Noncommutative evolution equations : two-sided multipliers, initial conditions and asymptotics.

Non-commutative differential equations
and (some) infinite dimensional Lie Groups.

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Hoang Ngoc Minh, Q.H. Ngô.

Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
N. Behr, K. A. Penson, C. Tollu.

Combinatorics and Arithmetic for Physics
Special days, IHES, 9-10 November 2017

Goal of this talk

In this talk, I will show tools and sketch proofs about Noncommutative Evolution Equations (in particular, preparing Hoang Ngoc Minh's talk about associators).

The main data (not to say the only one) is that of Noncommutative Formal Power Series with variable coefficients which allows explore in a compact and effective (in the sense of computability) way the Hausdorff group of Lie exponentials and group-like series.

In particular we have an analogue of Wei-Norman theorem for these groups allowing to understand some multiplicative renormalisations (as those of Drinfeld). Some parts of this work are connected with Dyson series and take place within the project:

Evolution Equations in Combinatorics and Physics.

J. A. Lappo-Danilevskij (J. A. Lappo-Danilevsky), Mémoires sur la théorie des systémes des équations différentielles linéaires. Vol. I, *Travaux Inst. Physico-Math. Stekloff*, 1934, Volume 6, 1–256

§ 2. HYPERLOGARITHMES 159

§ 2. Hyperlogarithmes. En abordant la résolution algorithmique du problème de Poincaré, nous introduisons le système des fonctions

$$L_b(a_{j_1}, a_{j_2}, \ldots, a_{j_\nu}|x), \qquad (j_1, j_2, \ldots, j_\nu = 1, 2, \ldots, m;\ \nu = 1, 2, 3 \ldots)$$

définies par les relations de récurrence:

$$L_b(a_{j_1}|x) = \int_b^x \frac{dx}{x - a_{j_1}} = \log \frac{x - a_{j_1}}{b - a_{j_1}};$$

(10)

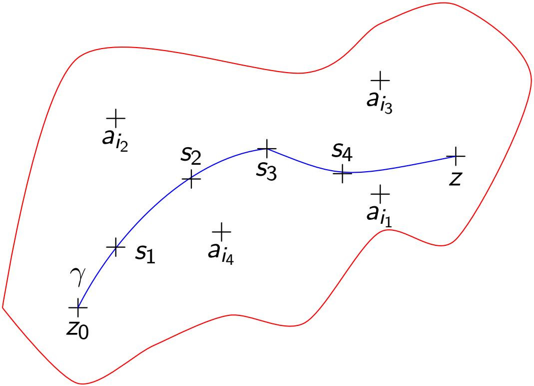
$$L_b(a_{j_1} a_{j_2} \ldots a_{j_\nu}|x) = \int_b^x \frac{L_b(a_{j_1} \ldots a_{j_{\nu-1}}|x)}{x - a_{j_\nu}}\, dx,$$

où b est un point fixe à distance finie, distinct des points $a_1, a_2, \ldots, a_m$. Ces fonctions seront nommées *hyperlogarithmes de la première espèce de la configuration* $a_1, a_2, \ldots, a_m$[1]). Chaque hyperlogarithme peut être traité

Let $(a_i)_{1\leq i\leq n}$ be a family of complex numbers (all different) and $z_0 \notin \{a_i\}_{1\leq i\leq n}$, then

Definition [Lappo-Danilevskij, 1928]

$$L(a_{i_1},\ldots,a_{i_n}|z_0 \overset{\gamma}{\rightsquigarrow} z) = \int_{z_0}^{z}\int_{z_0}^{s_n}\ldots\Big[\int_{z_0}^{s_1}\frac{ds}{s-a_{i_1}}\Big]\ldots\frac{ds_n}{s_n-a_{i_n}}.$$



Remarks

1. The result depends only on the homotopy class of the path and then the result is a holomorhic function on $\widetilde{B}$ ($B = \mathbb{C} \setminus \{a_1, \cdots, a_n\}$)
2. From the fact that they are holomorhic, we can also study them in an open (simply connected) subset like the cleft plane

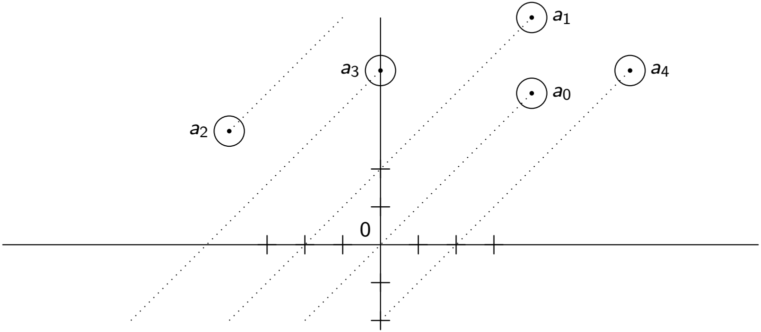


Figure: The cleft plane.

