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AERATED POISSON DISTRIBUTIONS AND THEIR CONTINUOUS APPROXIMANTS

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Plan

1) Combinatorial Numbers from Normal Ordering of Differential Operators (or Bosons)
2) Generalized Bell numbers as Stieltjes Moments
   a) continuous weights (*)
   b) discrete weights (**)
3) Aerated Poisson distributions
4) Their exact approximants
5) Problem of unicity of solutions

Collaborations on various related subjects:
P. Blasiak, G. Duchamp, K. Górska, A. Horzela, H. N. Minh

Normal Ordering of Differential Operators

(s. P. Blasiak, K. A. Penson & A. I. Solomon
Annals of Combinatorics 7, 127 (2003))

$$\left[\frac{d}{dx}, x\right] = 1 \longleftrightarrow [a, a^\dagger] = 1 \text{ (bosons)}$$

Definition: $r \geq s$ — integers

$$\left[x^r \frac{d^s}{dx^s}\right]^n = x^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n,k)\, x^k \left(\frac{d}{dx}\right)^k$$

normally ordered form

Once the integers $S_{r,s}(n,k)$ are known the normal order has been achieved

$$B_{r,s}(n) = \sum_{k=s}^{n \cdot s} S_{r,s}(n,k)$$

generalised Bell numbers

generalised Stirling numbers of second kind.

For all $r \geq s$ the closed form expressions for these numbers exist.

Our interest: generalised Bell numbers

$r \geq s$ integers:

• $\underline{r > s}$

$$B_{r,s}(n) = \frac{(r-s)^{s(n-1)}}{e} \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{j=1}^{s} \frac{\Gamma\left(n + \frac{k+j}{r-s}\right)}{\Gamma\left(1 + \frac{k+j}{r-s}\right)}$$

$n = 0, 1, \ldots$

•• $\underline{r = s} > 1$

$$B_{r,r}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{(k+r)!}{k!}\right]^{n-1}$$

••• $r = s = 1$

$$B_{1,1}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} \leftarrow \text{standard Bell no.}$$

Generalized Dobiński Relations:

Representations as infinite series (not power series)

Let us calculate some of these sequences for $\boxed{n=1\ldots.}$

$B_{1,1}(n) = 1, 2, 5, 15, 52, 203, 877, 4140\ldots$

A000110 (OEIS)

$B_{2,1}(n) = 1, 3, 13, 73, 501, 4051, 37633\ldots$

A000262 (OEIS)

$B_{2,2}(n) = 1, 7, 87, 1657, 43833, 1515903\ldots$

A020556 (OEIS)

$B_{3,3}(n) = 1, 34, 2971, 513559, 149670844, \ldots$

A069223 (OEIS)

⋮

OEIS: Online Encyclopedia of Integer Sequences (N.J.A. Sloane)

https://oies.org/

For all $r \geq s$ both $B_{r,s}(n)$ & $S_{r,s}(n,K)$ have representations in terms of generalized hypergeometric function

$${}_pF_q([\text{of } p \text{ "upper" par.}], [\text{of } q \text{ "lower" par.}], x)$$

list list

Example: $r = s = 1$

"Standard" Bell(n) & S2(n, K)

$$\underline{B_{1,1}(n)} \equiv \mathrm{Bell}(n) = \qquad (n = 1, 2, \dots)$$

$$= e^{-1} {}_{n-1}F_{n-1}([\underbrace{2,2,\dots 2}_{n-1}], [\underbrace{1,1,\dots 1}_{n-1}], 1)$$

$$B_{1,1}(0) \equiv 1$$

$$\underline{S_{1,1}(n,K)} = \mathrm{Stirling2}(n,K) =$$

$$= \frac{(-1)^{K+1}}{(K-1)!} {}_nF_{n-1}([-K+1, \underbrace{2,2\dots 2}_{n-1}], [\underbrace{1,1,\dots 1}_{n-1}], 1)$$

$$1 \leq K \leq n$$

... + more involved for $r, s > 1$

Potential Interest for Probability ?
(Flajolet, Bostan...)

