

Towards

An Institution for Graph Transformation

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institutions vs. DPO rewr.

— [Institutions: abstract model theory

— sentences, models (and satisfiability)

— [DPO rewriting: abstract rewriting formalism

— rules, rule applications (and replacement)

— lacking both sentences and models
(hence, lacking an entailment system)

shortly, the DPO approach

$$\text{Grp}(\Sigma) = \text{Grp} \downarrow \Sigma$$

a rule

arrows in $\text{Grp}(\Sigma)$

$$L \xleftarrow{l} K \xrightarrow{r} R$$

a derivation step

$$\begin{array}{ccccc} L & \xleftarrow{l} & K & \xrightarrow{r} & R \\ \downarrow m_L & & \downarrow m_K & & \downarrow m_R \\ G & \xleftarrow{l^*} & D & \xrightarrow{r^*} & H \end{array}$$

(1) (2)

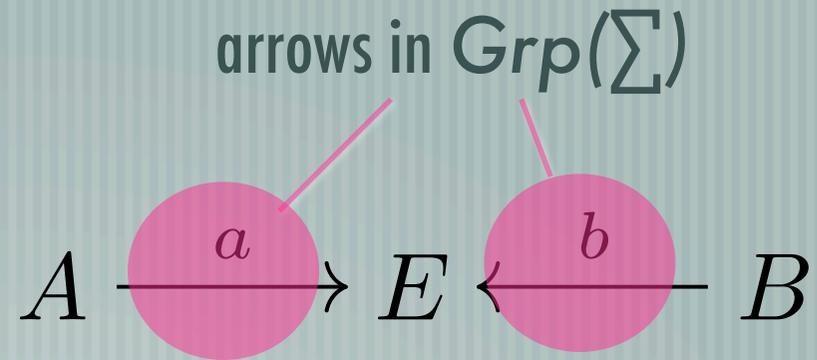
pushouts in $\text{Grp}(\Sigma)$

set of theoretical tools
(concurrency, mostly)
[holding for adhesive cats]

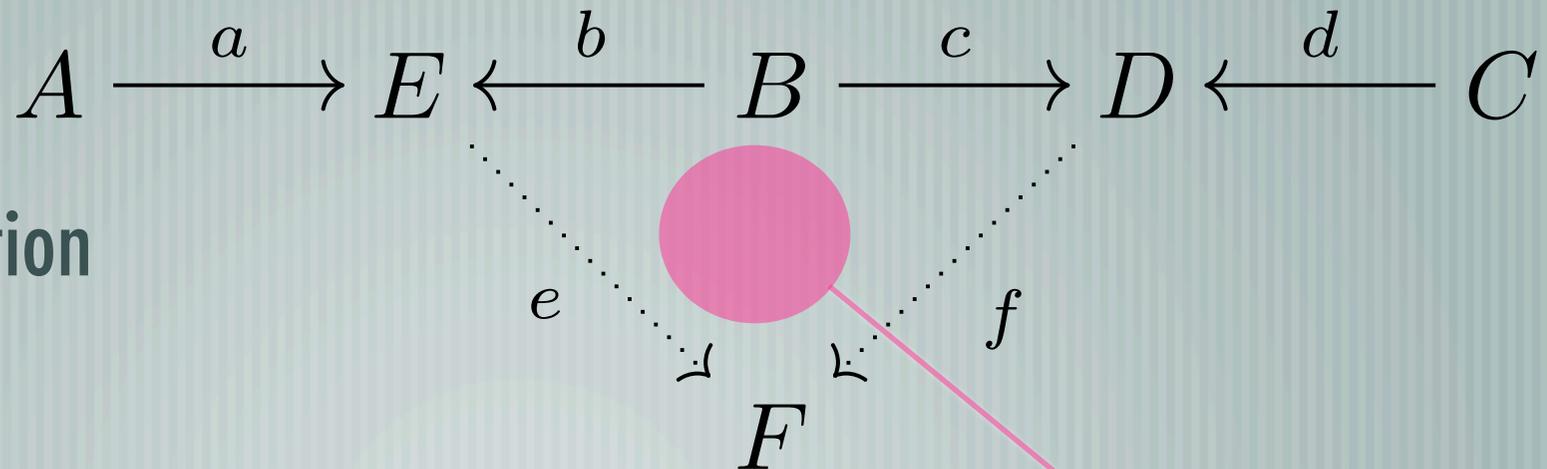
the CoSpan (bi-)category

[holding for any cocomplete cat]

