Towards An Institution for Graph Transformation

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institutions vs. DPO rewr.

- Institutions: abstract model theory
  - sentences, models (and satisfiability)
- DPO rewriting: abstract rewriting formalism
  - rules, rule applications (and replacement)
    - lacking both sentences and models
      (hence, lacking an entailment system)

shortly, the DPO approach

$Grp(\sum) = Grp\downarrow\sum$

arrows in $Grp(\sum)$

a rule

$$L \xleftarrow{l} K \xrightarrow{r} R$$

a derivation step

$$\begin{tikzcd}
L \arrow[d, "m_L"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "m_K"] \arrow[dl, phantom, "(1)"] \arrow[dr, phantom, "(2)"] & R \arrow[d, "m_R"] \\
G & D \arrow[l, "l^*"] \arrow[r, "r^*"'] & H
\end{tikzcd}$$

set of theoretical tools
(concurrency, mostly)
[holding for adhesive cats]

pushouts in $Grp(\sum)$

the CoSpan (bi-)category

[holding for any cocomplete cat]

arrows in $Grp(\Sigma)$

an arrow

$$A \xrightarrow{a} E \xleftarrow{b} B$$

composition

$$\begin{tikzcd}
A \arrow[r, "a"] & E \arrow[dr, dotted, "e"'] & B \arrow[l, "b"'] \arrow[r, "c"] & D \arrow[dl, dotted, "f"] & C \arrow[l, "d"'] \\
 & & F & &
\end{tikzcd}$$

powerful categorical tool
(cats of relations)

pushout in $Grp(\Sigma)$

connecting wth the DPO

$$
\begin{tikzcd}
L \arrow[d, "m_L"'] \arrow[dr, phantom, "(1)"] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "m_K"] \arrow[dr, phantom, "(2)"] & R \arrow[d, "m_R"] \\
G & D \arrow[l, "l^*"] \arrow[r, "r^*"'] & H
\end{tikzcd}
$$

a rule

a derivation step

operational vs.
induction-based

a “cell”

$$
\begin{tikzcd}
 & L \arrow[r, "m_L"] & G & & \\
\emptyset & \Downarrow & K \arrow[ul, "l"] \arrow[dl, "r"'] \arrow[r, "m_K"] \arrow[u, phantom, "(1)"] \arrow[d, phantom, "(2)"] & D \arrow[ul, "l^*"'] \arrow[dl, "r^*"] & \emptyset \\
 & R \arrow[r, "m_R"'] & H & &
\end{tikzcd}
$$

“whiskering”

DPO vs. CoSpans

- (A factorized sub-category of) Cospans over graphs (typed over $\sum$) form the free compact-closed category built from $\sum$ (with operators as basic arrows)
- The DPO approach is operational: search for the match, build the PO complement...
- The free construction (concretely, via cospans) is algebraic: inductive closure of a set of basic rules

first step: inductive sentences

- *DGSTh*(∑) is the self-dual, free symmetric (strict) monoidal category equipped with symmetric monoidal transformations

$$\nabla_a : a \to a \otimes a \qquad\qquad !_a : a \to e$$

- (intuitively representing pairing tuple <*x*,*x*> and empty tuple)
- plus two additional laws (relating transfs. and their dual)

main correspondence result

- I) (Isomorphic classes of) Cospans over graphs (typed on ∑) and sets of nodes as objects *1-1 correspond to* arrows in *DGSTh*(∑) (indeed, a categorical equivalence)
  - You abstract the identity of nodes not in the interface
  - ...but this way graphs get a “standard” notion of sentence
- II) The preorder on arrows obtained by replacing each DPO rule with an order on graphs *1-1 corresponds to* DPO rewrites
  - This way DPO rewriting gets an entailment system

very shortly, institutions

- A category *Sign* of specifications
- a functor *Sen*: *Sign* ➞ *Set* for sentences
- a functor *Mod*: *Sign* ➞ Cat^{op} for models
- a (satisfiability) relation $\models_\Sigma$ on $|Mod(\Sigma)| \times Sen(\Sigma)$
- a coherence axiom $\forall \phi : \Sigma \to \Sigma', e \in Sen(\Sigma), M' \in Mod(\Sigma')$

$$M' \models_{\Sigma'} Sen(\phi)(e) \Leftrightarrow Mod(\phi)(M') \models_\Sigma e$$

istitution for algebraic specs.

$$
\begin{tikzcd}
 & Sen(\phi){:}(s,t)\to(\overline{\phi}(s),\overline{\phi}(t)) \\
\phi{:}(\Sigma,E)\to(\Sigma',E') \arrow[ur, "Sen"] \arrow[dr, "Mod"'] & \\
 & Mod(\phi){:}\langle A,\iota_{\Sigma'}\rangle\to\langle A,\iota_{\Sigma'}\circ\phi\rangle
\end{tikzcd}
$$

$$\phi : \Sigma \hookrightarrow \Sigma'$$

$$\phi(E) \subseteq E'$$

istitution for rewriting specs.

$$\begin{tikzcd}
& Sen(\phi){:}(s,t)\rightarrow(\overline{\phi}(s),\overline{\phi}(t)) \\
\phi{:}(\Sigma,E,R)\rightarrow(\Sigma',E',R') \arrow[ur, "Sen"] \arrow[dr, "Mod"'] & \\
& Mod(\phi){:}\langle A,\iota_{\Sigma'},\leq_A\rangle\rightarrow\langle A,\iota_{\Sigma'}\circ\phi,\leq_A\rangle
\end{tikzcd}$$

$$[s,t] \rightarrow [\overline{\phi}(s),\overline{\phi}(t)]$$

$$\phi : \Sigma \hookrightarrow \Sigma'$$

$$\phi(E) \subseteq E'$$

$$\phi(R) \subseteq R'$$

the easy way out...