All $B_{r,s}(n)$ are positive definite, i.e. are n-th moments of positive functions on positive half-axis !!!

$$B_{r,s}(n) = \int_0^\infty x^n \cdot W_{r,s}(x)\,dx \qquad (*)$$

positive function (continuous or discrete)

How to obtain the "weights" $W_{r,s}(x)$? We have to solve the Stieltjes moment problem.

$W_{r,s}(x)$ are explicitly constructed:

$r > s$ $\Downarrow$ $W_{r,s}(x)$ continuous

$r = s$ $\Downarrow$ $W_{r,s}(x)$ discrete

by construction

Solution of the Stieltjes moment problem through the inverse Mellin transform, via Meijer G-functions

Integers: r, s , $r > s$

Known

$$W_{r,s}(x) = G^{s,0}_{0,s}\left(\frac{x}{(r-s)^s}\,\middle|\, \begin{matrix} \text{—} \\ \left\{\frac{j}{r-s}\right\}_{j=1}^{s} \end{matrix}\right) \cdot \frac{1}{x e (r-s)^s} \, * \, \underbrace{\sum_{k=0}^{\infty} \frac{\left[x(r-s)^{-s}\right]^{\frac{k}{r-s}}}{k!} \frac{1}{\prod_{j=1}^{s} \Gamma\left(1+\frac{k+j}{r-s}\right)}}_{\text{power series}}$$

- Can be tackled using Computer Algebra Systems
- Rapid convergence in x.
- Assumptions about r, s integers can be relaxed.....!

... but $r > s$ always

Example : $n = 1, 2, \ldots$

$$B_{2,1}(n) = 1,\ 3,\ 13,\ 73,\ 501,\ 4051 \ldots$$

$$B_{2,1}(n) = \int_0^{\infty} x^n \underbrace{\left[\exp(-x-1) \cdot \frac{I_1(2\sqrt{x})}{\sqrt{x}} \right]}_{W_{2,1}(x)} dx$$