Remarks/2

3. The set of functions

$$\alpha_{z_0}^{z}(\lambda) = L(a_{i_1}, \ldots, a_{i_n} | z_0 \overset{\gamma}{\rightsquigarrow} z) \text{ ; or "1" if the list is void.}$$

has a lot of nice combinatorial properties

- Noncommutative ED with left multiplier
- Linear independence
- Shuffle property
- Factorisation
- Possiblity of left or right multiplicative renormalization at a neighbourhood of the singularities
- Extension to rational functions

Coding by words

In order to use the rich allowance of notations invented by algebraists, computer scientists, combinatorialists and physicists about noncommutative power series we first (classically) code the lists by words.

$$(a_{i_1}, \ldots, a_{i_n}) \to w = x_{i_1} \ldots x_{i_n}$$

Lappo-Danilevskij recursion is from left to right, we will use here right to left indexing to match with[1]

1. P. Cartier, *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, Volume 30 (1987-1988) , Talk no. 687 , p. 31-52

2. V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.

3. H.J. Susmann, *A product expansion for Chen Series*, in Theory and Applications of Nonlinear Control Systems, C.I. Byrns and Lindquist (eds). 323-335, 1986

4. P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math, 163, Springer-Verlag (1970).

[1] Data structures are Letters (1,2), Vector fields (3) and Matrices (4).

Noncommutative generating series

Let X^* be the set of words constructed on the alphabet X. We now have a function $w \mapsto \alpha_{z_0}^{z}(w)$ which maps words to holomorphic functions on Ω. This is a noncommutative series of variables in X and coefficients in $\mathcal{H}(\Omega)$. It is convenient to use the "sum notation".

$$S = \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\, w$$

and it is not difficult to see that S is the unique solution of

$$\begin{cases} \mathbf{d}(S) & = & M.S \text{ with } M = \sum_{i=1}^{n} \frac{x_i}{z-a_i} \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{cases}$$

The series $S_{Pic}^{z_0}$

As S can be computed by Picard's process

$$S_0 = 1_{X^*} \ ; \ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n$$

we adopt the following notation

$$S_{Pic}^{z_0} = \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\, w$$

then, one has

Proposition

The series $S_{Pic}^{z_0}$ is the unique solution of

$$\begin{cases} \mathbf{d}(S) & = & M.S \text{ with } M = \sum_{i=1}^{n} \frac{x_i}{z-a_i} \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} \end{cases}$$

The complete set of solutions of $\mathbf{d}(S) = M.S$ is $S_{Pic}^{z_0}.\mathbb{C}\langle\langle X \rangle\rangle$.

Transition: Solutions as $\sqcup\!\sqcup$-characters with values in $\mathcal{H}(\omega)$

We have seen that (some) solutions of systems like that of Hyperlogarithms possess the shuffle property i.e. defining the shuffle product by the recursion

$$\begin{array}{rcl} u \sqcup\!\sqcup 1_{Y^*} & = & 1_{Y^*} \sqcup\!\sqcup u = u \text{ and} \\ au \sqcup\!\sqcup bv & = & a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) \end{array}$$

one has

$$\langle S_{Pic}^{z_0} | u \sqcup\!\sqcup v \rangle = \langle S_{Pic}^{z_0} | u \rangle \langle S_{Pic}^{z_0} | v \rangle \qquad (1)$$

(product in $\mathcal{H}(\omega)$).

Now it is not difficult to check that the characters of type (1) form a group (these are characters on a Hopf algebra, see below). I would be intersting to have at our disposal a system of local coordinates in order to perform estimates in neighbourhood of the singularities.