- exploit the categorical laws...
  - sentences as pairs of arrows in $DGSTh(\Sigma)$ (same homs.)
  - models as dgs-monoidal categories
    - obvious satisfiability
    - reductions via order enrichment
- unsatisfactory: looking for a "concrete" model characterisation, in terms of "classical" algebraic models (algebras for specs.)

a functorial detour

- The algebraic theory *Th*(∑) is concretely defined as
  - lists of vars as objects, (tuples of) typed terms as arrows
  - term substitution as composition
- (the theory is also the free cartesian category over ∑)
- Algebras over ∑ and axioms in E as functors

$$M \in [Th(\Sigma) \to Set]_E^{\times}$$

- product and axioms preserving (homs as natural transfs.)

a functorial detour, II

- PreAlgebras as rule-preserving functors

$$M \in [Th(\Sigma) \to Pre]^{\times}_{E,R}$$

$$s \to t \in R \Leftrightarrow \forall X.\, M(s) \leq M(t)$$

  - (still homomorphisms as natural transformations)

- How to generalize? Note that functors

$$M \in [Th(\Sigma) \to Rel]^{\times}_{E}$$

- still define algebras!!

alternative take on *Th*(∑)

- *Th*(∑) is the free symmetric (strict) monoidal category equipped with symmetric monoidal natural transformations

$$\nabla_a : a \to a \otimes a \qquad\qquad !_a : a \to e$$

- (intuitively representing pairing tuple <*x*,*x*> and empty tuple)

explicit definition of a theory

$$\begin{tikzcd}
a \arrow[r, "\nabla_a"] \arrow[d, "s"'] & a \otimes a \arrow[d, "s \otimes s"] \\
b \arrow[r, "\nabla_b"'] & b \otimes b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "!_a"] \arrow[d, "s"'] & e \arrow[d, "e"] \\
b \arrow[r, "!_b"'] & e
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[r, "\nabla_a"] \arrow[d, "\nabla_a"'] & a \otimes a \arrow[d, "a \otimes \nabla_a"] \\
a \otimes a \arrow[r, "\nabla_a \otimes a"'] & a \otimes a \otimes a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\nabla_a"] \arrow[dr, "\nabla_a"'] & a \otimes a \arrow[d, "\gamma_{a,a}"] \\
 & a \otimes a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \otimes b \arrow[r, "\nabla_a \otimes \nabla_b"] \arrow[dr, "\nabla_{a \otimes b}"'] & a \otimes a \otimes b \otimes b \arrow[d, "a \otimes \gamma_{a,b} \otimes b"] \\
 & a \otimes b \otimes a \otimes b
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[r, "\nabla_a"] \arrow[d, "a"'] & a \otimes a \arrow[d, "!_a \otimes a"] \\
a \arrow[r, "\equiv"'] & e \otimes a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \otimes b \arrow[r, "!_a \otimes !_b"] \arrow[dr, "!_{a \otimes b}"'] & e \otimes e \arrow[d, "\equiv"] \\
 & e
\end{tikzcd}$$

two alternative takes

$$\begin{tikzcd} a \arrow[r, "\nabla_a"] \arrow[d, "s"'] & a \otimes a \arrow[d, "s \otimes s"] \\ b \arrow[r, "\nabla_b"'] & b \otimes b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "!_a"] \arrow[d, "s"'] & e \arrow[d, "e"] \\ b \arrow[r, "!_b"'] & e \end{tikzcd}$$

GSTh(∑) *GTh(∑)*

$$\begin{tikzcd} a \arrow[r, "\nabla_a"] \arrow[d, "\nabla_a"'] & a \otimes a \arrow[d, "a \otimes \nabla_a"] \\ a \otimes a \arrow[r, "\nabla_a \otimes a"'] & a \otimes a \otimes a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\nabla_a"] \arrow[dr, "\nabla_a"'] & a \otimes a \arrow[d, "\gamma_{a,a}"] \\ & a \otimes a \end{tikzcd} \qquad \begin{tikzcd} a \otimes b \arrow[r, "\nabla_a \otimes \nabla_b"] \arrow[dr, "\nabla_{a \otimes b}"'] & a \otimes a \otimes b \otimes b \arrow[d, "a \otimes \gamma_{a,b} \otimes b"] \\ & a \otimes b \otimes a \otimes b \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, "\nabla_a"] \arrow[d, "a"'] & a \otimes a \arrow[d, "!_a \otimes a"] \\ a \arrow[r, "\equiv"'] & e \otimes a \end{tikzcd} \qquad \begin{tikzcd} a \otimes b \arrow[r, "!_a \otimes !_b"] \arrow[dr, "!_{a \otimes b}"'] & e \otimes e \arrow[d, "\equiv"] \\ & e \end{tikzcd}$$

