Periods in action

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IHES

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What is a period?

A **period** is the integral on a closed path of a rational function in one or several variables with *rational* coefficients.

"Rational coefficients" may mean

- coefficients in $\mathbb{Q}$
- coefficients in $\mathbb{C}(t)$, **the period is a function of** t.
  This is what we will be interested in.

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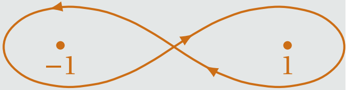

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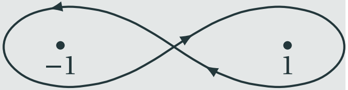
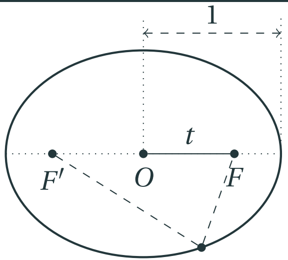


An ellipse

eccentricity	t
major radius	1
perimeter	$E(t)$



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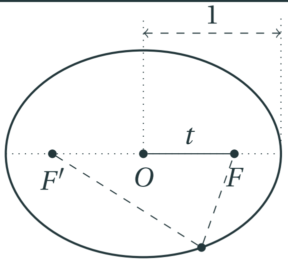
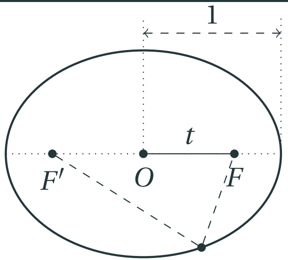


$$E(t) = 2\int_{-1}^{1} \sqrt{\frac{1 - t^2 x^2}{1 - x^2}}\,\mathrm{d}x$$

Periods with a parameter



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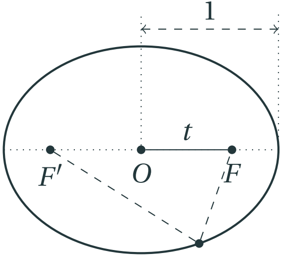

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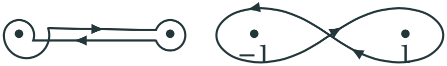

Euler (1733) $(t-t^3)E'' + (1-t^2)E' + tE = 0$

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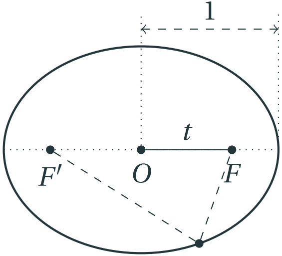

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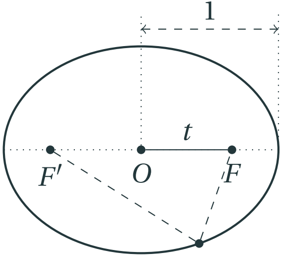

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since then Many applications in algebraic geometry (Gauß-Manin connection)

geometry of the cycles ↔ analytic properties of the periods

Content

1. Computing periods
2. Multiple binomial sums
3. Volume of semialgebraic sets

Computing periods

Differential equations as a data structure

Representation of algebraic numbers

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explicit $\sqrt{5+2\sqrt{6}}$ (also $\sqrt{2}+\sqrt{3}$)

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An example by Bostan, Chyzak, van Hoeij, and Pech (2011)

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explicit $1 + 6 \cdot \int_0^t \frac{{}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\,\middle|\, \frac{27w(2-3w)}{(1-4w)^3}\right)}{(1-4w)(1-64w)} \mathrm{d}w$

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implicit $$t(t-1)(64t-1)(3t-2)(6t+1)y''' + (4608t^4 - 6372t^3 + 813t^2 + 514t - 4)y'' + 4(576t^3 - 801t^2 - 108t + 74)y' = 0$$ (+ init. cond.)