Examples of $ш_\varphi$

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Local coordinates on (infinite dimensional) Lie groups, factorization of Riemann zeta functions

3

3

Given a (finite dimensional) Lie group G (real $k = \mathbb{R}$ or complex $k = \mathbb{C}$) and its Lie algebra $\mathfrak{g}$, one can prove (a basis $B = (b_i)_{1 \le i \le n}$ of $\mathfrak{g}$ being given) that there exists a neighbourhood W of 1_G (in G) and n *local coordinate* analytic functions

$$W \to k,\ (t_i)_{1 \le i \le n}$$

such that, for all $g \in W$

$$(*) \quad g = \prod_{1 \le i \le n}^{\rightarrow} e^{t_i(g) b_i} = e^{t_1(g) b_1} e^{t_2(g) b_2} \ldots e^{t_n(g) b_n}$$

to see this, just remark that

$$(t_1, t_2, \cdots t_n) \to exp(t_1 b_1) exp(t_2 b_2) \cdots exp(t_n b_n)$$

is a *local* diffeomorphism from k^n to G in a neighbourhood of 0 and take the inverse.

This is the local **Wei-Norman's theorem**.

My questions are the following

> Let us loosely call *infinite dimensional* a Lie group whose Lie algebra is not finite dimensional (this includes the example below and infinite dimensional Banach-Lie groups for instance).
>
> Q1) Can you provide examples of infinite dimensional Lie groups where the exponential map

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8 hours ago - Aaron Mazel-Gee

Schützenberger's (MRS) factorisation

This MRS[2] factorisation is, in fact, a resolution of the identity. It reads as follows

Theorem (Schützenberger, 1958, Reutenauer, 1988)

Let $\mathcal{D}_X := \sum_{w \in X^*} w \otimes w$. *Then* $\mathcal{D}_X = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{S_l \otimes P_l}$.

where the product laws is the shuffle on the left and concatenation on the right, $(P_l)_{l \in \mathcal{L}yn(X)}$ is an homogeneous basis of $\mathcal{L}\rangle]\langle X\rangle$ and $(S_l)_{l \in \mathcal{L}yn(X)}$, the "Lyndon part" of the dual basis of $(P_w)_{w \in X^*}$ which, given that is formed by

$$P_{l_1}^{\alpha_1} \ldots P_{l_n}^{\alpha_n} \text{ where } w = l_1^{\alpha_1} \ldots l_n^{\alpha_n} \text{ with } l_1 > \ldots > l_n \text{ (lexorder)}$$

[2] after Mélançon, Reutenauer, Schützenberger

Applying MRS to a shuffle character

Now, remarking that this factorization lives within the subalgebra

$$Iso(X) = \{T \in R\langle\langle X^* \otimes X^* \rangle\rangle | (u \otimes v \in supp(T) \Longrightarrow |u| = |v|)\}$$

if Z is any shuffle character, one has

$$Z = (Z \otimes Id)(\sum_{w \in X^*} w \otimes w) = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\langle Z | S_l \rangle P_l}$$

We would like to get such a factorisation at our disposal for other types of (deformed) shuffle products, this will be done in the second part of the talk. Let us first, with this factorization (MRS) at hand, construct explicitely Drinfeld's solution G_0.

Definition of Drinfeld's G_0

We give here a computational example of an asymptotic construction.
In his paper (2. above), V. Drinfel'd states that there is a unique solution (called G_0) of

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{\substack{z\to 0\\ z\in\Omega}} S(z)e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{cases}$$

and a unique solution (called G_1) of

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{\substack{z\to 1\\ z\in\Omega}} e^{x_1 log(1-z)} S(z) = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{cases}$$

Let us give here, as an example, a construction of G_0 (G_1 can be derived or checked by symmetry see also Minh's talk).

Explicit construction of Drinfeld's G_0

Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\alpha_0^z(w)=\begin{cases} 1_\Omega & \text{if} \quad w=1_{X^*}\\ \int_0^z \alpha_0^s(w)\frac{ds}{1-s} & \text{if} \quad w=x_1u\\ \int_0^z \alpha_1^s(w)\frac{ds}{s} & \text{if} \quad w=x_0u \text{ and } |u|_{x_1}=0\\ \int_0^z \alpha_0^s(w)\frac{ds}{s} & \text{if} \quad w=x_0u \text{ and } |u|_{x_1}>0 \end{cases}$$

one can check that (a) all the integrals are well defined (b) the series $T=(\sum_{w\in X^*}\alpha_0^z(w)\,w)e^{-x_0 log(z)}$ satisfies the two sided evolution equation (TSM)

$$\mathbf{d}T=(\frac{x_0}{z}+\frac{x_1}{1-z}).T+T.(-\frac{x_0}{z})$$

and, together with the asymptotic condition, we will see that this implies that the solution is group-like and hence so is

$$(\sum \alpha_0^z(w)\,w)=:\mathrm{L}=Te^{x_0 log(z)}\ .$$

The main theorem

Theorem

Let

$$(TSM) \qquad \mathbf{d}S = M_1 S + S M_2 \ . \tag{2}$$

with $S \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle,\ M_i \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle$