an arrow



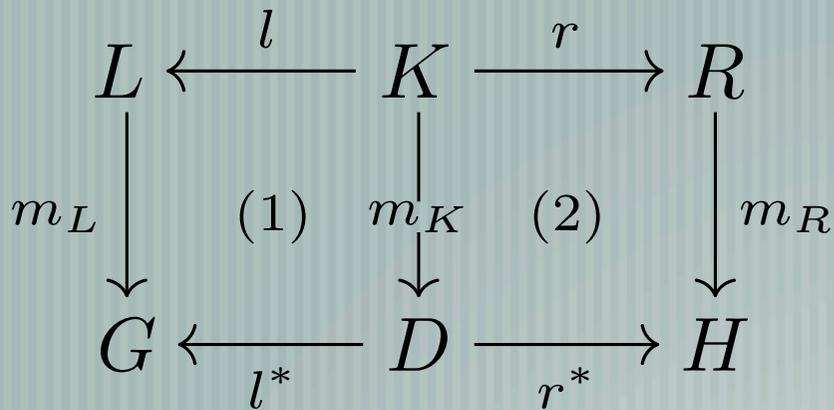
composition



powerful categorical tool
(cats of relations)

pushout in $Grp(\Sigma)$

connecting with the DPO



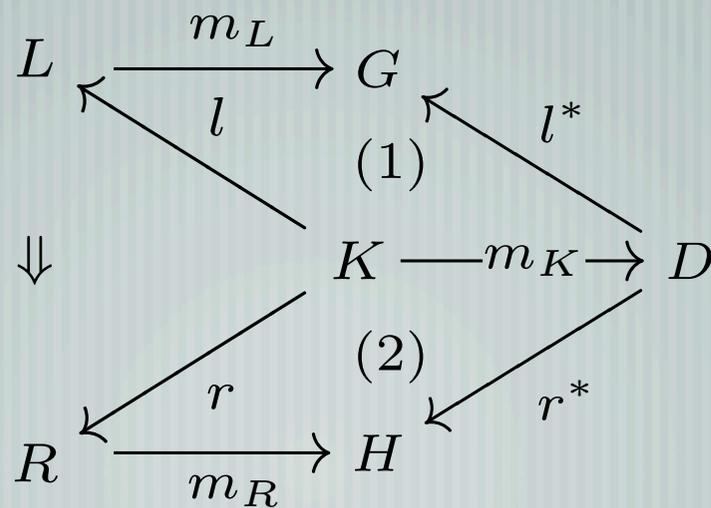
a rule

a derivation step

operational vs.
induction-based

a "cell"

\emptyset



\emptyset

"whiskering"

DPO vs. CoSpans

— [(A factorized sub-category of) Cospans over graphs (typed over Σ) form the free compact-closed category built from Σ (with operators as basic arrows)

— [The DPO approach is operational: search for the match, build the PO complement...

— [The free construction (concretely, via cospans) is algebraic: inductive closure of a set of basic rules

first step: inductive sentences

[$DGSTh(\Sigma)$ is the self-dual, free symmetric (strict) monoidal category equipped with symmetric monoidal transformations

$$\nabla_a : a \rightarrow a \otimes a \qquad !_a : a \rightarrow e$$

[(intuitively representing pairing tuple $\langle x, x \rangle$ and empty tuple)

[plus two additional laws (relating transfs. and their dual)

main correspondence result

I) (Isomorphic classes of) Cospans over graphs (typed on Σ) and sets of nodes as objects \ast 1-1 correspond to \ast arrows in $DGSTh(\Sigma)$ (indeed, a categorical equivalence)

— You abstract the identity of nodes not in the interface

— ...but this way graphs get a “standard” notion of sentence

II) The preorder on arrows obtained by replacing each DPO rule with an order on graphs \ast 1-1 corresponds to \ast DPO rewrites