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Differential equations as a data structure

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- addition, multiplication, composition with algebraic functions
- power series expansion
- **equality testing**, given differential equations and initial condtions
- **numerical analytic continuation** with certified precision

  (D. V. Chudnovsky and G. V. Chudnovsky 1990; van der Hoeven 1999; Mezzarobba 2010)

```
sage: from ore_algebra import *
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz
sage: dop.numerical_solution(ini=[0,1], path=[0,1])
                    [0.7853981633974483 +/- 1.08e-18]
sage: dop.numerical_solution(ini=[0,1], path=[0,i+1,2*i,i-1,0,1])
          [3.926990816987242 +/- 4.81e-17] + [+/- 4.63e-21]*I
```

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One equation fits all cycles, the **Picard-Fuchs equation**.

recall $E(t)=\oint\sqrt{\frac{1-t^2x^2}{1-x^2}}\mathrm{d}x=\frac{1}{2\pi i}\oint\overbrace{\frac{1}{1-\frac{1-t^2x^2}{(1-x^2)y^2}}}^{R(t,x,y)}\mathrm{d}x\mathrm{d}y$

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Computational proof

$$(t-t^3)\frac{\partial^2 R}{\partial t^2} + (1-t^2)\frac{\partial R}{\partial t} + tR = \frac{\partial}{\partial x}\left(-\frac{t(-1-x+x^2+x^3)y^2(-3+2x+y^2+x^2(-2+3t^2-y^2))}{(-1+y^2+x^2(t^2-y^2))^2}\right) + \frac{\partial}{\partial y}\left(\frac{2t(-1+t^2)x(1+x^3)y^3}{(-1+y^2+x^2(t^2-y^2))^2}\right)$$

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Problem (mostly) solved!

Multiple binomial sums

joint work with Alin Bostan and Bruno Salvy

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{(n!)^3} \qquad \text{(Dixon)}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \qquad \text{(Strehl)}$$

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2$$

$$\sum_{r \geqslant 0} \sum_{s \geqslant 0} (-1)^{n+r+s} \binom{n}{r} \binom{n}{s} \binom{n+s}{s} \binom{n+r}{r} \binom{2n-r-s}{n} = \sum_{k \geqslant 0} \binom{n}{k}^4$$

$$\sum_i \sum_j \binom{2n}{n+i}\binom{2n}{n+j} \left| i^3 j^3 (i^2 - j^2) \right| = \frac{2n^4(n-1)(3n^2-6n+2)}{(2n-3)(2n-1)} \binom{2n}{n}^2$$

Conjectured by Brent, Ohtsuka, Osborn, and Prodinger (2014)

$$1 + F_n^{-1,-1} + 2F_n^{0,0} - F_n^{0,1} + F_n^{1,0} - 3F_n^{1,1} + F_n^{1,2} - F_n^{3,1} + 3F_n^{3,2}$$
$$- F_n^{3,3} - 2F_n^{4,2} + F_n^{4,3} - F_n^{5,2} = \sum_{m=0}^{n} \frac{\binom{n+2}{m}\binom{n+2}{m+1}\binom{n+2}{m+2}}{\binom{n+2}{1}\binom{n+2}{2}},$$

where $F_n^{a,b} = \sum_{d=0}^{n-1} \sum_{c=0}^{d-a} \binom{d-a-c}{c}\binom{n}{d-a-c}\left(\binom{n+d+1-2a-2c+2b}{n-a-c+b} - \binom{n+d+1-2a-2c+2b}{n+1-a-c+b}\right)$

Conjectured by Le Borgne

Both proved using periods!

The not so formal grammar of binomial sums

$\bigcirc \to$ integer linear combination of the variables

$\square \to \binom{\bigcirc}{\bigcirc}$ $\quad$ $\square \to \mathrm{Cst}^{\bigcirc}$

$\square \to \square + \square$ $\quad$ $\square \to \square \cdot \square$ $\quad$ $\square \to \sum_{n=\bigcirc}^{\bigcirc} \square$

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summation $y(t) = \frac{1}{(2i\pi)^3} \oint \frac{\left(x_1 x_2 x_3 - t \prod_{i=1}^3 (1+x_i)^2\right) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3}{\left(x_1^2 x_2^2 x_3^2 - t \prod_{i=1}^3 (1+x_i)^2\right)\left(1 - t \prod_{i=1}^3 (1+x_i)^2\right)}$

where $y(t)$ is the generating function of the l.h.s.