(i) *Solutions of* (TSM) *form a* $\mathbb{C}$*-vector space.*

(ii) *Solutions of* (TSM) *have their constant term (as coefficient of* 1_{X^*}*) which are constant functions (on* Ω*); there exists solutions with constant coefficient* 1_Ω *(hence invertible).*

(iii) *If two solutions coincide at one point* $z_0 \in \Omega$ *(or asymptotically), they coincide everywhere.*

Theorem (cont'd)

(iv) *Let be the following one-sided equations*

$$(LM_1) \quad \mathbf{d}S = M_1 S \qquad (RM_2) \quad \mathbf{d}S = SM_2. \tag{3}$$

and let S_1 *(resp.* S_2*) be a solution of* (LM_1) *(resp.* (LM_2)*), then* S_1S_2 *is a solution of* (TSM)*. Conversely, every solution of* (TSM) *can be constructed so.*

(v) *Let* $S^{z_0}_{Pic,LM_1}$ *(resp.* $S^{z_0}_{Pic,RM_2}$*) the unique solution of* (LM_1) *(resp.* (RM_2)*) s.t.* $S(z_0) = 1_{\mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle}$ *then, the space of all solutions of* (TSM) *is*

$$S^{z_0}_{Pic,LM_1}.\mathbb{C}\langle\langle X\rangle\rangle.S^{z_0}_{Pic,RM_2}$$

(vi) *If* $M_i,\ i = 1,2$ *are primitive for* $\Delta_{ш}$ *and if* S*, a solution of* (TSM)*, is group-like at one point (or asymptotically), it is group-like everywhere (over* Ω*).*

Other combinatorial instances of MRS factorisation

Shuffle product governs Poly- and Hyper- logarithms, stuffle governs Harmonic functions and one can see that other forms of perturbated shuffles govern other types of special functions.
In combinatorics (and computer science), one often uses products[3] defined by recursions on words of the form

$$\begin{aligned} u \,ш_{\varphi} 1_{Y^*} &= 1_{Y^*} \,ш_{\varphi} u = u \text{ and} \\ au \,ш_{\varphi} bv &= a(u \,ш_{\varphi} bv) + b(au \,ш_{\varphi} v) + \varphi(a,b)(u \,ш_{\varphi} v) \end{aligned}$$

where $\varphi : R.X \otimes R.X \to R.X$ is some associative law.

[3] as shuffle, stuffle, infiltration, q-infiltration.

G.H.E. Duchamp, J.-Y. Enjalbert, H. N. Minh, C. Tollu, *The mechanics of shuffle products and their siblings*, Discrete Mathematics, **340** (Sep. 2017)

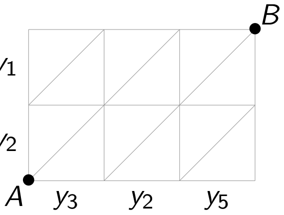
Examples of $\shuffle_\varphi$

Name	Formula (recursion)	φ	Type
Shuffle [21]	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$	$\varphi \equiv 0$	I
Stuffle [19]	$x_i u \stuffle x_j v = x_i(u \stuffle x_j v) + x_j(x_i u \stuffle v) + x_{i+j}(u \stuffle v)$	$\varphi(x_i, x_j) = x_{i+j}$	I
Min-stuffle [7]	$x_i u \ministuffle x_j v = x_i(u \ministuffle x_j v) + x_j(x_i u \ministuffle v) - x_{i+j}(u \ministuffle v)$	$\varphi(x_i, x_j) = -x_{i+j}$	III
Muffle [14]	$x_i u \muffle x_j v = x_i(u \muffle x_j v) + x_j(x_i u \muffle v) + x_{i\times j}(u \muffle v)$	$\varphi(x_i, x_j) = x_{i\times j}$	I
q-shuffle [3]	$x_i u \stuffle_q x_j v = x_i(u \stuffle_q x_j v) + x_j(x_i u \stuffle_q v) + q x_{i+j}(u \stuffle_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	III
q-shuffle$_2$	$x_i u \stuffle_q x_j v = x_i(u \stuffle_q x_j v) + x_j(x_i u \stuffle_q v) + q^{i.j} x_{i+j}(u \stuffle_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$	II
LDIAG$(1, q_s)$ [10] (non-crossed, non-shifted)	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v) + q_s^{\lvert a\rvert\lvert b\rvert} a.b(u \shuffle v)$	$\varphi(a,b) = q_s^{\lvert a\rvert\lvert b\rvert}(a.b)$	II
q-Infiltration [12]	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b} a(u \uparrow v)$	$\varphi(a,b) = q\delta_{a,b} a$	III
AC-stuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a,b)(u \shuffle_\varphi v)$	$\varphi(a,b) = \varphi(b,a)$ $\varphi(\varphi(a,b),c) = \varphi(a,\varphi(b,c))$	IV
Semigroup-stuffle	$x_t u \shuffle_\perp x_s v = x_t(u \shuffle_\perp x_s v) + x_s(x_t u \shuffle_\perp v) + x_{t\perp s}(u \shuffle_\perp v)$	$\varphi(x_t, x_s) = x_{t\perp s}$	I
φ-shuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a,b)(u \shuffle_\varphi v)$	$\varphi(a,b)$ law of AAU	V