— This way DPO rewriting gets an entailment system

very shortly, institutions

— [A category $Sign$ of specifications

— [a functor $Sen: Sign \rightarrow Set$ for sentences

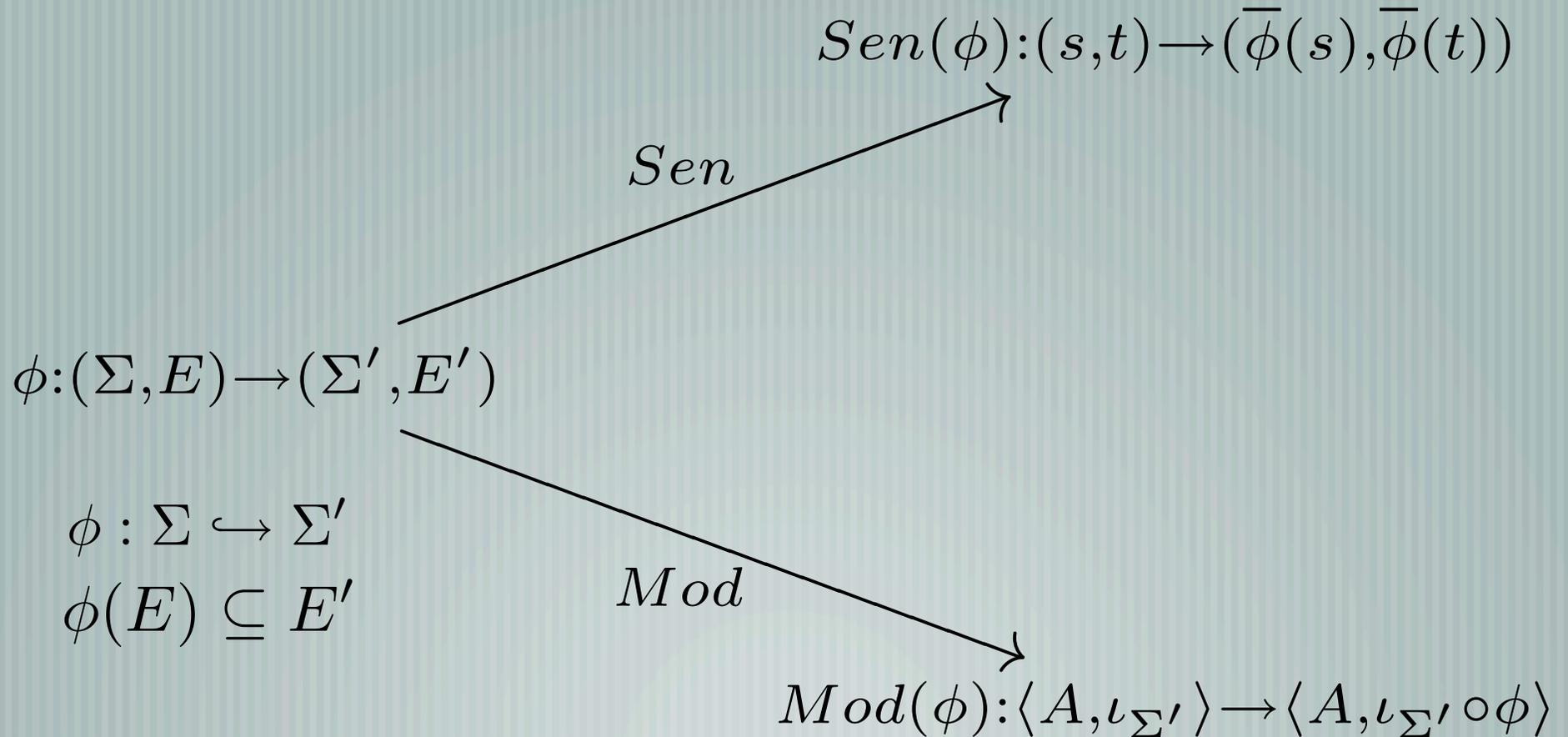
— [a functor $Mod: Sign \rightarrow Cat^{op}$ for models