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simplification $\displaystyle y(t) = \frac{1}{(2i\pi)^2} \oint \frac{x_1 x_2 \mathrm{d}x_1 \mathrm{d}x_2}{x_1^2 x_2^2 - t(1+x_1)^2(1+x_2)^2(1-x_1 x_2)^2}$

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integration $t(27t+1)y'' + (54t+1)y' + 6y = 0$, i.e. $3(3n+2)(3n+1)u_n + (n+1)^2 u_{n+1} = 0$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3}$$

basic block $\displaystyle \binom{n}{k} = \frac{1}{2\pi i} \oint \frac{(1+x)^n}{x^k} \frac{\mathrm{d}x}{x}$

product $\displaystyle \binom{2n}{k}^3 = \frac{1}{(2\pi i)^3} \oint \frac{(1+x_1)^{2n}}{x_1^k} \frac{(1+x_2)^{2n}}{x_2^k} \frac{(1+x_3)^{2n}}{x_3^k} \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \frac{\mathrm{d}x_3}{x_3}$

summation $\displaystyle y(t) = \frac{1}{(2i\pi)^3} \oint \frac{\left(x_1 x_2 x_3 - t \prod_{i=1}^3 (1+x_i)^2\right) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3}{\left(x_1^2 x_2^2 x_3^2 - t \prod_{i=1}^3 (1+x_i)^2\right)\left(1 - t \prod_{i=1}^3 (1+x_i)^2\right)}$

where $y(t)$ is the generating function of the l.h.s.

simplification $\displaystyle y(t) = \frac{1}{(2i\pi)^2} \oint \frac{x_1 x_2 \mathrm{d}x_1 \mathrm{d}x_2}{x_1^2 x_2^2 - t(1+x_1)^2(1+x_2)^2(1-x_1x_2)^2}$

integration $t(27t+1)y'' + (54t+1)y' + 6y = 0$, i.e. $3(3n+2)(3n+1)u_n + (n+1)^2 u_{n+1} = 0$

conclusion *Generating functions of binomial sums are periods!*

Computing binomial sums with periods

- Many related works on multiple sums (Chyzak, Egorychev, Garoufalidis, Koutschan, Sun, Wegshaider, Wilf, Zeilberger, etc)

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- ~~“This approach, while it is explicit in principle, in fact yields an infeasible algorithm.”~~ —Wilf and Zeilberger, 1992
- Excellent running times, thanks to **simplification** and better algorithms for **integration**

Binomial sums are diagonals of rational functions

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Theorem (Bostan, Lairez, and Salvy 2016)

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  The converse does not hold, but...
- If $(u_n)_{n\geqslant 0}$ is a binomial sum, then $\sum_n u_n t^n$ is algebraic modulo p for all prime p (but finitely many).

$$y(t) \triangleq \sum_n \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 t^n$$

$$= \operatorname{diag} \frac{1}{(1-x-y)(1-z-w) - wxyz} \qquad \text{(Straub 2014)}$$

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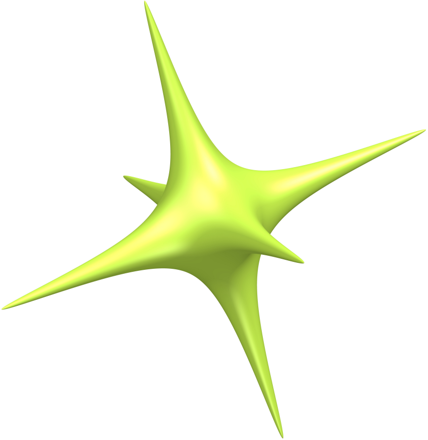
and of course $t^2(t^2-34t+1)y''' + 3t(2t^2-51t+1)y'' + (7t^2-112t+1)y' + (t-5)y = 0.$

Volume of semialgebraic sets

joint work with Mohab Safey El Din

A numeric integral

$$\{x^2 + y^2 + z^2 \leqslant 1 - 2^{10}\left(x^2 y^2 + y^2 z^2 + z^2 x^2\right)\}$$



What is the volume of this shape?