e^{-1}

```
> plot(exp(-x-1)*BesselI(1,2*sqrt(x))/sqrt(x),x=0.00..6,axes=boxed);
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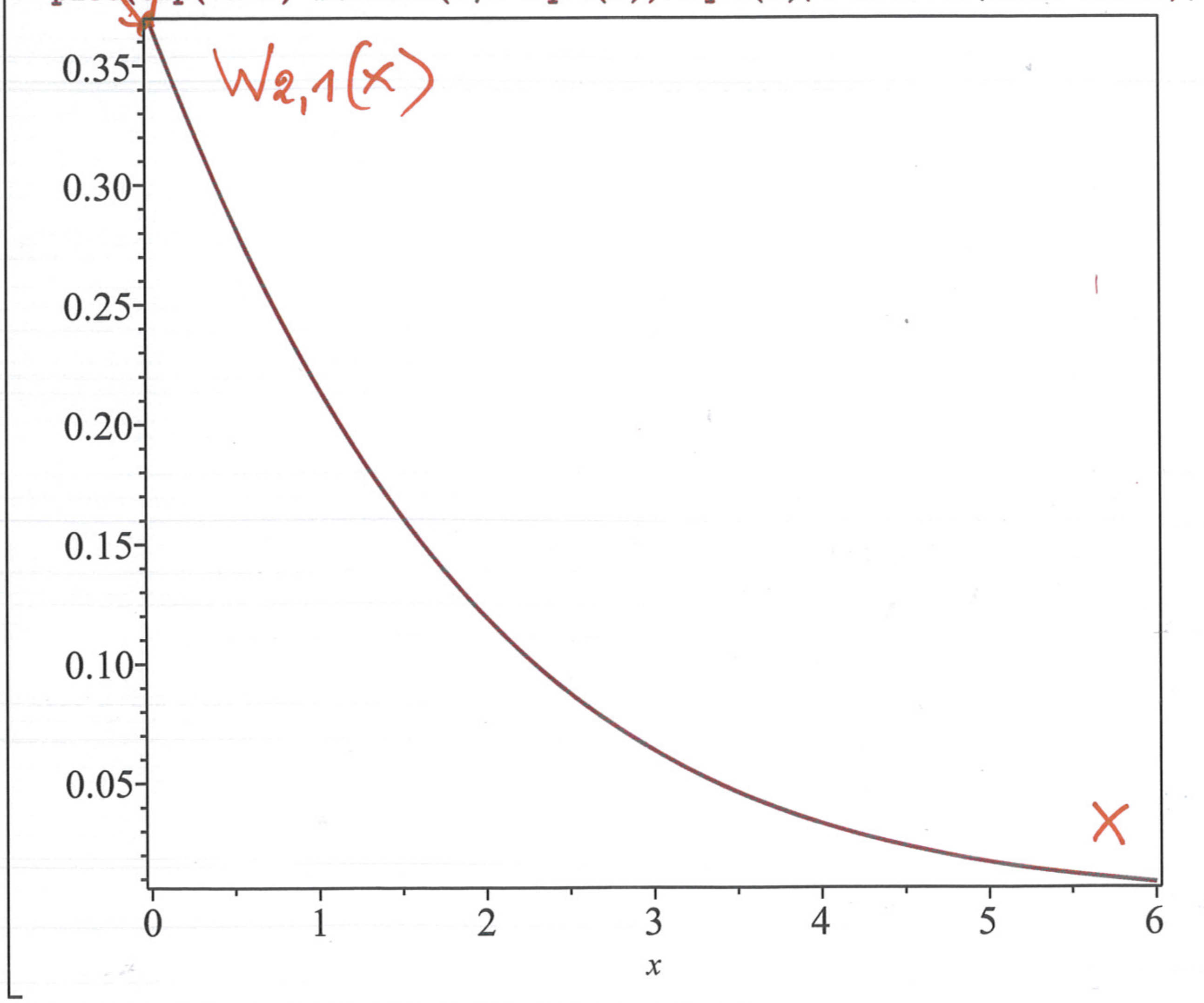


Problem: $B_{2,1}(0) = \frac{1}{e} < 1 \ldots$

Problem with $n=0$ moments:

$$B_{r,s}(n=0) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+s+1)} =$$

$$= 1 - \frac{\Gamma(s,1)}{\Gamma(s)} < 1 \quad !!!$$

$\Gamma(s,1) =$ incomplete gamma function

$$= \int_1^{\infty} e^{-t} t^{s-1}\, dt$$

$n=0$ moment $\longrightarrow$ normalisation of probability

is not OK ...

A) The deficit of probability does not depend on r;

B) $\lim_{s\to\infty} \frac{\Gamma(s,1)}{\Gamma(s)} = 1$

Solution of the deficit $n=0$
(missing probability)

1. Combinatorists:

   Set by definition $B_{r,s}(n=0) \equiv 1$