— [a (satisfiability) relation \models_{Σ} on $|Mod(\Sigma)| \times Sen(\Sigma)$

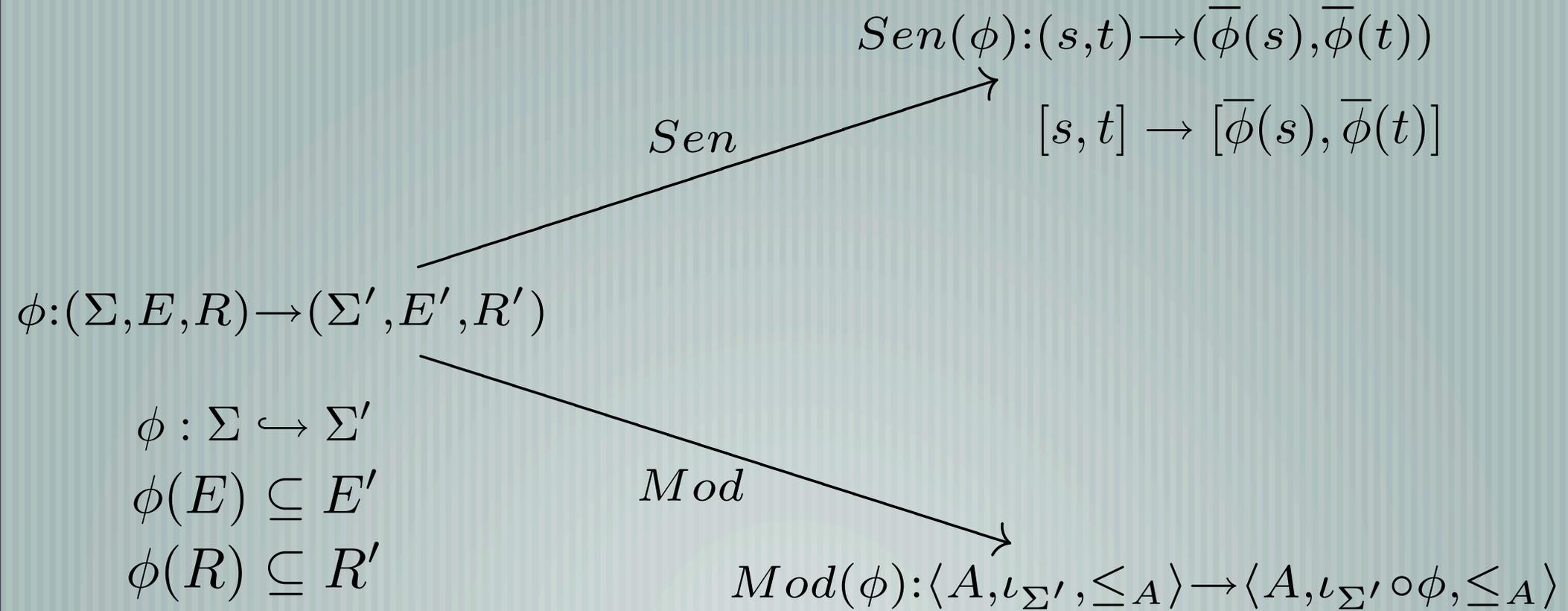
— [a coherence axiom $\forall \phi: \Sigma \rightarrow \Sigma', e \in Sen(\Sigma), M' \in Mod(\Sigma')$

$$M' \models_{\Sigma'} Sen(\phi)(e) \Leftrightarrow Mod(\phi)(M') \models_{\Sigma} e$$

institution for algebraic specs.



Institution for rewriting specs.



the easy way out...

- [exploit the categorical laws...

- sentences as pairs of arrows in $DGSTh(\Sigma)$ (same homs.)

- models as dgs-monoidal categories

- obvious satisfiability

- reductions via order enrichment

- [unsatisfactory: looking for a “concrete” model characterisation, in terms of “classical” algebraic models (algebras for specs.)

a functorial detour

— [The algebraic theory $Th(\Sigma)$ is concretely defined as

— lists of vars as objects, (tuples of) typed terms as arrows

— term substitution as composition

— [(the theory is also the free cartesian category over Σ)

— [Algebras over Σ and axioms in E as functors

$$M \in [Th(\Sigma) \rightarrow Set]_E^\times$$

— [product and axioms preserving (homs as natural transfs.)

a functorial detour, II

— [PreAlgebras as rule-preserving functors

$$M \in [Th(\Sigma) \rightarrow Pre]_{E,R}^{\times}$$

$$s \rightarrow t \in R \Leftrightarrow \forall X. M(s) \leq M(t)$$

— (still homomorphisms as natural transformations)

— [How to generalize? Note that functors

$$M \in [Th(\Sigma) \rightarrow Rel]_E^{\times}$$

— [still define algebras!!

alternative take on $Th(\Sigma)$

— [$Th(\Sigma)$ is the free symmetric (strict) monoidal category equipped with symmetric monoidal natural transformations

$$\nabla_a : a \rightarrow a \otimes a \qquad !_a : a \rightarrow e$$

— [(intuitively representing pairing tuple $\langle x, x \rangle$ and empty tuple)

explicit definition of a theory

$$\begin{array}{ccc}
 a & \xrightarrow{\nabla_a} & a \otimes a \\
 \downarrow s & & \downarrow s \otimes s \\
 b & \xrightarrow{\nabla_b} & b \otimes b
 \end{array}$$

$$\begin{array}{ccc}
 a & \xrightarrow{!_a} & e \\
 \downarrow s & & \downarrow e \\
 b & \xrightarrow{!_b} & e
 \end{array}$$