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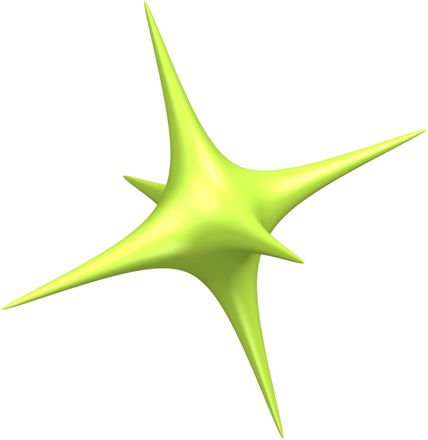

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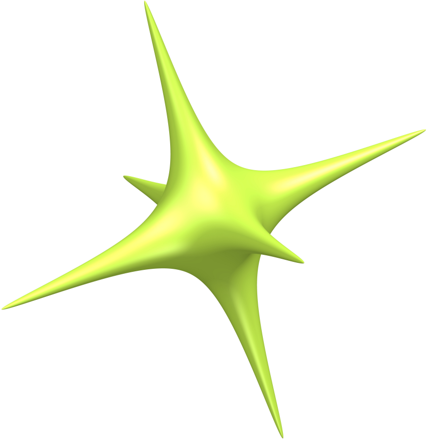

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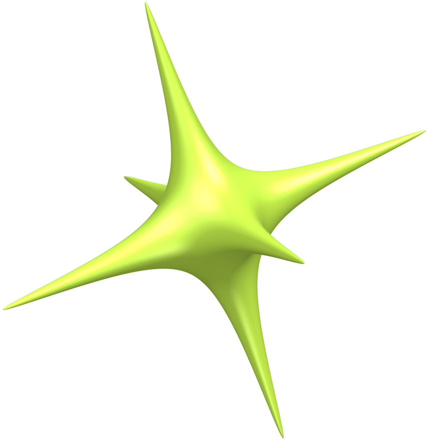

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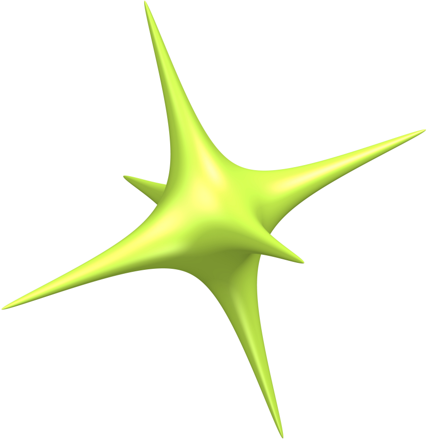

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Volumes are periods

Proposition

For any generic $f \in \mathbb{R}[x_1, \ldots, x_n]$,

$$\operatorname{vol}\{f \leqslant 0\} \triangleq \int_{\{f \leqslant 0\}} \mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{1}{2\pi i} \oint_{\mathrm{Tube}\{f=0\}} \frac{x_1}{f} \frac{\partial f}{\partial x_1} \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

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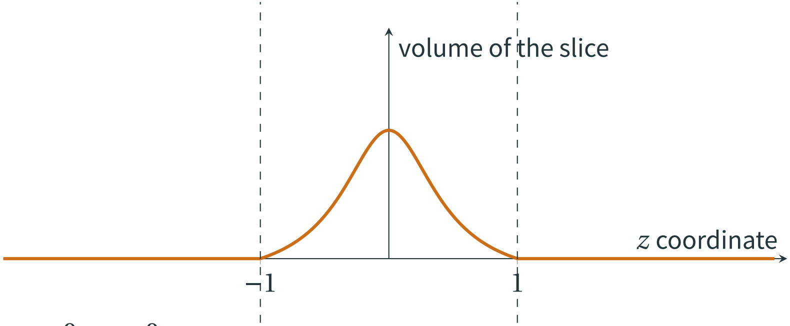
NB. $\operatorname{vol}\{f \leqslant 0\} = \int_{-\infty}^{\infty} \operatorname{vol}\{f \leqslant 0\} \cap \{x_n = t\} \mathrm{d}t$

The “volume of a slice” function

$\{y_1, y_2\}$, basis of the solution space of the Picard-Fuchs equation



$0 \cdot y_1 + 0 \cdot y_2$

$1.0792353\ldots \cdot y_1 - 40.100605\ldots \cdot y_2$

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input $f \in \mathbb{R}[x_1, \ldots, x_n]$ generic

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The complexity is quasi-linear with respect to the precision!

(To get twice as many digits, you need only twice as much time.)

A hundred digits (within a minute)

vol () = 0.108575421460360937739503
395994207619810917874446
607475444475822993285360
673032928194943474414064
066136624234627959808778
1034932346781568...

A hundred digits (within a minute)

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395994207619810917874446
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066136624234627959808778
103493234678156 8...

Questions?

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