2. Probabilists:

   Compensate by adding an extra term in the weight $W_{r,s}(x)$:

$$\frac{\Gamma(s,1)}{\Gamma(s)}\,\delta(x=0)$$

$\Longrightarrow$ For the combinatorial numbers in question there is always a Dirac delta pic at $x=0$.

Conventional Poisson distribution

Let $\lambda > 0$, X_λ is the random variable with the Poisson distribution $(\lambda > 0)$

$$\left[P(X_\lambda = K) = e^{-\lambda} \frac{\lambda^K}{K!} \underset{\lambda \equiv 1}{=} \frac{1}{e} \frac{1}{K!} \quad \text{for } K = 0, 1, \ldots \right]$$

Therefore, as a function of $\underline{x}$ this is the discrete distribution

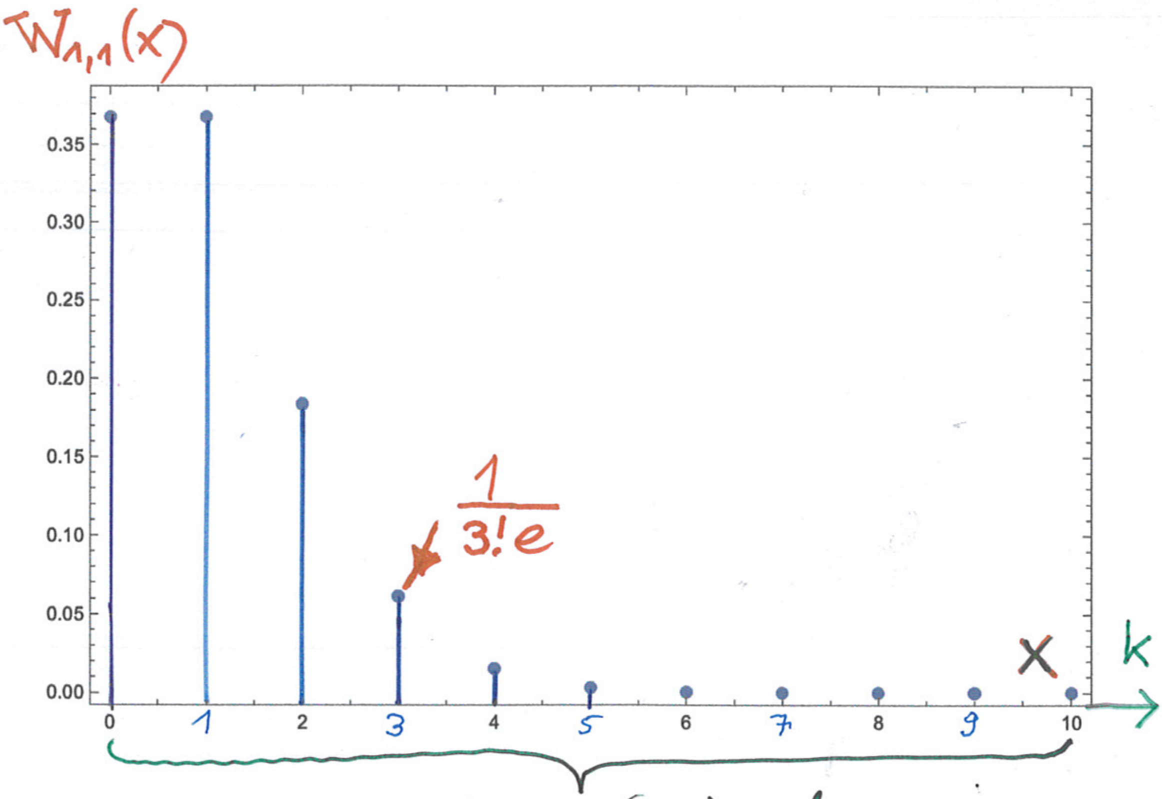
$$W_{1,1}(x) = \frac{1}{e} \sum_{K=0}^{\infty} \frac{\delta(x-K)}{K!}$$

with moments