$$\begin{array}{ccc}
 a & \xrightarrow{\nabla_a} & a \otimes a \\
 \downarrow \nabla_a & & \downarrow a \otimes \nabla_a \\
 a \otimes a & \xrightarrow{\nabla_a \otimes a} & a \otimes a \otimes a
 \end{array}$$

$$\begin{array}{ccc}
 a & \xrightarrow{\nabla_a} & a \otimes a \\
 \searrow \nabla_a & & \downarrow \gamma_{a,a} \\
 & & a \otimes a
 \end{array}$$

$$\begin{array}{ccc}
 a \otimes b & \xrightarrow{\nabla_a \otimes \nabla_b} & a \otimes a \otimes b \otimes b \\
 \searrow \nabla_{a \otimes b} & & \downarrow a \otimes \gamma_{a,b} \otimes b \\
 & & a \otimes b \otimes a \otimes b
 \end{array}$$

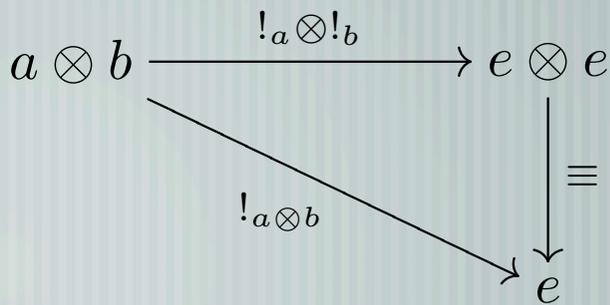
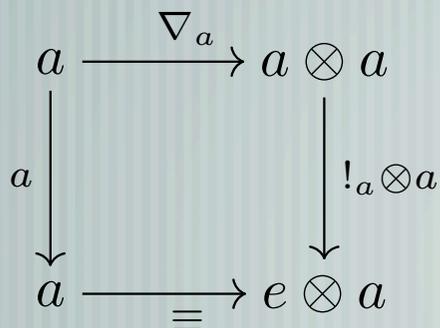
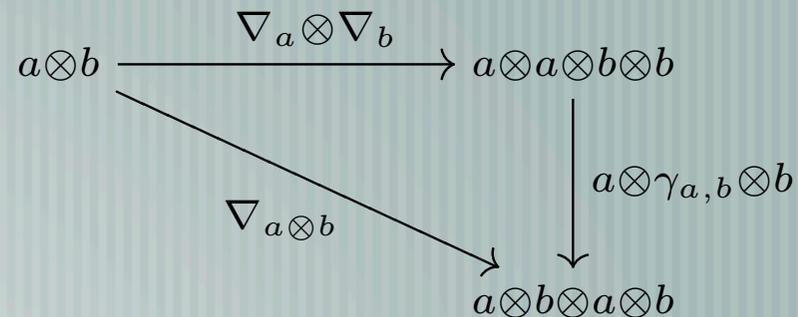
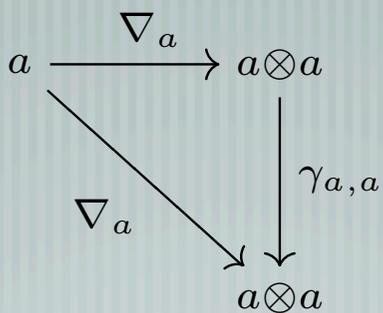
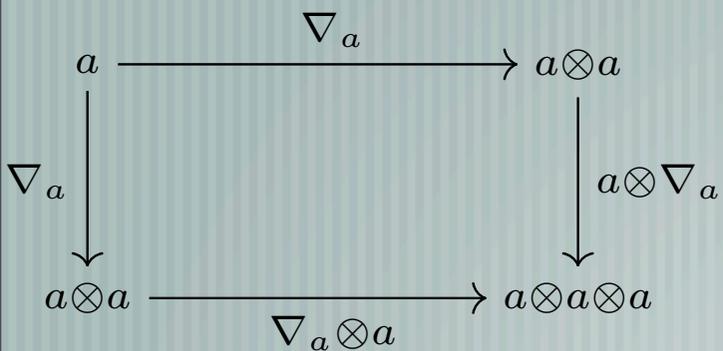
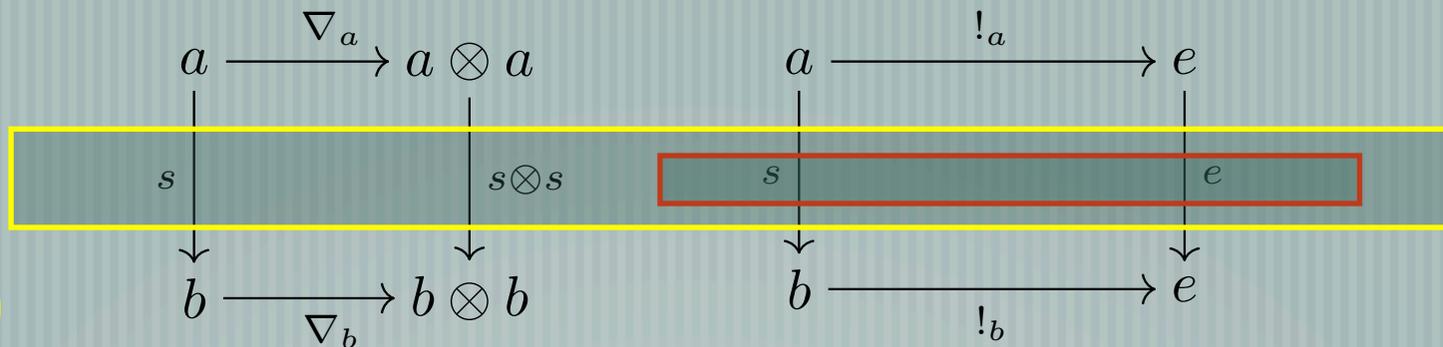
$$\begin{array}{ccc}
 a & \xrightarrow{\nabla_a} & a \otimes a \\
 \downarrow a & & \downarrow !_a \otimes a \\
 a & \xrightarrow{\equiv} & e \otimes a
 \end{array}$$