Of course, the q-shuffle is equal to the (classical) shuffle when $q = 0$. As for the q-infiltration, when $q = 1$, one recovers the infiltration product defined in [6].

Many shuffle products arise in number theory when one studies polylogarithms, harmonic sums and polyzêtas; it was in order to study all these products

One can see the product $u \,\text{ш}_{\varphi}\, v$ as a sum indexed by paths (with right-up-diagonal(ne) steps) within the grid formed by the two words (u horizontal and v vertical, the diagonal steps corresponding to the factors $\varphi(a, b)$)



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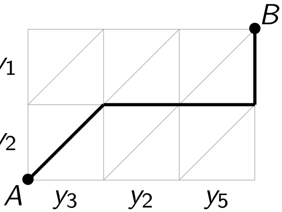
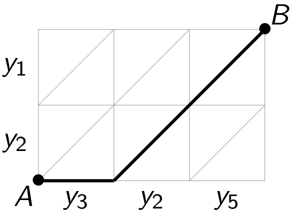
For example,

the path



reads $\varphi(y_3, y_2)y_2y_5y_1$

the path



reads $y_3\varphi(y_2,y_2)\varphi(y_5,y_1)$. We have the following

Theorem (Radford theorem for $ш_\varphi$)

Let R be a $\mathbb{Q}$-algebra (associative, commutative with unit) such that

$$\varphi : R\langle X\rangle \otimes R\langle X\rangle \to R\langle X\rangle$$

is associative.

If X is totally ordered by $<$, then $(\mathcal{L}yn(X)^{ш_\varphi \alpha})_{\alpha\in\mathbb{N}^{(\mathcal{L}yn(X))}}$ is a linear basis of $R\langle X\rangle$.

In particular, if moreover φ is commutative, then $(R\langle X\rangle, ш_\varphi, 1_{X^*})$ is a polynomial algebra with $\mathcal{L}yn(X)$ as transcendence basis.

Bialgebra structure

Theorem

Let R be a commutative ring (with unit). We suppose that the product φ is associative and commutative, then the algebra $(R\langle X\rangle, ш_{\varphi}, 1_{X^*})$ can be endowed with the comultiplication Δ_{conc} dual to the concatenation

$$\Delta_{conc}(w) = \sum_{uv=w} u \otimes v \tag{4}$$

and the "constant term" character $\varepsilon(P) = \langle P|1_{X^*}\rangle$.

(i) *With this setting, we have a bialgebra* [a].

$$\mathcal{B}_{\varphi} = (R\langle X\rangle, ш_{\varphi}, 1_{X^*}, \Delta_{conc}, \varepsilon) \tag{5}$$

(ii) *The bialgebra (eq. 6) is, in fact, a Hopf Algebra.*

[a] Commutative and, when $|X| \geq 2$, noncocommutative.

Dualizability

If one considers φ as defined by its structure constants

$$\varphi(x,y)=\sum_{z\in X}\gamma_{x,y}^{z}\,z$$

one sees at once that $ш_\varphi$ is dualizable within $R\langle X\rangle$ iff the tensor $\gamma_{x,y}^{z}$ is locally finite in its contravariant place "z" i.e.

$$(\forall z\in X)(\#\{(x,y)\in X^2|\gamma_{x,y}^{z}\neq 0\}<+\infty)\ .$$

Remark

Shuffle, stuffle, infiltration are dualizable. The comultiplication associated with Generalized Lerch Functions and $\top$ are not (see HNM's talk).