$$\int_0^\infty x^n W_{1,1}(x)\,dx = \frac{1}{e} \sum_{K=0}^{\infty} \frac{K^n}{K!} \equiv B_{1,1}(n)$$

For $\lambda = 1$, the moments of Poisson distributions are the Bell numbers.

Discrete Poisson Distribution



Equidistant (all) integers

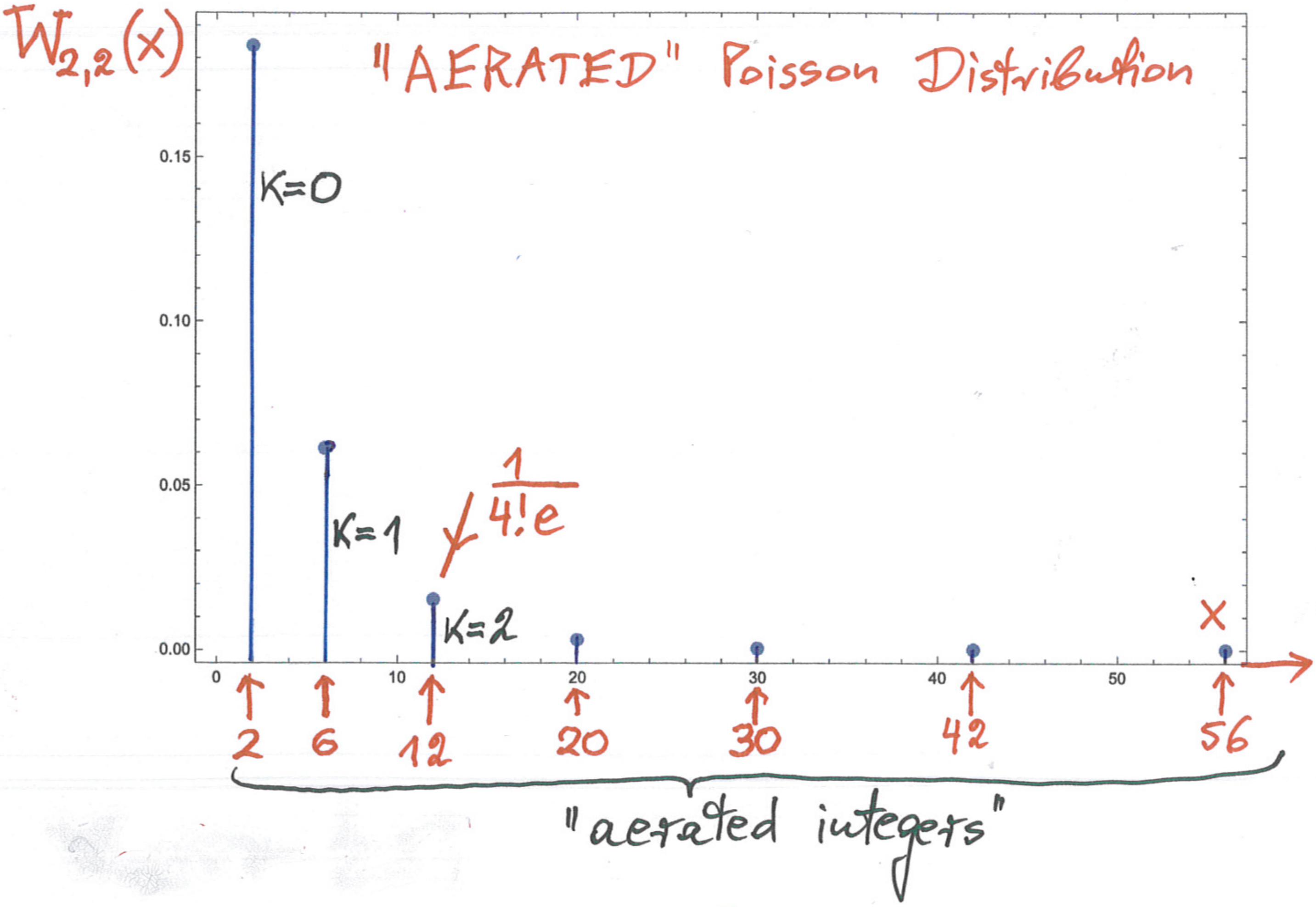
$$W_{1,1}(x) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{\delta(x-k)}{k!}$$

$$\int_0^{\infty} x^n\left[\frac{1}{e}\sum_{k=0}^{\infty}\frac{\delta(x-k)}{k!}\right]dx = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!} = B_{1,1}(n)$$

(Dobiński formula)

Normal order of $\left(x\frac{d}{dx}\right)^n$ furnishes Poisson distribution whose n-th moment is equal $B_{1,1}(n) = \mathrm{Bell}(n) = 1, 2, 15, 52\ldots$

$$\left(x^2 \frac{d^2}{dx^2}\right)^n \to B_{2,2}(n)$$



