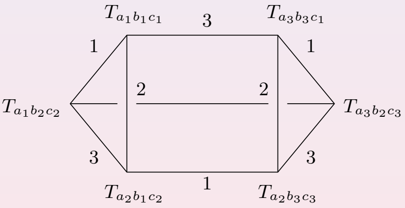
$$= M_c(\mu) \left(1 + N^{-\frac{1}{4}} \sqrt{3} \frac{t_c(\mu)^{\frac{1}{4}}}{(1+6t_c(\mu))^{\frac{1}{2}}} \frac{1 - \sqrt{1-4\kappa(\mu)}}{2\kappa(\mu)^{\frac{1}{2}}} \right)$$

convergent for $\kappa(\mu) \leq \frac{1}{4}$.

tensor double scailing limit is summable
(different behaviour with respect to the celebrated matrix models case)

The prismatic tensor model

$O(N)^3$-invariance, 6th order interaction



T. Krajewski, T. Muller and A. T. arXiv:2301.02093

Definition of the model

model introduced in

S. Giombi, I. Klebanov, F. Popov, S. Prakash, G. Tarnopolsky, *Phys. Rev.* **D** (2018)

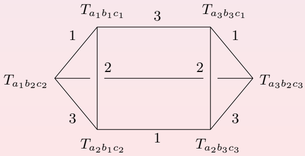
$O(N)^3$ invariance

$$T_{i_1 i_2 i_3} = O^{(1)}_{i_1 j_1} O^{(2)}_{i_1 j_1} O^{(3)}_{i_1 j_1} T_{j_1 j_2 j_3}$$

The action

$$S(T) = -\frac{1}{2} \sum_{i,j,k} T_{ijk} T_{ijk}$$

$$+\frac{tN^{-3}}{6} \sum_{a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3} T_{a_1 b_1 c_1} T_{a_1 b_2 c_2} T_{a_2 b_1 c_2} T_{a_3 b_3 c_1} T_{a_3 b_2 c_3} T_{a_2 b_3 c_3}$$



(generalization of S_{CTKT})

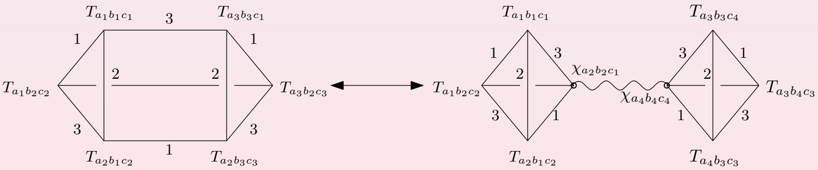
Intermediate field method

the prismatic interaction term rewrites

$$\int \frac{[d\chi]}{(2\pi)^{N^3/2}} e^{-\frac{1}{2}\sum_{i,j,k=1}^{N} \chi_{ijk}\chi_{ijk} + \sqrt{\frac{2tN^{-\alpha}}{6}}\tilde{I}_t(T,\chi)}, \qquad (5)$$

where

$$\tilde{I}_t(T,\chi) = \sum_{a_1,a_2,b_1,b_2,c_1,c_2=1}^{N} T_{a_1b_1c_1} T_{a_1b_2c_2} T_{a_2b_1c_2} \chi_{a_2b_2c_1}. \qquad (6)$$

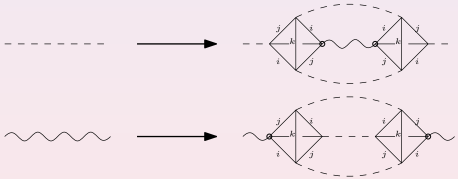


tetrahedric representation (of the prismatic model)

Melonic insertions in the tetrahedric representation

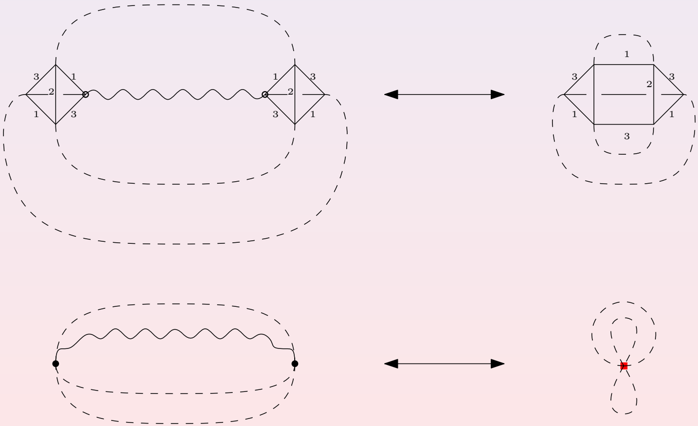
vacuum elementary melon:

2 types of melonic insertions:



Leading order graphs in the tetrahedric representation

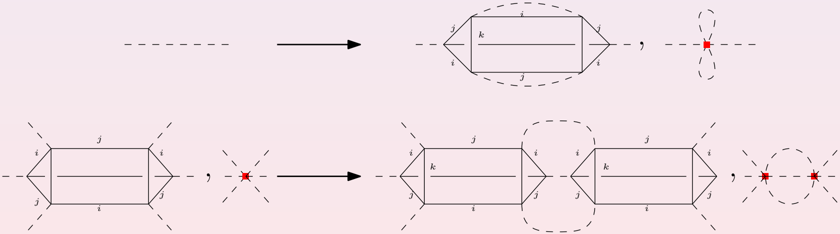
elementary melon of the tetrahedric representation
$\rightarrow$ *elementary triple tadpole*



Melonic moves in the prismatic representation

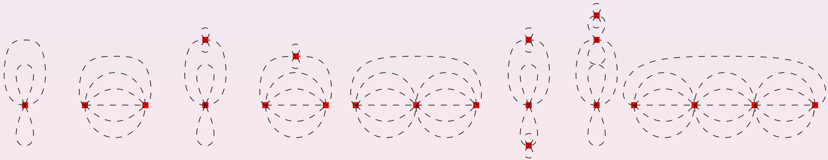
insertion on a T propagator
$\rightarrow$ insertion of a 2-point double tadpole

insertion on χ propagator $\rightarrow$ insertion at the level of a prismatic vertex(split a vertex into 2vertices)



same result as in S. Prakash and R. Sinha, *Phys. Rev.* **D** (2020)
(where no intermediate field approach was used)

Examples of LO graphs in the prismatic representation



Implementation of the double scaling limit mechanism

T. Krajewski, T. Muller and A. T. arXiv:2301.02093[hep-th]

use of the tetrahedric representation

much more teadious then for S_{CTKT}:

- 5 types of dipoles
- a bunch of types of chains
- much more involved structure of the schemes

double scaling parameter

$$\kappa(t, N) = \frac{I(t_c)L(t_c)}{4NM_{T,c}^2 K^2(1 - \frac{t}{t_c})}$$

2-point function in the double scaling limit

$$G_{2,DS}(t,N) = M_{T,c} + \sum_{\omega>0} N^{-\omega} G_{\omega,\text{dom}}$$
$$= M_{T,c} + M_{T,c} N^{-\frac{1}{2}} \left(\frac{L(t_c)\kappa(t,N)}{I(t_c)}\right)^{1/2} \sum_{\omega\in\mathbb{N}^*} \text{Cat}_{\omega-1}\kappa(t,N)^{\omega}$$
$$= M_{T,c}\left(1 + N^{-\frac{1}{2}}\left(\frac{L(t_c)}{I(t_c)}\right)^{1/2} \frac{1-\sqrt{1-4\kappa(t,N)}}{2\kappa(t,N)^{1/2}}\right) \quad (7)$$

Some final comments

- contributions of all degrees, and not just from the vanishing degree (the higher it is the degree of the graph, the greater it is the contribution from the respective degree)

- in the limit $\kappa \to 0$ the large N limit is recovered.

- the double scaling limit series is convergent (differnce wrt matrix models)

Implementation of this approach for multi-matrix models

double/triple-scaling limit mechanism

- $U(N)^2 \times O(D)$, tetrahedric interation, multi-matrix models

  F. Ferrari, arXiv:1701.01171, *Annales IHP* **D** *Comb., Phys. and their Interactions*

  D. Benedetti *et. al., Annales IHP* **D** *Comb., Phys. and their Interactions* 2022)

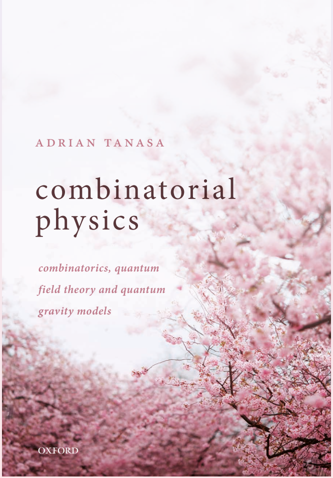
- generalized interactions (all invariant quartic interactions) for multi-matrix models

  V. Bonzom, V. Nador, A. T., arXiv:2209.02026*J. Phys.* **A** (2023)

Take away message

- purely combinatorial techniques can be used to study physical mechanisms, such as the double scaling limit for various tensor and multi-matrix models

A very good book on all these topics



A. T., "Combinatorial Physics", Oxford Univ. Press (2021)

Je vous remercie pour votre attention !

Vă mulțumesc pentru atenție!

Comparison with the colored case

The dominant schemes differ:

for the colored model, for degree $\omega \in \mathbb{Z}_+$, the dominant schemes are associated to rooted binary trees with $\omega + 1$ leaves (and $\omega - 1$ inner nodes), where the root-leaf is occupied by a root-melon, while the ω non-root leaves are occupied by the unique scheme of degree 1.