another alternative take

- $DGSTh(\sum)$ as self-dual $GSTh(\sum)$ satisfying

$$
\begin{tikzcd}
a \otimes a \arrow[r, "\nabla_a^{op}"] \arrow[d, "\nabla_a \otimes a"'] & a \arrow[d, "\nabla_a"] \\
a \otimes a \otimes a \arrow[r, "a \otimes \nabla_a^{op}"'] & a \otimes a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\nabla_a"] \arrow[dr, "a"'] & a \otimes a \arrow[d, "\nabla_a^{op}"] \\
& a
\end{tikzcd}
$$

some characterization results

- arrows in $\mathit{DGSTh}(\Sigma)$ are (isomorphic classes of) cospans of graphs (typed over Σ)
- arrows in $\mathit{GSTh}(\Sigma)$ are (isomorphic classes of) cospans of term graphs (typed over Σ)
- arrows in $\mathit{GTh}(\Sigma)$ are conditioned terms $s \mid D$ (over Σ)
  - s a term (the functional)
  - D a sub-term closed set of terms (the domain restriction)

functorial characterizations

- Partial algebras with $\perp$-preserving operators, tight homomorphisms and conditioned Kleene (in)equations
- Multialgebras with tight point-to-set operators, tight point-to-point homomorphisms and “term graph” (in)equations
- Multialgebras with tight point-to-set operators, tight point-to-point homomorphisms and “graph” (in)equations

$$[GTh(\Sigma) \to Set_{\perp}]_E^{\times} \qquad [DGSTh(\Sigma) \to 2^{Set}]_E^{\times}$$

$$[GSTh(\Sigma) \to 2^{Set}]_E^{\times}$$

back to institutions

$$\begin{tikzcd}
 & Th(\phi)^2 : Th(\Sigma)^2 \to Th(\Sigma')^2 \\
\phi:(\Sigma,E)\to(\Sigma',E') \arrow[ur, "Sen"] \arrow[dr, "Mod"] & \\
 & Th(\phi);_ : [Th(\Sigma')\to Set]^{\times}_{E'} \to [Th(\Sigma)\to Set]^{\times}_{E}
\end{tikzcd}$$

or

$$[GTh(\Sigma) \to Set_{\perp}]^{\times}_{E}$$

or $$[(D)GSTh(\Sigma) \to 2^{Set}]^{\times}_{E}$$

on entailment systems

- Claim: complete entailment system for partial algebras

$$\frac{s \mid D_s \equiv t \mid D_t}{s \mid D_s \cup D \equiv t \mid D_t \cup D} \qquad \frac{u_i \mid D_u \qquad (s \mid D_s, t \mid D_t) \in E}{s[\overline{u}/\overline{x}] \mid D_s[\overline{u}/\overline{x}] \cup D_u \equiv t[\overline{u}/\overline{x}] \mid D_t[\overline{u}/\overline{x}] \cup D_u}$$

- Conjecture: complete entailment system for multi-algebras

$GSTh(\sum)$ plus

$$\begin{tikzcd} a \arrow[r, "\nabla_a"] \arrow[d, "s"'] & a \otimes a \arrow[d, "s \otimes s"] \\ b & b \otimes b \arrow[l, "!_a \otimes a"] \end{tikzcd}$$

back to insts. on preorders

$$
\begin{tikzcd}[column sep=small]
 & 2Th(\phi)^2 \colon 2Th(\Sigma)^2 \to 2Th(\Sigma')^2 \\
\phi \colon (\Sigma, E, R) \to (\Sigma', E', R') \arrow[ur, "Sen"] \arrow[dr, "Mod"'] & \\
 & Th(\phi);\_ \colon [Th(\Sigma') \to Set]^{\times}_{E',R'} \to [Th(\Sigma) \to Set]^{\times}_{E,R}
\end{tikzcd}
$$

or

$$[GTh(\Sigma) \to Pre_{\perp}]^{\times}_{E,R}$$

NOT! the bottom of the preorder

Smyth power-domain $X \leq Y \Leftrightarrow \forall y \in Y. \exists x \in X. x \leq y$

or $[(D)GSTh(\Sigma) \to 2^{Pre}]^{\times}_{E,R}$

preliminary conclusions

- uniform presentation of institutions for the DPO rewriting formalism over various graph-like structures
- sound and complete “abstract” entailment systems
- sound (possibly complete) “concrete” entailment systems

to be addressed...

- completeness for the entailment system
  - (rewriting) interpretation for up-to garbage law
- tackling hyper-graphs and hyper-signature
  - (singular vs plural) interpretation for hyper-operators
- considering cospans of adhesive categories
  - free construction for suitable algebraic varieties