Dualizability/2

In case $\shuffle_\varphi$ is dualizable, one has a comultiplication

$$\Delta_{\shuffle_\varphi} : R\langle X\rangle \to R\langle X\rangle \otimes R\langle X\rangle$$

(with structure constants the transpose of the tensor $\gamma_{x,y}^{z}$). The following

$$\mathcal{B}_\varphi^\vee = (R\langle X\rangle, conc, 1_{X^*}, \Delta_{\shuffle_\varphi}, \varepsilon) \tag{6}$$

is a bialgebra in duality with $\mathcal{B}_\varphi$ (not always a Hopf algebra although the letter was so → ex. $\shuffle_\varphi = \uparrow_q$ i.e. the q-infiltration).

Associative commutative φ-deformed shuffle products

Theorem (CAP 2015)

Let us suppose that φ is associative and dualizable. We still denote the dual law of $ш_\varphi$ by $\Delta_{ш_\varphi} : RY \longrightarrow RY \otimes RY$, $\mathcal{B}_\varphi^\vee := (RY, \mathrm{conc}, 1_{Y^*}, \Delta_{ш_\varphi}, \varepsilon)$ is a bialgebra. Moreover, if φ is commutative the following conditions are equivalent

i) $\mathcal{B}_\varphi^\vee$ *is an enveloping bialgebra.* (**CQMM** *theorem*)

ii) $\mathcal{B}_\varphi^\vee$ *is isomorphic to $(RY, \mathrm{conc}, 1_{Y^*}, \Delta_ш, \epsilon)$ as a bialgebra.*

iii) *For all $y \in Y$, the following series is a polynomial.*

$$\pi_1(y) = y + \sum_{l \geq 2} \frac{(-1)^{l-1}}{l} \sum_{x_1,\ldots,x_l \in Y} \langle y | \varphi(x_1 \ldots x_l) \rangle \, x_1 \ldots x_l.$$

In the previous equivalent cases, φ is called moderate.

In this case, one can straighten the $ш_\varphi$ product and imitate Lyndon basis computation in order to get a basis of the primitive elements and then have an effective calculus for Schützenberger factorisation.

Linear independence without monodromy

Theorem

Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit $(ch(k) = 0)$ and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS \; ; \; \langle S|1 \rangle = 1 \tag{7}$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle \; . \tag{8}$$

The following conditions are equivalent :

1. *The family $(\langle S|w \rangle)_{w \in X^*}$ of coefficients of S is free over $\mathcal{C}$.*

Linear independence without monodromy cont'd

Theorem (cont'd)

2. *The family* $(\langle S|w\rangle)_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.
3. *The family of coefficients* $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.
4. *The family* $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$

$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{9}$$

5. *The family* $(u_x)_{x\in X}$ *is free over* k *and*

$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{10}$$

Linear independence without monodromy: theorem

Theorem (DDMS, 2011)

Let $S \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ be a solution of the (LM) equation

$$\mathbf{d}(S) = MS \;;\; \langle S|1_{X^*}\rangle = 1.$$

The following are equivalent :

i) *the family $(\langle S|w\rangle)_{w\in X^*}$ of coefficients is independant (linearly) over $\mathcal{C}$.*

ii) *the family of coefficients $(\langle S|x\rangle)_{x\in X\cup\{1_{X^*}\}}$ is independant (linearly) over $\mathcal{C}$.*

iii) *the family $(u_x)_{x\in X}$ is such that, for $f \in \mathcal{C}$ et $\alpha_x \in \mathbb{C}$*

$$\mathrm{d}(f) = \sum_{x\in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0).$$

Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics, MD, G. H. E. Duchamp V. Hoang Ngoc Minh and A. I. Solomon, **4th International Conference on Algebraic Informatics**, Linz (2011), Proceedings, Lecture Notes in

Conclusion

- For Series with variable coefficients, we have a theory of Noncommutative Evolution Equation sufficiently powerful to cover iterated integrals and multiplicative renormalisation
- MRS factorisation provides an analogue of the (local) theorem of Wei-Norman and allows to remove singualrities with simple counterterms
- MRS factorisation can be performed in many other cases (like stuffle for harmonic functions)
- Use of combinaorics on words gives a necessary and sufficient condition on the "inputs" to have linear independance of the solutions over higher function fields.