$$W_{2,2}(x) = \frac{1}{e} \sum_{K=0}^{\infty} \frac{1}{(K+2)!} \delta\big(x - (K+1)(K+2)\big)$$

Dirac Delta pics situated on points $(K+1)(K+2)$

$K = 0, 1, \ldots$

with strenght

$$\frac{1}{e(K+2)!}$$

$$\vdots$$

$$W_{3,3}(x) = \frac{1}{e} \sum_{K=0}^{\infty} \frac{1}{(K+3)!} \delta\big(x - (K+1)(K+2)(K+3)\big)$$

etc.

$W(1,1,x) \longrightarrow$ Delta pics at $x = 0, 1, 2, \ldots$
Poisson distribution

Approximating by $1+\frac{1}{n^2}$

$W(1+\frac{1}{n^2}, 1, x)$ for $n = 5, 6, 7, 8$; <u>continuous</u>

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> plot([seq(Wmod(1+1/n^2,1,x,24),n=5..8)],x=0..6,color=[red, blue,
  green,magenta,black],axes=boxed);
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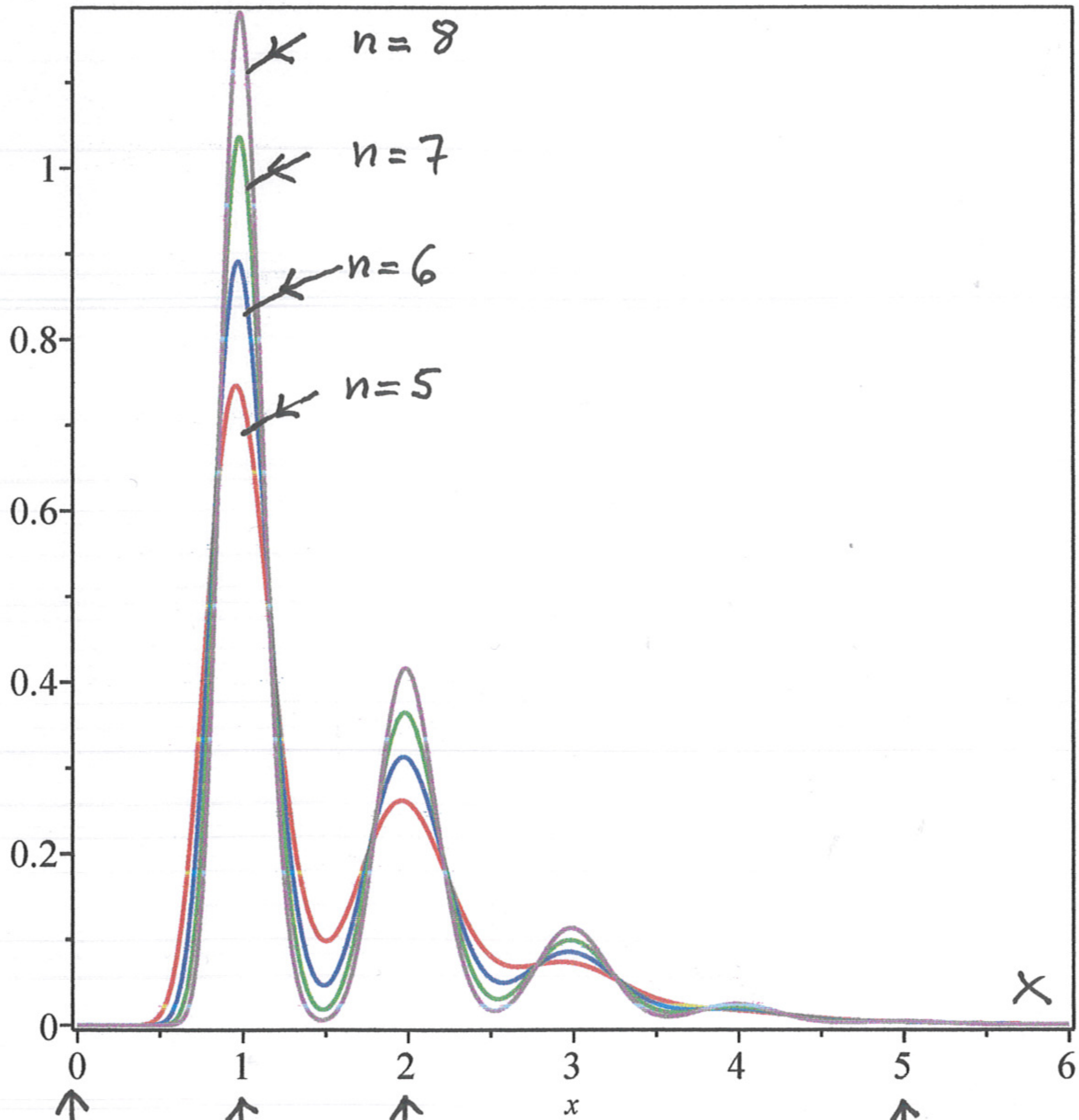