$$\begin{array}{ccc}
 a \otimes b & \xrightarrow{!_a \otimes !_b} & e \otimes e \\
 \searrow !_a \otimes b & & \downarrow \equiv \\
 & & e
 \end{array}$$

two alternative takes

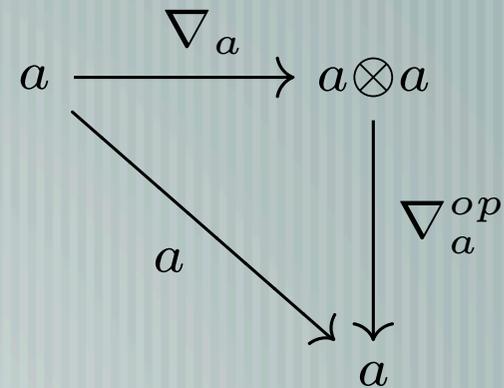
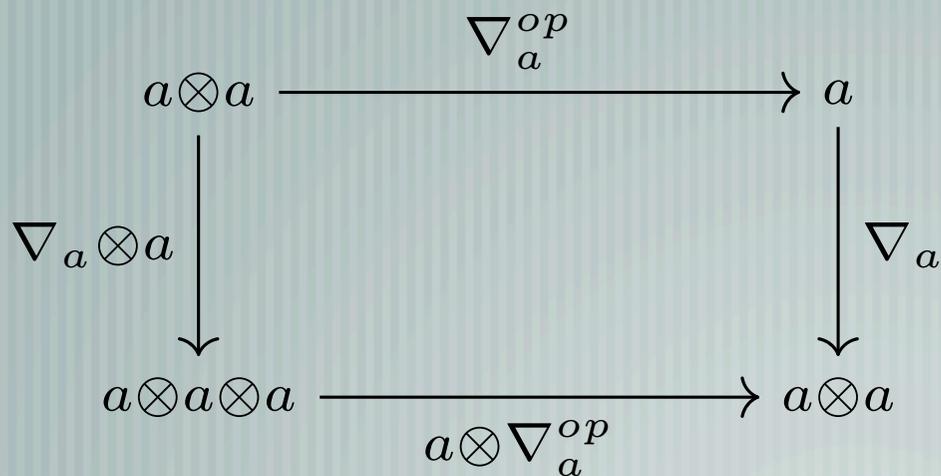
$GSTh(\Sigma)$

$GTh(\Sigma)$



another alternative take

[$DGSTh(\Sigma)$ as self-dual $GSTh(\Sigma)$ satisfying



some characterization results

— [arrows in $DGSTh(\Sigma)$ are (isomorphic classes of) cospans of graphs (typed over Σ)

— [arrows in $GSTh(\Sigma)$ are (isomorphic classes of) cospans of term graphs (typed over Σ)

— [arrows in $GTh(\Sigma)$ are conditioned terms $s \mid D$ (over Σ)

— s a term (the functional)