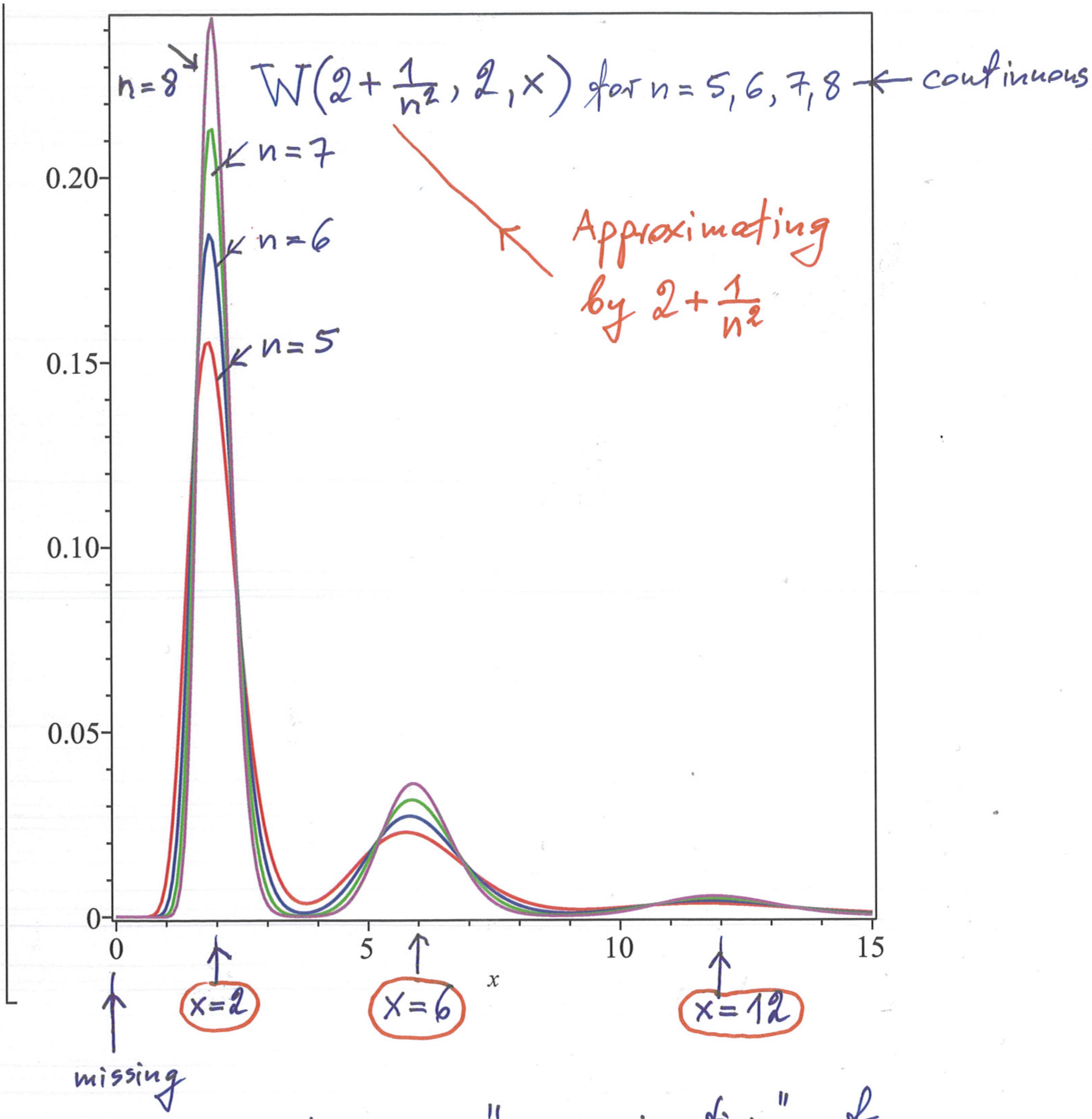


missing

$\Longrightarrow$ EXACT continuous "approximation" of classical Poisson distribution

$W(2,2,x) \to$ Delta pics at $x = 2, 6, 12, \ldots$



$\Rightarrow$ EXACT continuous "approximation" of discrete "aerated" Poisson distribution

Question of unicity of these distributions:

A) Aerated moments grow very strongly
→ conditions for unicity of discrete distribution ?

B) Unicity of continuous distributions
$W_{r,s}(x)$, $r > s$:

Krein Criterion: (sufficient)

if $$\int_0^\infty \frac{-\ln[W_{r,s}(x^2)]}{1+x^2} < \infty \quad (*)$$

$\Longrightarrow W_{r,s}(x)$ non-unique (**)

Only numerical check of (*) possible

C) Generation of non-unique solutions.
(Some ideas under study...)