— D a sub-term closed set of terms (the domain restriction)

functorial characterizations

Partial algebras with \perp -preserving operators, tight homomorphisms and conditioned Kleene (in)equations

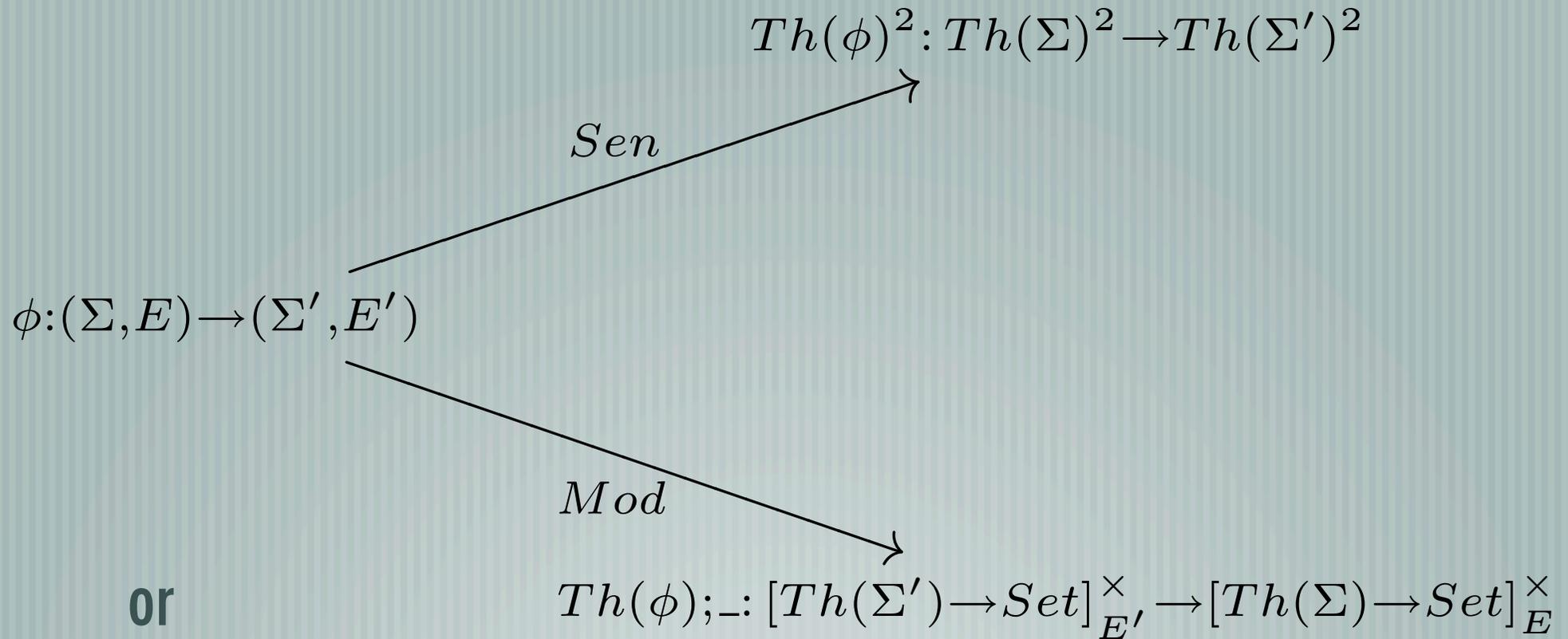
Multialgebras with tight point-to-set operators, tight point-to-point homomorphisms and “term graph” (in)equations

Multialgebras with tight point-to-set operators, tight point-to-point homomorphisms and “graph” (in)equations

$$[GTh(\Sigma) \rightarrow Set_{\perp}]_E^{\times} \qquad [DGSTh(\Sigma) \rightarrow 2^{Set}]_E^{\times}$$

$$[GSTh(\Sigma) \rightarrow 2^{Set}]_E^{\times}$$

back to institutions



or

$$[GTh(\Sigma) \rightarrow Set_{\perp}]_E^{\times}$$

or

$$[(D)GSTh(\Sigma) \rightarrow 2^{Set}]_E^{\times}$$

on entailment systems

— [Claim: complete entailment system for partial algebras

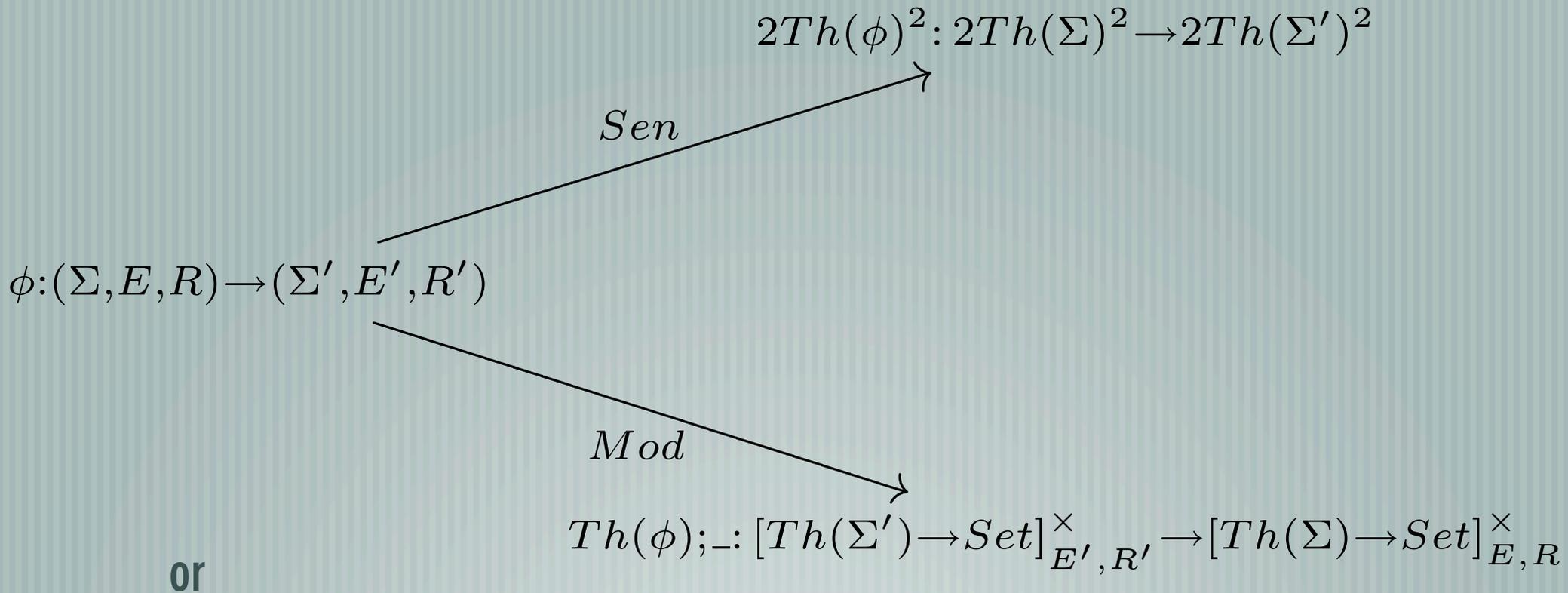
$$\frac{s \mid D_s \equiv t \mid D_t}{s \mid D_s \cup D \equiv t \mid D_t \cup D} \quad \frac{u_i \mid D_u \quad (s \mid D_s, t \mid D_t) \in E}{s[\bar{u}/\bar{x}] \mid D_s[\bar{u}/\bar{x}] \cup D_u \equiv t[\bar{u}/\bar{x}] \mid D_t[\bar{u}/\bar{x}] \cup D_u}$$

— [Conjecture: complete entailment system for multi-algebras

$GSTh(\Sigma)$ plus

$$\begin{array}{ccc} a & \xrightarrow{\nabla_a} & a \otimes a \\ s \downarrow & & \downarrow s \otimes s \\ b & \xleftarrow{!_a \otimes a} & b \otimes b \end{array}$$

back to insts. on preorders



$[GTh(\Sigma) \rightarrow Pre_{\perp}]_{E, R}^\times$

NOT! the bottom
 of the preorder

Smyth
 power-domain

$X \leq Y \Leftrightarrow \forall y \in Y. \exists x \in X. x \leq y$

or $[(D)GSTh(\Sigma) \rightarrow 2^{Pre}]_{E, R}^\times$

preliminary conclusions

— [uniform presentation of institutions for the DPO rewriting formalism over various graph-like structures

— [sound and complete “abstract” entailment systems

— [sound (possibly complete) “concrete” entailment systems

to be addressed...

- [completeness for the entailment system

- (rewriting) interpretation for up-to garbage law

- [tackling hyper-graphs and hyper-signature

- (singular vs plural) interpretation for hyper-operators

- [considering cospans of adhesive categories

- free construction for suitable algebraic varieties